PHYSICAL REVIEW A VOLUME 29, NUMBER 3 MARCH 1984

Time-dependent behavior of one-dimensional many-fermion models: Comparison with two- and three-dimensional models

M. Howard Lee
Department of Physics, University of Georgia, Athens, Georgia 30602

J. Hong
Department of Physics Education, Seoul National University, Seoul 151, Korea

N. L. Sharma*
Department of Physics, University of Georgia, Athens, Georgia 30602
(Received 28 September 1983)

Time-dependent behavior of one-dimensional (1D) many-fermion models is obtained by a method of recurrence relations. The Hilbert space of the density-fluctuation operator is two dimensional ($2d$), resulting in a time-independent generalized random force. The relevant Hilbert spaces of 2D and 3D many-fermion models, however, are infinite dimensional and the generalized random forces are consequently time dependent. The structure of these Hilbert spaces provides a picture of time-dependent behavior for 1D models which is fundamentally very different from that for 2D or 3D models.

The Tomonaga model is perhaps the best known strictly one-dimensional (1D) many-fermion model.[1] It is an exactly soluble model. The exact solutions for the Tomonaga model are not generalizable to higher dimensions.[2] Hence this model is not so useful for understanding the importance of dimensionality in many-body problems. The standard electron gas model, when restricted to electron-hole scattering, is often referred to as the Sawada mode.[3] The Sawada model is not exactly soluble, but one can show that the Sawada model in 1D reduces to the Tomonaga model if the electron-hole excitations are further confined to the vicinity of the Fermi surface. Hence one may regard the Sawada model as a generalizable version of the Tomonaga model.

The time-dependent behavior of the Tomonaga or 1D Sawada model is basically very different from that of the Sawada model in 2D or 3D. If a dense electron gas in the ground state is slightly perturbed momentarily, the system will undergo a relaxation process. In 1D the relaxation process will be purely oscillatory. It will remain oscillatory even when the electron-hole interaction is removed, i.e., when the system becomes an ideal, degenerate electron gas. This is because the 1D system has only one degree of freedom in momentum space whether there is an interaction or not. In 2D or 3D the system has infinitely many degrees of freedom and its relaxation process is richer, reflecting the two distinctly different single-particle and collective modes.[4]

The different time-dependent behavior of the Sawada model in 1D and 2D or 3D is especially apparent if we study the time evolution in this model via the method of recurrence relations.[5] Consider the density fluctuations at wave vector k,

$$\rho_k = \sum_p c_p^\dagger c_{p+k} \;,$$

where $c_k^\dagger$ and c_k are, respectively, the creation and annihilation operators at wave vector k. We shall confine our consideration to $|k/k_F| \ll 1$, where k_F is the Fermi wave vector and $\hbar = 1$. The time-dependent behavior of this system can be completely determined by $\rho_k(t)$ and its generalized random force $\mathscr{F}_k(t)$. According to the method of recurrence relations, $\rho_k(t)$ and $\mathscr{F}_k(t)$ are given by

$$\rho_k(t) = \sum_{\nu=0}^{d-1} a_\nu(t) f_\nu \;, \tag{1a}$$

$$\mathscr{F}_k(t) = \sum_{\nu=1}^{d-1} b_\nu(t) f_\nu \;, \tag{1b}$$

where $\{f_\nu\}$ is a set of basis vectors in $\mathscr{S}$ the Hilbert space of ρ_k, $\{a_\nu\}$ and $\{b_\nu\}$ are sets of real time-dependent functions, and d is the dimensionality of $\mathscr{S}$. If the space $\mathscr{S}$ is realized by the Kubo scalar product and $\{f_\nu\}$ is orthogonal in that space, the Hilbert space is spanned by $\{f_\nu\}$.

