Dielectric response of quantum plasmas in thermal equilibrium

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The dielectric function $\epsilon(k,\omega)$ of an electron plasma in thermal equilibrium is calculated for all degrees of plasma degeneracy. Previous results for degenerate and nondegenerate plasmas, based on quantum-mechanical, classical, or semiclassical approaches, are contained in this analysis. Expressions for the real and imaginary parts of $\epsilon(k,\omega)$ are given for the cases of low and high frequencies and long and short wavelengths. In particular, the roles of thermal and quantum effects on screening, collective plasma resonance, and single-particle behavior are discussed.

I. INTRODUCTION

Theoretical studies of the dielectric response of plasmas are usually concentrated in the following two well-separated domains of plasma physics. (a) Dense plasmas at low temperatures, usually described with degenerate electron-gas models and with the use of quantum-mechanical methods. Such is the case of solid-state plasmas. (b) Dilute plasmas at high temperatures, where a classical description of the plasma is normally used. This description applies in a wide range of densities and temperatures.[1] In particular, this includes the case of high-temperature plasmas of interest for fusion research.

The purpose of this paper is to study the transition between these cases, through a calculation of the plasma dielectric function for all degrees of plasma degeneracy. Previous results from classical, semiclassical, and quantum approximations will be contained in our analysis. Here we sketch briefly the ideas that give rise to these various approximations.

In the *classical* description the response of the system to an external field is calculated using the Boltzmann-Vlasov equation,[2] and it gives the following expression for the dielectric function

$$\epsilon(k,\omega)=1+\frac{4\pi e^2}{k^2}\int d^3v\frac{\vec{k}\cdot\vec{\nabla}(\partial f/\partial E)}{\omega-\vec{k}\cdot\vec{v}+i\delta}\ , \tag{1}$$

in terms of the wave vector $\vec{k}$ and frequency ω. Here $f\equiv f(\vec{v})$ is the thermal (Maxwell-Boltzmann) distribution of electron velocities $\vec{v}$ and $E=\frac{1}{2}mv^2$ is the corresponding kinetic energy. The normalization adopted here is $\int f(\vec{v})d^3v=n$, where n is the electron density.

This description is sometimes extended to the case of degenerate or semidegenerate plasmas, using the so-called *semiclassical* approximation.[3] This consists of using the classical expression of Eq. (1), but introducing a quantum-mechanical expression for the distribution function $f(\vec{v})$, namely, the Fermi-Dirac distribution. This approach allows for an extended description, which includes the conditions (a) and (b) mentioned at the beginning, and in many cases it gives quite satisfactory results.

A more general solution, however, is provided by a fully consistent quantum-mechanical analysis, where the response of the system is calculated with perturbation theory (in the context of solid-state physics this approach is referred to as the random-phase approximation or the self-consistent field approach).[4] The analysis leads in this case to the expression[5]

$$\epsilon(k,\omega)=1+\frac{e^2}{\pi^2k^2}\int d^3k'\frac{\hat{f}(\vec{k}+\vec{k}')-\hat{f}(\vec{k}')}{\hbar\omega+i\delta-(E_{\vec{k}+\vec{k}'}-E_{\vec{k}'})}, \tag{2}$$

where $E_{\vec{q}}=\hbar^2q^2/2m$ and $\hat{f}(\vec{k})$ is the Fermi-Dirac function [its relation to the normalized distribution $f(\vec{v})$ in Eq. (1) is thus $\hat{f}(\vec{k})=(2\pi\hbar)^3f(\vec{v})/2m^3$, see also Eq. (3)]. All classical and semiclassical results are contained in this formula, in correspondence with the long-wavelength (and low-frequency) limit. This can be seen immediately by approximating Eq. (2) for $\vec{k}\to 0$, with $\hat{f}(\vec{k}+\vec{k}')-\hat{f}(\vec{k}')\cong\vec{k}\cdot\vec{\nabla}_{\vec{k}'}\hat{f}(\vec{k}')=\hbar\vec{k}\cdot\vec{\nabla}(\partial\hat{f}/\partial E)$, and $E_{\vec{k}+\vec{k}'}-E_{\vec{k}'}\cong\vec{k}\cdot\vec{\nabla}_{\vec{k}'}E_{\vec{k}'}=\hbar\vec{k}\cdot\vec{v}$. In this limit we retrieve Eq. (1) exactly.

In consequence, the additional content of the quantum-mechanical analysis pertains to the domain of short-range phenomena. This is physically obvious since one expects the quantum properties of the plasma to appear in the region of wavelengths shorter than the wavelengths of the electrons in the plasma,[5] as the analysis will show in detail.

Then, the applicability of either classical, semiclassical, or quantum response functions for the plasma depends on the plasma temperature (i.e., its degeneracy), and on the region of frequencies and wavelengths that is important for the description of a particular phenomenon.

The transition from degenerate to nondegenerate plasmas in the range of high densities ($n \sim 10^{23}$–$10^{27} e/\mathrm{cm}^3$) is a subject of much interest for current studies of inertial-confinement fusion. The approach to those extreme conditions is being tested using laser and ion beams. A reliable description of the plasma properties at all degrees of degeneracy is therefore important for the studies of heating and confinement of dense plasmas.

In this work we obtain expressions for the dielectric function of plasmas of all degeneracies, incorporating both thermal and quantum effects. In Sec. II we develop analytical expressions for the real and imaginary parts of the dielectric function in terms of ω and k, and give expansions for low and high frequencies and for long and short wavelengths. Some differences between quantum and classical results are discussed. They are applied in Sec. III to the calculation of the energy-loss function $\mathrm{Im}[-1/\epsilon(k,\omega)]$, in the domains of low and high frequencies of interest for particle-plasma interactions. Section IV contains a summary of our results.

II. TEMPERATURE-DEPENDENT DIELECTRIC FUNCTION

We develop in this section the dielectric function $\epsilon(k,\omega)$ for plasmas of all degeneracies, in terms of the wave number k and of the frequency ω. The basic formulation of Eq. (2) permits one to calculate the real and imaginary parts of $\epsilon(k,\omega)=\epsilon_1(k,\omega)+i\epsilon_2(k,\omega)$. The temperature dependence is incorporated in Eq. (2) through the Fermi-Dirac function

$$\hat{f}(\vec{k})=\{1+\exp[\beta(E_k-\mu)]\}^{-1}, \tag{3}$$

where $\beta=1/k_BT$, $E_k=\hbar^2k^2/2m$, and μ is the chemical potential of the plasma with electron density n and temperature T. In the absence of collisions we assume that the damping constant approaches the value $\delta \rightarrow 0^+$.