If one chooses $f_0 = \rho_k$, the time-dependent functions then satisfy the initial condition:

$$a_0(0) = 1, \quad a_\nu(0) = 0 \text{ if } \nu \geq 1 \;,$$

$$b_1(0) = 1, \quad b_\nu(0) = 0 \text{ if } \nu \geq 2 \;.$$

In addition, these functions are connected by a recurrence relation[5]

$$\Delta_{\nu+1} a_{\nu+1}(t) = -\dot{a}_\nu(t) + a_{\nu-1}(t), \quad 0 \leq \nu \leq d-1 \;, \tag{2}$$

where

$$a_{-1} \equiv 0, \quad \dot{a}_\nu = da_\nu/dt, \quad \Delta_\nu = (f_\nu, f_\nu)/(f_{\nu-1}, f_{\nu-1})$$

with $\Delta_0 \equiv 1$. The inner product denotes the Kubo scalar product. We shall term Δ_ν the νth *recurrant*. Also, $\{b_\nu\}$ satisfies exactly the same recurrence relation but starting with $\nu = 1$ and $b_0 \equiv 0$. Hence Δ_1 does not appear in the recurrence relation for $b_\nu(t)$. Every b_ν is, furthermore, relatable to a_ν via a convolution,[5]

$$a_\nu(t) = \int_0^t dt' a_0(t-t') b_\nu(t'), \quad 1 \leq \nu \leq d-1 \;. \tag{3}$$

To realize the recurrence relation, one must know the recurrants, which are functions of the static properties of the model. Since the a_ν's are coefficients associated with the orthogonal basis vectors spanning the Hilbert space, the structure of the Hilbert space already implies the form of the time evolution. The structure is shaped by the recurrants. Observe also that the generalized random force $\mathscr{F}_k(t)$ extends in a space $\mathscr{S}_1$ spanned by $f_1, f_2, \ldots, f_{d-1}$, which is thus a linear manifold of the Hilbert space $\mathscr{S}$. We shall denote the dimensionality of $\mathscr{S}_1$ by d_1. This subdimensionality d_1 is related to dimensionality d by $d_1 = d-1$. We shall refer to $\{a_\nu\}$ and $\{b_\nu\}$, respectively, as the relaxation and memory functions.

For the Tomonaga model,

$$\ddot{\rho}_k = [H, [H, \rho_k]] = -\omega_k^2 \rho_k \ , \tag{4}$$

where

$$\omega_k^2 = (k\eta)^2 + k^2 \eta v_k \ , \tag{4a}$$

where k is measured in units of the Fermi vector k_F, $\eta = \pi^2\rho^2/m$, ρ is the number density, and v_k is the interaction of the Tomonaga Hamiltonian H. Observe that the above relation (4) holds in the ideal case. For the Sawada model, Eq. (4) is valid if and only if $|k| \to 0$. Now, given $f_0 = \rho_k$, we then find that

$$f_1 = \dot{\rho}_k = i[H, \rho_k] \ , \tag{5a}$$

$$f_2 = f_3 = \ldots = 0 \ . \tag{5b}$$

Also,

$$\Delta_1 = (f_1, f_1)/(f_0, f_0)$$

$$= (\dot{\rho}_k, \dot{\rho}_k)/(\rho_k, \rho_k) = -(\ddot{\rho}_k, \rho_k)/(\rho_k, \rho_k) = \omega_k^2 \tag{6a}$$

and

$$\Delta_\nu = 0, \quad \nu \geq 2 \ . \tag{6b}$$

The dimensionality d of the Hilbert space $\mathscr{S}$ for the 1D model is thus two. The time evolution in such a finite space can only be oscillatory and there can be no time evolution at all in its subspace $\mathscr{S}_1$. The time behavior follows directly from the recurrence relation (2).

The recurrence relation for $d=2$, i.e., $\Delta_1 = \omega_k^2 \neq 0$ and $\Delta_2 = \Delta_3 = \Delta_4 = \cdots = 0$, becomes

$$\Delta_1 a_1(t) = -\dot{a}_0(t) \tag{7}$$

and

$$\dot{a}_1(t) = a_0(t) \ . \tag{8}$$

Hence we obtain

$$\rho_k(t) = a_0(t)\rho_k + a_1(t)\dot{\rho}_k \tag{9}$$

and

$$a_0(t) = \cos(\omega_k t), \quad a_1(t) = \sin(\omega_k t)/\omega_k \ . \tag{10}$$

Similarly, the recurrence relation for $d_1 = 1$ gives

$$\dot{b}_1(t) = 0 \ . \tag{11}$$

Hence, $b_1(t) = \text{const}$. But since $b_1(0) = 1$, the constant must be unity. Thus

$$\mathscr{F}_k(t) = \dot{\rho}_k \tag{12}$$

at all time, being independent of time and there is no time evolution. We observe that the general structure of $\rho_k(t)$ is preserved when $v_k \to 0$. This ideal limit is also attained when $\rho \to \infty$, where ρ is the electron number density.[3]