We measure the electron interaction relative to the electron kinetic energy at $T=0$ through the parameter[1]

$$\chi_0^2=\frac{3}{16}\left[\frac{\hbar\omega_P}{E_F}\right]^2=\frac{1}{\pi k_F a_0}\simeq\frac{r_s}{6.03}, \tag{4}$$

where a_0 is Bohr's radius, $\omega_P=(4\pi ne^2/m)^{1/2}$ is the plasma frequency, $E_F=\frac{1}{2}mv_F^2$ is the Fermi energy in terms of the velocity v_F, and $k_F=mv_F/\hbar=(3\pi^2n)^{1/3}$ is the corresponding wave number. It is also convenient to introduce the one-electron radius r_s (in a.u.), which is related to the electron density n by $(4\pi/3)na_0^3r_s^3=1$. The range of r_s values of interest for plasma research is shown in Fig. 1 of Ref. 1.

The degeneracy of the plasma is measured through the reduced temperature θ, or equivalently, the degeneracy parameter D, as

$$\theta\equiv\frac{1}{D}=\frac{k_BT}{E_F}, \tag{5}$$

so that degenerate and nondegenerate plasmas correspond to $\theta \ll 1$ and $\theta \gg 1$, respectively.

The chemical potential μ depends on $\theta=D^{-1}$ through the expression

$$\tfrac{2}{3}D^{3/2}=F_{1/2}(\eta)=\int_0^\infty \frac{x^{1/2}}{1+e^{x-\eta}}dx, \tag{6}$$

where $F_{1/2}(\eta)$ is the Fermi integral of order $\frac{1}{2}$ and $\eta=\beta\mu=\mu/k_BT$.

In the following we discuss the calculation of the dielectric function for all plasma degeneracies. Our results for the functions $\epsilon_1(k,\omega)$ and $\epsilon_2(k,\omega)$ are summarized in Tables I and II, respectively. We begin now by analyzing the behavior of $\epsilon_1(k,\omega)$.

A. Real part of $\epsilon(k,\omega)$

A closed analytical expression for $\epsilon_1(k,\omega)$ based on Eq. (2) is not possible. Efforts have been made to find solutions in terms of infinite series expansions.[6,7] We proceed in a somewhat different manner to develop approximate expressions that contain some well-known results in various limiting cases, as well as some other new ones.

The function $\epsilon_1(k,\omega)$ can be written in the form

$$\epsilon_1(k,\omega)=1+\frac{\chi_0^2}{4z^3}[g(u+z)-g(u-z)], \tag{7}$$

which follows directly from Eq. (2), after some lengthy integrations, and where the function $g(x)$ is given by

$$g(x)=-g(-x)=\int_0^\infty \frac{y\,dy}{e^{Dy^2-\eta}+1}\ln\left|\frac{x+y}{x-y}\right|. \tag{8}$$

Here we have introduced the usual reduced variables[5]

$$u=\omega/kv_F,\quad z=k/2k_F. \tag{9}$$

With regard to plasma degeneracy, Eq. (8) attains the following limits:

$$g(x)\cong g_0(x)=x+\tfrac{1}{2}(1-x^2)\ln\left|\frac{1+x}{1-x}\right|,\quad \theta \ll 1 \tag{10a}$$

$$g(x)\cong\tfrac{2}{3}D^{1/2}\Phi(D^{1/2}x),\quad \theta \gg 1 \tag{10b}$$

where

$$\Phi(s)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}dz\frac{e^{-z^2}}{s-z} \tag{11}$$

is the "plasma dispersion function."[8] This assures the retrieval of all previous results for the case of classical plasmas.

On the other hand, when Eq. (10a) is inserted in Eq. (7) we recuperate the Lindhard dielectric function for a degenerate plasma.[5]

At intermediate temperatures $g(x)$ changes as indicated with solid lines in Fig. 1. At $\theta\cong 0$ excitations out of the Fermi sphere (through virtual processes) give rise to a pronounced peak in $g(x)$ for $x\cong 0.8$. At high temperatures, $\theta \gtrsim 1$, the maximum of $g(x)$ shifts to larger x values, because the electron velocities v_e become $v_e\sim\theta^{1/2}v_F$, and the peak broadens correspondingly. The difference between

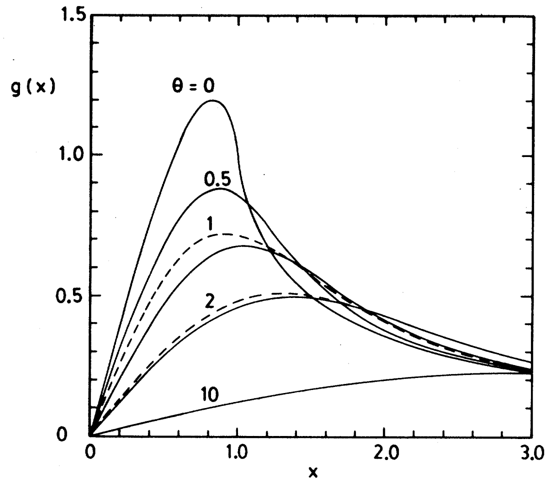

FIG. 1. Function $g(x)$ pertaining to the calculation of $\epsilon_1(k,\omega)$, Eq. (7). Solid lines give the values of $g(x)$ as calculated from Eq. (8) for reduced temperatures $\theta \equiv k_B T/E_F = 0$, 0.5, 1.0, 2.0, and 10. Dashed lines for $\theta = 1$ and 2 correspond to the classical approximation, $g(x) = \frac{2}{3} D^{1/2} \Phi(D^{1/2} x)$, in terms of the plasma dispersion function of Eq. (11) (for $\theta = 10$ the dashed and solid lines coincide in this graph).

the solid lines representing Eq. (8) and the dashed curves corresponding to the approximation for $\theta \gg 1$, Eq. (10b), becomes negligible for $\theta \gtrsim 2$.

Irrespective of the plasma degeneracy we can expand $g(x)$ as follows.

Case (i), $x \rightarrow 0$. By expanding the logarithm in Eq. (8) for $x \ll y$ we obtain

$$g(x) \cong 2x H_1(D) - \tfrac{2}{3} x^3 H_2(D) , \tag{12}$$

where the functions $H_i(D)$ are given by

$$H_1(D) = \frac{1}{2D^{1/2}} F_{-1/2}(\eta) = \int_0^\infty \hat{f}(y) dy , \tag{13}$$

$$H_2(D) = \int_0^\infty \frac{dy}{y^2} [\hat{f}(0) - \hat{f}(y)] , \tag{14}$$

with the abbreviation, in analogy to Eq. (3),

$$\hat{f}(y) = \frac{1}{1 + \exp(Dy^2 - \eta)} , \tag{15}$$

and we denote by $F_\nu(\eta)$ the Fermi integral of order ν. The properties of these integrals are given in the Appendix.