Now we shall compare the above with the time evolution in 2D and 3D. The recurrants for the 2D *ideal* system are found to be[6]

$$\Delta_1^{(0)} = 2\Delta \ , \tag{13a}$$

$$\Delta_\nu^{(0)} = \Delta, \quad 1 \leq \nu \leq \infty \ , \tag{13b}$$

where $\Delta = k^2\epsilon_F^2$, with ϵ_F the 2D Fermi energy. For the interacting system, only the first recurrant is modified:

$$\Delta_1 = 2\Delta + \Gamma \ , \tag{14a}$$

$$\Delta_\nu = \Delta, \quad 1 \leq \nu \leq \infty \ , \tag{14b}$$

where $\Gamma^{1/2} = \omega_p^{\text{cl}}$, the classical plasma frequency. The above results are valid for $|k| \ll 1$ only. Hence in 2D, $d \to \infty$. The recurrence relation for such a Hilbert space can be realized. For the ideal system,

$$a_\nu^{(0)}(t) = 2^\nu \mu^{-\nu} J_\nu(\mu t), \quad 0 \leq \nu \leq \infty \ , \tag{15}$$

where $\mu = 2\Delta^{1/2} = 2k\epsilon_F$ and J_ν is the cylindrical Bessel function of order ν.

The relaxation functions for the interacting gas can be obtained from (15), noting that the difference between the ideal and interacting cases is in Δ_1 only. In particular,

$$a_0(t) = A_s \sum_{r=0}^{\infty} (-\alpha)^r \left(\frac{d}{\mu\, dt}\right)^{2r} \frac{J_1(\mu t)}{\mu t} + A_p \cos(\omega_p t) \ , \tag{16}$$

where

$$\alpha = (x^2 + \tfrac{1}{4})/(x^2 + \tfrac{1}{2})^2, \quad x = \omega_p^{\text{cl}}/\mu, \quad \omega_p = \alpha^{-1/2}\mu \ ,$$

$$A_s = 1 - (1-\alpha)^{1/2}, \quad A_p = 2[(1-\alpha)^{1/2} - (1-\alpha)]/\alpha \ .$$

The other a_ν's may be obtained from the recurrence relation using (16). Observe that unlike in 1D, the general structure of the relaxation functions in 2D is not preserved when the interaction is introduced. Also, the time evolution in this infinite-dimensional Hilbert space is one in which the amplitude decreases with oscillations. Also, the subdimensionality $d_1 \to \infty$. For the 2D system, ideal or interacting, we have

$$b_\nu(t) = \nu 2^\nu \mu^{-\nu+1} J_\nu(\mu t)/\mu t, \quad 1 \leq \nu \leq \infty \ , \tag{17}$$

since Δ_1 does not enter into the recurrence relation for b_ν's. The random force is now itself time dependent.

The recurrants for the 3D ideal system are found to be[7]

$$\Delta_\nu^{(0)} = \mu^2\nu^2(4\nu^2-1)^{-1}, \quad 1 \leq \nu \leq \infty \ , \tag{18}$$

where $\mu = 2k\epsilon_F$, with ϵ_F the 3D Fermi energy. For the interacting system, only the first recurrant is modified:

$$\Delta_1 = \Delta_1^{(0)} + \Gamma \ , \tag{19a}$$

$$\Delta_\nu = \Delta_\nu^{(0)}, \quad 2 \leq \nu \leq \infty \ , \tag{19b}$$

where $\Gamma^{1/2} = \omega_p^{\text{cl}}$, the 3D classical plasma frequency. The above result is valid for $|k| \ll 1$ only. Thus, as in 2D, $d \to \infty$ for both ideal and interacting. The recurrence relation for these Hilbert spaces can be realized. For the ideal system, the relaxation functions are

$$a_\nu^{(0)}(t) = \frac{2^\nu[(2\nu+1)!!]j_\nu(\mu t)}{\mu^\nu(2\nu!!)}, \quad 0 \leq \nu \leq \infty \ , \tag{20}$$

where j_ν is the spherical Bessel function of order ν.