Case (ii), $x \rightarrow \infty$. Using a complementary expansion of Eq. (8) for $y \ll x$ we get

$$g(x) = \frac{2}{3x} + \frac{F_{3/2}(\eta)}{3D^{5/2} x^3} + \cdots + \frac{F_{l/2}(\eta)}{l D^{1+l/2} x^l} + \cdots \quad \text{(with } l \text{ odd)} \tag{16}$$

where in the first term we have used the fact that $F_{1/2}(\eta) = \frac{2}{3} D^{3/2}$ (see Appendix).

We can now obtain expressions for $\epsilon_1(k,\omega)$ in various limits of u and z that are of interest in the description of plasma properties. This follows from cases (i) and (ii), when they are applied to calculate the difference $\Delta g(u,z) \equiv g(u+z) - g(u-z)$ in Eq. (7). Thus, for $u,z \ll 1$ we can use Eq. (12) and obtain

$$\Delta g(u,z) \cong [2x H_1(D) - \tfrac{2}{3} x^3 H_2(D)] \Big|_{x=u-z}^{x=u+z} . \tag{17}$$

For large values of u and z we instead use Eq. (16) and calculate Δg from the appropriate Taylor expansion:

$$\Delta g(u,z) \cong \begin{cases} 2z g'(u) + \frac{1}{3} z^3 g'''(u), & z \ll u \quad (18a) \\ 2g(z) + u^2 g''(z), & u \ll z . \quad (18b) \end{cases}$$

The formulas obtained from this analysis are collected in Table I. The first column of this table indicates various limiting conditions of interest. The second column contains expressions for $\epsilon_1(k,\omega)$ that apply to all values of $\theta = k_B T/E_F$.

Known formulas in the limits $\theta \ll 1$ and $\theta \gg 1$ can be culled from the expressions given in the second column by using the following limits: for $\theta \ll 1$,

$$H_1(D) \cong 1, \quad H_2(D) \cong 1, \quad F_\nu(\eta) \cong D^{\nu+1}/(\nu+1) ; \tag{19a}$$

for $\theta \gg 1$,

$$H_1(D) \cong \tfrac{2}{3} D, \quad H_2(D) \cong \tfrac{4}{3} D^2 , \qquad F_\nu(\eta) \cong \frac{2}{3} \frac{\Gamma(\nu+1)}{\Gamma(3/2)} D^{3/2} . \tag{19b}$$

The results are also listed in Table I, columns 3 and 4, and coincide with expressions given by Lindhard[5] (for degenerate plasmas, $\theta \ll 1$) and by Pines[9] (nondegenerate plasmas, $\theta \gg 1$)— also see Ref. 10.

Thus, the equations in the second column of Table I, denoted expressions (2,1)–(2,4) (with the first number in parentheses giving the column of the expression and the second the row) span the range of plasmas of arbitrary degeneracies.

Specifically, expression (2,1) of Table I is pertinent to the screening of a static impurity in the plasma and to the slowing down of low-velocity particles. It can be written (for $\chi_0^2 \ll 1$) in the familiar form

$$\epsilon_1(k,\omega) \cong 1 + \frac{k_s^2}{k^2}, \quad \omega \rightarrow 0 \tag{20}$$

where k_s is a temperature-dependent screening wave vector given by

$$k_s^2 = 4 k_F^2 \chi_0^2 H_1(D) = \tfrac{1}{2} k_{TF}^2 \theta^{1/2} F_{-1/2}(\eta) . \tag{21}$$

Using the limiting values of $F_{-1/2}(\eta)$ (see Appendix) we find that $k_s \rightarrow k_{TF} = \sqrt{3}\omega_P/v_F$ for $\theta \ll 1$ (Thomas-Fermi approximation) and that $k_s \rightarrow k_D = (4\pi n e^2/k_B T)^{1/2}$ for $\theta \gg 1$ (Debye approximation), as indicated in Table I by expressions (3,1) and (4,1). For a stationary or slowly moving ion in the plasma, with effective charge Ze, the screened field would be of the form $(Ze/r)\exp(-k_s r)$. This corresponds to a screening length $\lambda_s = k_s^{-1}$ that increases with plasma temperature.

Expression (2,2) of Table I gives access to the long-wavelength plasma resonance through the condition

TABLE I. Results for the real part of the dielectric function, $\epsilon_1(k,\omega)$, for a plasma of density n and temperature T. Expressions are given for various domains of frequencies ω and wave vectors k, as indicated in the first column in terms of $u=\omega/kv_F$ and $z=k/2k_F$. Results in the second column apply for all degrees of degeneracy, and those in the third and fourth columns apply to low and high temperatures, respectively. Degeneracy is measured by the parameter $\theta\equiv 1/D\equiv k_BT/E_F$, where $E_F=\frac{1}{2}mv_F^2$ is the Fermi energy. Other quantities here are $\eta=\beta\mu=\mu/k_BT$, $\chi_0^2=1/(\pi k_F a_0)$, $k_{\rm TF}^2=3\omega_P^2/v_F^2=6\pi ne^2/E_F$, $k_D^2=m\omega_P^2/k_BT=4\pi ne^2/k_BT$, and $E_k=\hbar^2k^2/2m$. Integrals $H_1(D)$ and $H_2(D)$ are defined by Eqs. (13) and (14), whereas the Fermi integrals $F_\nu(\eta)$ are defined in the Appendix, Eq. (A1). Its limiting values for $\theta\ll 1$ and $\theta\gg 1$ are also given in the Appendix. Expressions in this table are referred to as expression (c,r) where c and r are the numbers of the column and row of the expression, respectively.

	$\epsilon_1(k,\omega)$		
Cases	All values of $\theta=k_BT/E_F$	$\theta\ll 1$	$\theta\gg 1$
$u,z\ll 1$	$1-\frac{\chi_0^2}{3}H_2(D)+\frac{\chi_0^2}{z^2}[H_1(D)-u^2H_2(D)]$	$1-\frac{\chi_0^2}{3}+\frac{k_{\rm TF}^2}{k^2}\left[1-\frac{\omega^2}{k^2v_F^2}\right]$	$1-\frac{1}{12}\left[\frac{\hbar\omega_P}{k_BT}\right]^2+\frac{k_D^2}{k^2}\left[1-\frac{m\omega^2}{k_BTk^2}\right]$
$1,z\ll u$	$1-\frac{\omega_P^2}{\omega^2}\left[1+\frac{z^2}{u^2}+\frac{3}{2}\frac{F_{3/2}(\eta)}{u^2D^{5/2}}+\frac{3}{2}\frac{F_{5/2}(\eta)}{u^4D^{7/2}}+\cdots\right]$	$1-\frac{\omega_P^2}{\omega^2}\left[1+\frac{z^2}{u^2}+\frac{3}{5u^2}+\frac{3}{7u^4}\right]$	$1-\frac{\omega_P^2}{\omega^2}\left[1+\frac{z^2}{u^2}+\frac{3}{2}\frac{\theta}{u^2}+\frac{15}{4}\frac{\theta^2}{u^4}\right]$
$1,u\ll z$	$1+\left[\frac{\hbar\omega_P}{E_k}\right]^2\left[1+\frac{u^2}{z^2}+\frac{1}{2}\frac{F_{3/2}(\eta)}{z^2D^{5/2}}+\frac{3}{10}\frac{F_{5/2}(\eta)}{z^4D^{7/2}}+\cdots\right]$	$1+\left[\frac{\hbar\omega_P}{E_k}\right]^2\left[1+\frac{u^2}{z^2}+\frac{1}{5z^2}+\frac{3}{35z^4}\right]$	$1+\left[\frac{\hbar\omega_P}{E_k}\right]^2\left[1+\frac{u^2}{z^2}+\frac{1}{2}\frac{\theta}{z^2}+\frac{3}{4}\frac{\theta^2}{z^4}\right]$
$u=z$ single-particle ridge	$1+2\chi_0^2\left[\frac{k_F}{k}\right]^3 g(x)\,,\quad x=\frac{k}{k_F}=\left[\frac{\hbar\omega}{E_F}\right]^{1/2}$	same as (2,4) with $g(x)\cong x+\frac{1}{2}(1-x^2)\ln\left\lvert\frac{1+x}{1-x}\right\rvert$	same as (2,4) with $g(x)\cong\frac{2}{3}D^{1/2}\Phi(D^{1/2}x)$

$\epsilon(k,\omega)=0$. For $\epsilon_2(k,\omega)\ll 1$ the resonance frequency is then given by the root of

$$\omega_k^2=\omega_P^2\left[1+\left[\frac{\hbar k^2}{2m\omega_k}\right]^2+\tfrac{3}{2}\theta^{5/2}F_{3/2}(\eta)k^2v_F^2/\omega_k^2 + \tfrac{3}{2}\theta^{7/2}F_{5/2}(\eta)k^4v_F^4/\omega_k^4+\cdots\right] . \quad (22)$$

When the perturbation of the plasma is due to a fast charged particle this resonance leads to the excitation of "wakes" in the plasma and gives place to collective energy losses.

Finally, expression (2,3) of Table I pertains to the dielectric behavior in the short-range limit ($k\gg k_F$ and $\hbar k^2\gg 2m\omega$), whereas expression (2,4) of Table I applies to the single-particle ridge, namely, $\omega=\hbar k^2/2m$. Applications of these results will be considered in Sec. III.

B. Imaginary part of $\epsilon(k,\omega)$

The imaginary part of the dielectric function, $\epsilon_2(k,\omega)$ can be obtained by direct integration of Eq. (2) for $\delta\to 0^+$, with the following result:

$$\epsilon_2(k,\omega)=\frac{\pi\chi_0^2}{8z^3}\theta\ln\left[\frac{1+\exp[\eta-D(u-z)^2]}{1+\exp[\eta-D(u+z)^2]}\right] , \quad (23)$$

and it agrees with previous calculations by Khanna and Glyde.[6] This is an exact result, applicable to plasmas of all degeneracies. It has limiting values which apply to plasmas under various conditions, of much interest for the analysis in this paper. These cases are indicated in Table II. Some of them can be found in the literature. We now consider some examples.

(a) Small ω, all θ. We expand Eq. (23) in powers of $u=\omega/kv_F$, and in the first nonvanishing order we obtain [see expression (2,3) in Table II]

$$\epsilon_2(k,\omega)\cong\frac{\pi\chi_0^2}{8z^3}\theta\frac{4Duz}{[1+\exp(Dz^2-\eta)]} \cong\frac{2m^2e^2\omega}{(\hbar k)^3}\left[1+\exp\left[\frac{\hbar^2k^2}{8mk_BT}-\eta\right]\right]^{-1} . \quad (24)$$

In the limit $\theta\gg 1$, moreover, $e^{\eta}\cong 4D^{3/2}/3\pi^{1/2}\ll 1$ (cf. the Appendix), and Eq. (24) reduces to

$$\epsilon_2(k,\omega)\cong\frac{ne^2\omega}{mk^3}\left[\frac{2\pi m}{k_BT}\right]^{3/2}\exp\left[-\frac{\hbar^2k^2}{8mk_BT}\right] , \quad \theta\gg 1 . \quad (25)$$

The forms of Eqs. (24) and (25) are central to the treatment of low-velocity stopping power and energy straggling in plasmas.[1]

(b) Degenerate plasma ($\theta\ll 1$), all k,ω. For $\theta\ll 1$, $\mu\cong E_F$ and thus $e^{\eta}=e^{\beta\mu}\gg 1$. Equation (23) then becomes

$$\epsilon_2(k,\omega)\cong\frac{\pi\chi_0^2}{8z^3}\theta\ln\left[\frac{1+\exp\{D[1-(u-z)^2]\}}{1+\exp\{D[1-(u+z)^2)]\}}\right] , \quad (26)$$

so that

$$\lim_{D\to\infty}\epsilon_2(k,\omega)=\begin{cases}\dfrac{\pi}{8}\dfrac{\chi_0^2}{z^3}\dfrac{\hbar\omega}{E_F}, & (u\pm z)^2<1 \quad (27a)\\ \dfrac{\pi\chi_0^2}{8z^3}[1-(u-z)^2], & (u-z)^2<1<(u+z)^2 \quad (27b)\\ 0, & 1<(u-z)^2 . \quad (27c)\end{cases}$$

In this case we retrieve the dielectric function of a degenerate plasma.[5]

(c) Nondegenerate plasma ($\theta\gg 1$), all k,ω. Here $e^{\eta}\cong 4D^{3/2}/3\pi^{1/2}\ll 1$ (cf. the Appendix), and thus $\exp[\eta-D(u\pm z)^2]\ll 1$. Hence we expand the logarithm in Eq. (23) to obtain

$$\epsilon_2(k,\omega)\cong\frac{\pi\chi_0^2}{8z^3}\theta\{\exp[\eta-D(u-z)^2] -\exp[\eta-D(u+z)^2]\} \cong\frac{m\omega_P^2}{\hbar k^3}\left[\frac{2\pi m}{k_BT}\right]^{1/2}\sinh\left[\frac{\hbar\omega}{2k_BT}\right] \times\exp\left[-(u^2+z^2)\frac{E_F}{k_BT}\right] . \quad (28)$$

The first form of this equation exhibits an important property of $\epsilon_2(k,\omega)$, in that it has a maximum when $u=z$, i.e., when $\omega=\hbar k^2/2m$. This defines the single-particle ridge, which corresponds to the region of excitation of single particles by the external field. This ridge was already contained in the case of the low-temperature limit of Eq. (27b). Moreover, we see from Eq. (28) that for high temperatures the spectrum of excitations assumes the shape of a broad Gaussian in the variable $|u-z|$, of width $\sim(k_BT/E_F)^{1/2}\gg 1$. Hence, the shape of this spectrum mirrors the distribution of particle velocities in the plasma.