For the interacting system, the leading relaxation function is

$$a_0(t) = A_s \sum_{r=0}^{\infty} (-\epsilon)^r \left(\frac{d}{\mu\, dt} \right)^{2r} b_1(t) + A_p \cos(\omega_p t) \quad , \tag{21}$$

where

$$\epsilon = 3s(1-s)^{-4}, \quad s \equiv \Delta_1^{(0)}/\Delta_1 = (1+3x^2)^{-1}, \quad x \equiv \omega_p^{\rm cl}/\mu \quad ,$$

$$A_s = \mu\epsilon/3, \quad A_p = 2s(x_0^2 - 1)/(1-s)(1-sx_0^2), \quad x_0 \equiv \omega_p/\mu \quad ,$$

where ω_p is the solution of $\tanh[\mu/\omega_p(1-s)] = \mu/\omega_p$. The above solution is valid for $s \ll 1$. For other values of s, it is possible to give analytic solutions.[7] Other relaxation functions can be obtained from (21) via the recurrence relation. The above solution (21) is given in terms of $b_1(t)$, the leading memory function, which as in 2D does not depend on the interaction. We find that, unlike in 2D, $b_1(t) \approx 3j_1(\mu t)/\mu t$ even though the subdimensionality $d_1 \to \infty$.[7]

The different time-dependent behavior between 1D and 2D or 3D is formally attributable to the dimensionality of the Hilbert spaces. For the 1D electron gas, we have $d=2$ and $d_1=1$. Hence the random force, being time independent, is like a constant imposed force and is not driven in time. Also, since the random force is orthogonal, the dynamical variable (i.e., the density fluctuations) can only rotate simply in the 2D Hilbert space, resulting in a periodic oscillation. Physically it means that the 1D electron gas at long wavelengths has but one macroscopic or giant mode and it is thus unable to redistribute the perturbation energy imparted. This behavior is the same as the dynamic response in single-spin models[8] and many-body models in dynamic mean-field approximation.[9] That is, their time-dependent and dynamic behavior is formally identical.

In 2D and 3D, the random force is not constant. Hence the dynamical variable is continuously driven towards higher dimensions spanning the Hilbert space, resulting in a complex rotationary motion. This behavior denotes the existence of infinitely many multiply degenerate modes excited by the perturbation. Hence the time-dependent behavior in 1D is fundamentally very different from that in 2D or 3D.

Finally, we note that in 1D

$$(\dot\rho_k, \dot\rho_k) = \rho k^2 \eta \quad .$$

Hence with the definition for Δ_1 and our result [Eq. (6a)], we obtain the static susceptibility

$$(\rho_k, \rho_k) = \rho(\eta + v_k)^{-1}$$

in agreement with the result of Mattis and Lieb.[10]

This work is supported in part by the U. S. Department of Energy, Office of Basic Energy Sciences, under Contract No. DE-AS09-77ER01023 and also by the Korea Science and Engineering Foundation.

---

*On leave of absence from the University of Rajasthan, Jaipur, Rajasthan, India.

1S. Tomonaga, Prog. Theor. Phys. (Kyoto) 5, 544 (1950). Also, see J. M. Luttinger, J. Math. Phys. 4, 1154 (1963).

2See, e.g., M. H. Lee, Prog. Theor. Phys. (Kyoto) 40, 990 (1968).

3K. Sawada, Phys. Rev. 106, 372 (1957). Also, see R. Brout, *Lectures on the Many-Electron Problem* (Interscience, New York, 1963); M. H. Lee and J. Hong, Phys. Rev. B 26, 2227 (1982) (see especially Ref. 7).

4In 1D, the single-particle and collective excitations are never resolved, giving the appearance of one giant collectivelike mode. As the interaction is turned off, the collective excitation disappears but the picture of one giant mode still persists.

5M. H. Lee, Phys. Rev. B 26, 2547 (1982); Phys. Rev. Lett. 49, 1072 (1982).

6M. H. Lee and J. Hong, Phys. Rev. Lett. 48, 634 (1982).

7M. H. Lee and J. Hong (unpublished).

8M. H. Lee, Can. J. Phys. 61, 428 (1983); A. S. T. Pires and M. E. DeGouvea, *ibid.* 61, 1475 (1983).

9M. H. Lee, Phys. Rev. B 8, 3290 (1973); in *Magnetism and Magnetic Materials—1973,* edited by C. D. Graham and J. J. Rhyne, AIP Conf. Proc. No. 18 (AIP, New York, 1974), p. 714.

10D. C. Mattis and E. H. Lieb, J. Math. Phys. 6, 304 (1965) [see especially Eq. (6.1)].