In addition, we note that Eq. (28) reduces to the quantum form of Eq. (25) in the limit of small frequencies, and that it approaches the classical limit, $\hbar\to 0$, as

TABLE II. Results for the imaginary part of the dielectric function, $\epsilon_2(k,\omega)$, for various domains of frequencies ω and wave vectors k, as indicated in the first column in terms of the variables $u=\omega/kv_F$ and $z=k/2k_F$. Results in the second column apply for all degrees of plasma degeneracy, and those in the third and fourth columns apply to low and high temperatures. Degeneracy is measured by the parameter $\theta\equiv 1/D\equiv k_BT/E_F$, where $E_F=\frac{1}{2}mv_F^2$ is the Fermi energy. Other quantities here are $\eta=\beta\mu=\mu/k_BT$, $\chi_0^2=1/(\pi k_F a_0)$, and $E_k=\hbar^2k^2/2m$. Expressions are referenced in the same manner as Table I.

	$\epsilon_2(k,\omega)$		
Cases	All values of $\theta=k_BT/E_F$	$\theta\ll 1$	$\theta\gg 1$
low frequencies long wavelengths $u,z\ll 1$	$\dfrac{2m^2e^2\omega}{(\hbar k)^3}\dfrac{1}{(1+e^{-\eta})}$	$\dfrac{2m^2e^2\omega}{(\hbar k)^3}$	$\dfrac{ne^2\omega}{mk^3}\left[\dfrac{2\pi m}{k_BT}\right]^{3/2}$
long wavelengths $z\ll 1$ (all frequencies)	$\dfrac{2m^2e^2\omega}{(\hbar k)^3}\dfrac{1}{1+\exp\left[\left[\dfrac{E_F}{k_BT}\right]\left[\dfrac{\omega}{kv_F}\right]^2-\eta\right]}$	$\begin{cases}\dfrac{2m^2e^2\omega}{(\hbar k)^3}, & \omega<kv_F\\ 0, & \omega>kv_F\end{cases}$	$\dfrac{ne^2\omega}{mk^3}\left[\dfrac{2\pi m}{k_BT}\right]^{3/2}\exp\left[-\dfrac{m\omega^2}{2k_BTk^2}\right]$
low frequencies $u\ll 1$ (all wavelengths)	$\dfrac{2m^2e^2\omega}{(\hbar k)^3}\dfrac{1}{1+\exp\left[\dfrac{E_k}{4k_BT}-\eta\right]}$	$\begin{cases}\dfrac{2m^2e^2\omega}{(\hbar k)^3}, & k<2k_F\\ 0, & k>2k_F\end{cases}$	$\dfrac{ne^2\omega}{mk^3}\left[\dfrac{2\pi m}{k_BT}\right]^{3/2}\exp\left[-\dfrac{E_k}{4k_BT}\right]$
single-particle ridge $u\cong z\gg 1$	$\dfrac{\pi\chi_0^2}{8z^3}\theta\ln\{1+\exp[\eta-D(u-z)^2]\}$	$\dfrac{\pi\chi_0^2}{8z^3}[1-(u-z^2)]$	$\dfrac{2\pi ne^2}{\hbar k^3}\left[\dfrac{2\pi m}{k_BT}\right]^{1/2}\exp\left[-\dfrac{(\hbar\omega-E_k)^2}{4k_BTE_k}\right]$

$$\epsilon_2(k,\omega)\cong\frac{ne^2\omega}{mk^3}\left[\frac{2\pi m}{k_BT}\right]^{3/2}\exp\left[-\frac{m\omega^2}{2k_BTk^2}\right]\times\left[1-\frac{(\hbar k)^2}{8mk_BT}+\frac{1}{24}\left[\frac{\hbar\omega}{k_BT}\right]^2+\cdots\right],\quad \theta\gg 1\ . \qquad (29)$$

This illustrates how the classical result is obtained when k_BT is large compared with the values of E_F, $\hbar\omega$, and $\hbar^2k^2/2m$. These conditions ($\theta\gg1$ *and* $\hbar\to0$) define the classical domain of the plasma response.

In Fig. 2 we show the results of $\epsilon_2(k,\omega)$, for $k=0.2$ and $v_F=1$, as a function of the frequency ω, for various reduced temperatures. We see that for low frequencies the classical approximation [dashed lines, from Eq. (29) with $\hbar=0$] gives values higher than the exact result [solid lines, from Eq. (23)], and that it shifts to lower values at high frequencies, as predicted by Eq. (29) (notice the different signs of the two correction terms in order $\hbar^2$).

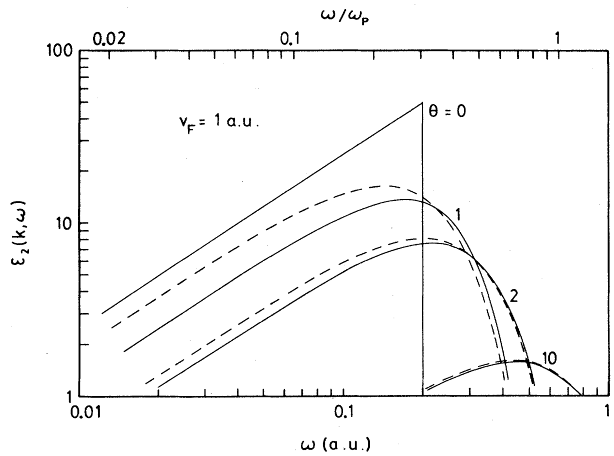

FIG. 2. Results for $\epsilon_2(k,\omega)$ as a function of the frequency ω, for reduced temperatures $\theta=0$, 1, 2, and 10, for $v_F=1$ a.u. ($n=2.28\times10^{23}$ cm^{-3}) and $k/k_F=0.2$. Solid lines show the exact results of Eq. (23), whereas the dashed lines give the classical approximation for $\epsilon_2(k,\omega)$—Eq. (29) with $\hbar=0$.

III. ENERGY-LOSS FUNCTION

The energy-loss function, defined as

$$S(k,\omega)=\mathrm{Im}\left[\frac{-1}{\epsilon(k,\omega)}\right]=\frac{\epsilon_2(k,\omega)}{|\epsilon(k,\omega)|^2}\ , \qquad (30)$$

is the crucial quantity which describes the spectrum of excitations in the plasma in terms of the momentum transfer $\hbar k$ and of the energy transfer $\hbar\omega$. Thus, for instance, the inelastic scattering of charged particles or photons in the plasma is governed by a scattering rate $R\propto N(\omega)S(k,\omega)$, where $N(\omega)=[\exp(\beta\hbar\omega)-1]^{-1}$ is the Planck function indicating the equilibrium of excitations in the medium.[1,4]

For finite temperatures $S(k,\omega)$ shows changes in the spectrum that can be characterized as a thermal redistribution of the oscillator strength, when compared to that in a cold plasma. It can be evaluated exactly through Eqs. (7), (8), and (23), or with the use of some of the approximate results of Sec. II.

We now consider the domains of direct interest for experiments of particle or light scattering in plasmas. (a) First the domain of $u\ll1$ (and all z values), which comprises the low-frequency behavior of plasmas and is characterized by the strong screening when $k\to0$. In the region of high frequencies, $u\gg1$, we can distinguish the following domains: (b) the single-particle ridge, $u\cong z\gg1$, which is important for short-range excitations, and (c) the plasma resonance [corresponding to the condition $\epsilon(k,\omega)=0$], for long wavelengths and high frequencies, $z\ll1\ll u$, which corresponds to collective excitations.

We now use the results applicable to these three domains, and calculate the energy-loss function through Eq. (30).

(a) Low frequencies. Using the results for $\epsilon_1(k,\omega)$ and $\epsilon_2(k,\omega)$ given in Eqs. (20) and (24) for $u=\omega/kv_F\ll1$ [cf. Tables I and II, expressions (2,1) and (2,3) respectively] we approximate

$$\mathrm{Im}\left[\frac{-1}{\epsilon(k,\omega)}\right]\cong\frac{\epsilon_2(k,\omega)}{\epsilon_1^2(k,\omega)}\cong\frac{2m^2e^2k\omega}{\hbar^3(k^2+k_s^2)^2}\left[1+\exp\left[\frac{\hbar^2k^2}{8mk_BT}-\eta\right]\right]^{-1}. \qquad (31)$$

The screening constant k_s depends on the plasma temperature as indicated by Eq. (21). However, using the analytical approximation for $F_{-1/2}(\eta)$ given in the Appendix, Eq. (A7), we can approximate

$$k_s^2\cong k_{\mathrm{TF}}^2/(1+\tfrac{9}{4}\theta^2)^{1/2}=k_D^2/(1+\tfrac{4}{9}D^2)^{1/2}\ , \qquad (32)$$

which gives the proper limits $k_s\cong k_{\mathrm{TF}}=\sqrt{3}\omega_p/v_F$ for $k_BT\ll E_F$, and $k_s\cong k_D=\omega_P(m/k_BT)^{1/2}$ for $k_BT\gg E_F$. One can also write Eq. (32) in the form $k_s^{-4}=k_{\mathrm{TF}}^{-4}+k_D^{-4}$, as in the approximation of Brysk *et al.*[11,12] Equation (31) then yields the following limits.

(i) $\theta\ll1$. Here,

$$S(k,\omega)\cong\frac{2m^2e^2k\omega}{\hbar^3(k^2+k_{\mathrm{TF}}^2)^2},\quad \text{for } k\le 2k_F \qquad (33)$$

and $S(k,\omega)=0$ for $k>2k_F$.

(ii) $\theta\gg1$. Here,

$$S(k,\omega)\cong\frac{nm^2e^2k\omega}{(k^2+k_D^2)^2}\left[\frac{2\pi}{mk_BT}\right]^{3/2}\exp\left[-\frac{\hbar^2k^2}{8mk_BT}\right]. \qquad (34)$$

Equation (33) corresponds to absorption of small amounts of energy $\hbar\omega\ll E_F$ by a degenerate electron gas. Owing to the exclusion principle only those electrons close to the Fermi surface can participate. Thus the momentum transfer $\hbar k$ can never be larger than $2\hbar k_F$.

This restriction is relaxed for high temperatures, as shown by Eq. (34), where excitations with small ω but large k values can occur—they involve electrons in the tail of the Maxwell-Boltzmann distribution, and thus contribute with exponentially decaying probability. Yet this is a characteristic quantum effect, as indicated clearly by the factor $\exp(-\hbar^2k^2/8mk_BT)$ which replaces the analogous factor $\exp(-m\omega^2/2k_BTk^2)$ arising in classical theories[9] [cf. also expressions (4,2) and (4,3) of Table II].

(b) Single-particle ridge. Let us now look at the region where $u \cong z \gg 1$, i.e., $\hbar\omega \cong \hbar^2k^2/2m$, corresponding to short-range excitations of individual electrons in the plasma.

Using Table I we obtain, from expression (2,4),

$$\epsilon_1(k,\omega) \cong 1 + O(k_F/k)^4 \;, \tag{35}$$

and from expression (2,4) of Table II,

$$\epsilon_2(k,\omega) \cong \frac{\pi\chi_0^2}{8z^3}\theta \ln\{1+\exp[\eta - D(u-z)^2]\} \ll 1 \;. \tag{36}$$

Equation (35) indicates that one can neglect screening effects in this region of high frequencies and short wavelengths. Then we set $\epsilon_1 \cong 1$ and approximate

$$S(k,\omega) \cong \epsilon_2(k,\omega)$$
$$\cong \left[\frac{k_BT}{E_F}\right]\frac{k_F^2}{a_0k^3}\ln\left[1+\exp\left[\frac{\eta-\beta(\hbar\omega-E_k)^2}{4E_k}\right]\right], \tag{37}$$

where $E_k = \hbar^2k^2/2m$ and $a_0 = \hbar^2/me^2$ (Bohr radius). This yields the following limits.

(i) $\theta \ll 1$. Here,

$$S(k,\omega) \cong \frac{k_F^2}{a_0k^3}\left[1-\frac{(\hbar\omega-E_k)^2}{4E_FE_k}\right], \tag{38}$$

(ii) $\theta \gg 1$. Here,

$$S(k,\omega) \cong \frac{2\pi ne^2}{\hbar k^3}\left[\frac{2\pi m}{k_BT}\right]^{1/2}\exp\left[-\frac{\beta(\hbar\omega-E_k)^2}{4E_k}\right]. \tag{39}$$

In all cases the locus of the ridge is given by the quantum condition $\hbar\omega = E_k = \hbar^2k^2/2m$, and the width is determined by the initial distribution of electron velocities. This explains the parabolic profile at low temperatures, Eq. (38), and the Gaussian shape for $\theta \gg 1$, Eq. (39).

The existence of the single-particle ridge is another well-defined quantum feature, with no correspondence in the classical or semiclassical treatments of the dielectric function.

(c) Plasma resonance. As a last example we now turn our attention to the collective resonance of the plasma. It is given by the condition $\epsilon(k,\omega)=0$, which defines the modes of collective motion of the electrons. Using the appropriate long-wavelengths expansion of $\epsilon_1(k,\omega)$, expression (2,2) of Table I, and considering $\epsilon_2(k,\omega) \ll 1$, we find the frequency of the resonance ω_k given as in Eq. (22). The solution of this equation constitutes a dispersion relation of the form

$$\omega_k^2 \cong \omega_P^2\left[1+\langle v^2\rangle\frac{k^2}{\omega_P^2}+\langle v^4\rangle\frac{k^4}{\omega_P^4}+\left[\frac{\hbar}{2m}\right]^2\frac{k^4}{\omega_P^2}+\cdots\right], \tag{40}$$

up to terms in k^4. In particular, this gives the limits

$$\omega_k^2 \cong \omega_P^2\left[1+\tfrac{3}{5}v_F^2\frac{k^2}{\omega_P^2}+\tfrac{3}{7}v_F^4\frac{k^4}{\omega_P^4}+\left[\frac{\hbar}{2m}\right]^2\frac{k^4}{\omega_P^2}\right], \quad \theta \ll 1 \tag{40'}$$

$$\omega_k^2 \cong \omega_P^2\left[1+\frac{3k_BT}{m}\frac{k^2}{\omega_P^2}+15\left[\frac{k_BT}{m}\right]^2\frac{k^4}{\omega_P^4}+\left[\frac{\hbar}{2m}\right]^2\frac{k^4}{\omega_P^2}\right], \quad \theta \gg 1 \;. \tag{40''}$$

The terms in $\langle v^2\rangle$ and $\langle v^4\rangle$ in Eq. (40) correspond to those in $F_{3/2}(\eta)$ and $F_{5/2}(\eta)$ in Eq. (22), as follows from Eq. (A2) in the Appendix. The last term in Eq. (40) represents a contribution due to single-particle dispersion. The ratio between the two terms in k^4 is $(\hbar/2m)^2\omega_P^2/\langle v^4\rangle$, which is of order $(\hbar\omega_P/E_F)^2 \sim 1/v_F$ for $\theta \ll 1$, and of order $(\hbar\omega_P/k_BT)^2 \ll 1$ for $\theta \gg 1$. Thus, for dilute plasmas where $\theta \gg 1$ the single-particle term in order k^4 can be neglected.

By neglecting both terms of order k^4 in Eq. (40) we obtain $\omega_k^2 \cong \omega_P^2 + \langle v^2\rangle k^2$, with $\langle v^2\rangle = \tfrac{3}{2}v_F^2\theta^{5/2}F_{3/2}(\eta)$, which is in agreement with the Bohm-Gross dispersion relation in the corrected form given by Gouedard and Deutsch.[7]

Moreover, introducing the approximation for $F_{3/2}(\eta)$ obtained in the Appendix [Eq. (A8)] here, we can write

$$\langle v^2\rangle \cong \tfrac{3}{5}v_F^2(1+\tfrac{25}{4}\theta^2)^{1/2} = \left[\tfrac{9}{25}v_F^4 + 9\left[\frac{k_BT}{m}\right]^2\right]^{1/2}, \tag{41}$$

which permits a fast calculation of $\langle v^2\rangle$ for all degeneracies ($\langle v^2\rangle \cong \tfrac{3}{5}v_F^2$ for $\theta \ll 1$ and $\langle v^2\rangle \cong 3k_BT/m$ for $\theta \gg 1$). The energy-loss function then becomes

$$S(k,\omega) = \frac{\epsilon_2}{\epsilon_1^2+\epsilon_2^2} \cong \frac{\omega_k^4\epsilon_2(k,\omega_k)}{(\omega^2-\omega_k^2)^2+\omega_k^4\epsilon_2^2(k,\omega_k)} \tag{42}$$

and its limits for low and high temperatures are as follows.

(i) $\theta \ll 1$. Since $\epsilon_2(k,\omega)=0$ over the region $\omega > kv_F$ (see Table II) we obtain, here, a sharp plasma resonance

$$S(k,\omega) \cong \frac{\pi\omega_P^2}{2\omega_k}\delta(\omega-\omega_k) \;. \tag{43}$$

(ii) $\theta \gg 1$. In this case $\epsilon_2(k,\omega)$ is given by expression (4,2) of Table II, i.e.,

$$\epsilon_2(k,\omega) \cong \frac{ne^2\omega}{mk^3}\left[\frac{2\pi m}{k_BT}\right]^{3/2}\exp\left[-\frac{m}{2k_BT}\frac{\omega^2}{k^2}\right]. \tag{44}$$

This is now a fully classical result. The fact that $\epsilon_2(k,\omega)$ has a finite value for $\omega \cong \omega_k$ gives rise to the damping of the plasma resonance: electrons in the tail of the thermal distribution, with velocities close to the phase velocity of the plasma wave, $v_e \sim \omega_P/k$, absorb energy from the wave. Such is the phenomenon of "Landau damping" as is known in the context of classical plasmas.[2,3] Inspection of Eq. (44) shows that the damping becomes important for wavelengths shorter than the Debye length; i.e., $k^{-1} < \lambda_D = \omega_P(m/k_BT)^{1/2}$.

With increasing values of $\theta = k_BT/E_F$ the transition between the previous cases occurs as illustrated in Fig. 3, where we show the "line-shape function" obtained from $S(k,\omega)$ in Eq. (42), normalized to one, with $\epsilon_2(k,\omega)$ given by [cf. Table II, expression (2,2)],

$$\epsilon_2(k,\omega) \cong \frac{2m^2e^2\omega}{(\hbar k)^3}\left[1+\exp\left[\frac{m\omega^2}{2k_BTk^2}-\eta\right]\right]^{-1} . \tag{45}$$

The resonance develops a width (damping) that increases with temperature. By comparison with Fig. 2 we see that the increasing damping of the plasma waves occurs when the tail of the function $\epsilon_2(k,\omega)$ reaches the resonance at $\omega_k \cong \omega_P$. The effect is specially important in the classical domain and for $k\lambda_D > 1$.

In this particular case the transition from quantum to classical behavior coincides with the transition from degenerate to nondegenerate plasmas.

IV. SUMMARY

We have obtained expressions for the dielectric function of quantum plasmas of arbitrary degeneracies which connect previous results for degenerate and nondegenerate systems. They incorporate thermal and quantum effects on $\epsilon(k,\omega)$, and include the results from classical and semiclassical approximations as well as those of quantum calculations for a degenerate plasma. This was illustrated with a few examples: the low-frequency behavior with reference to screening effects, the single-particle ridge, and the plasma resonance.

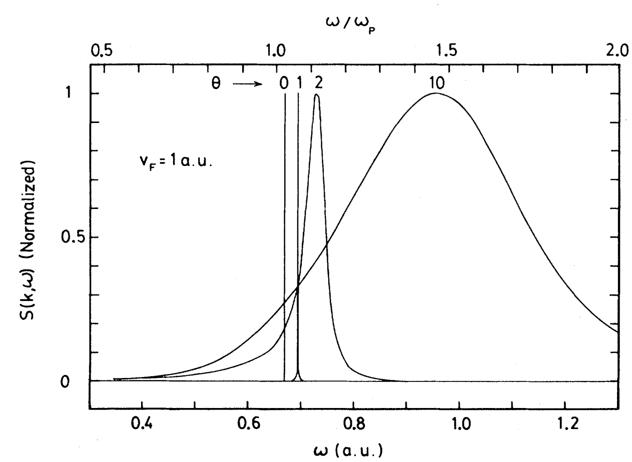

FIG. 3. Thermal broadening of the plasma resonance as obtained from the energy-loss function $S(k,\omega)$ in Eq. (42), after normalization to the peak values for the cases $\theta=0$, 1, 2, and 10. As in Fig. 2, these results correspond to $v_F = 1$ a.u. ($\omega_P = 0.651$ a.u.) and $k/k_F = 0.2$.

For nondegenerate plasmas the quantum-mechanical results agree with the classical results provided that the wavelengths and frequencies of interest are such that $\hbar^2k^2/2m \ll k_BT$ and $\hbar\omega \ll k_BT$. In particular, the single-particle behavior for $\hbar\omega \sim \hbar^2k^2/2m$ cannot be described by classical or semiclassical expressions.

An important application of these results is the calculation of energy-loss rates for protons, α particles, and other ions in plasmas of various degrees of degeneracy. Inclusion of quantum effects in the excitation of the plasma is important here for a proper description of short-wavelength phenomena, and to avoid divergent behaviors. In a forthcoming paper the results of this work will be applied to the calculation of plasma stopping powers.

ACKNOWLEDGMENTS

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APPENDIX

We summarize here some of the properties of the Fermi integrals and give approximate expressions to calculate $F_\nu(\eta)$ for the cases $\nu = -\frac{1}{2}$ and $\frac{3}{2}$ appearing in the text. First we write $F_\nu(\eta)$ in the form

$$F_\nu(\eta) = \int_0^\infty \frac{x^\nu dx}{1+e^{x-\eta}} . \tag{A1}$$

The statistical average of powers of the electron velocities, $\langle v^\alpha \rangle$, is then given by

$$\langle v^\alpha \rangle = \int_0^\infty \frac{v^\alpha \rho(E)dE}{1+e^{\beta(E-\mu)}} = \tfrac{3}{2} v_F^\alpha \theta^{(\alpha+3)/2} F_{(\alpha+1)/2}(\eta) , \tag{A2}$$

where $E = \frac{1}{2}mv^2$, $\rho(E) = \frac{3}{2}E^{1/2}/E_F^{3/2}$ is the density of states, $\theta = k_BT/E_F$, and $\eta = \beta\mu = \mu/k_BT$.

The normalization of the distribution function, $\int f(v)d^3v = n$—with $f(v) = 2m^3\hat{f}(k)/(2\pi\hbar)^3$, and $\hat{f}(k)$ given by Eq. (3)—is equivalent to the condition $\langle v^0 \rangle = 1$, which yields

$$F_{1/2}(\eta) = \tfrac{2}{3}\left[\frac{E_F}{k_BT}\right]^{3/2} . \tag{A3}$$

This is the usual relation that gives the chemical potential $\mu = \eta k_BT$ in terms of $D = E_F/k_BT = 1/\theta$.

In the limiting cases $\theta \ll 1$ and $\theta \gg 1$ we obtain the following.

(i) $k_BT \ll E_F$. Here,

$$F_\nu(\eta) \cong \frac{\eta^{(\nu+1)}}{(\nu+1)} \cong \frac{1}{(\nu+1)}\left[\frac{E_F}{k_BT}\right]^{\nu+1} , \tag{A4}$$

$$\eta \cong E_F/k_BT . \tag{A4'}$$

(ii) $k_BT \gg E_F$. Here,

TABLE III. Numerical results for the functions $F_{-1/2}(\eta)$, $F_{1/2}(\eta)$, and $F_{3/2}(\eta)$, and values of $D=E_F/k_BT$, for a range of η values, $-5\le\eta\le10$. Limiting expressions of Eqs. (A4)—(A5′) can be used outside this range.

$\eta=\mu/k_BT$	$D=E_F/k_BT$	$F_{-1/2}(\eta)$	$F_{1/2}(\eta)$	$F_{3/2}(\eta)$
-5	4.31×10^{-2}	1.19×10^{-2}	5.96×10^{-3}	8.95×10^{-3}
-4	8.36×10^{-2}	3.20×10^{-2}	1.61×10^{-2}	2.43×10^{-2}
-3	0.1617	8.53×10^{-2}	4.34×10^{-2}	6.56×10^{-2}
-2	0.3091	0.2192	0.1146	0.1758
-1	0.5747	0.5212	0.2905	0.4608
0	1.011	1.072	0.6780	1.153
1	1.637	1.820	1.396	2.662
2	2.415	2.595	2.502	5.537
3	3.289	3.285	3.977	10.35
4	4.216	3.874	5.771	17.63
5	5.170	4.383	7.838	27.80
6	6.140	4.834	10.14	41.26
7	7.120	5.242	12.66	58.34
8	8.104	5.617	15.38	79.35
9	9.092	5.967	18.28	104.6
10	10.08	6.297	21.34	134.3

$$F_\nu(\eta)\cong\Gamma(\nu+1)e^{\eta}\cong\frac{4\Gamma(\nu+1)}{3\sqrt{\pi}}\left[\frac{E_F}{k_BT}\right]^{3/2}, \quad \text{(A5)}$$

$$\eta\cong\ln\left[\frac{4}{3\sqrt{\pi}}\left[\frac{E_F}{k_BT}\right]^{3/2}\right]<0\,, \quad \text{(A5′)}$$

where $\Gamma(x)$ denotes the gamma function.

In Table III we show numerical results for the functions $F_\nu(\eta)$ in the cases of $\nu=-\frac{1}{2}$, $\frac{1}{2}$, and $\frac{3}{2}$, which are of interest for the analysis give in the text.

Analytical approximations

From the limiting results of Eqs. (A4) and (A5), for $\nu=-\frac{1}{2}$ and $\frac{3}{2}$, we find the following.

(i) For $\theta\ll1$,

$$F_{-1/2}(\eta)\cong2D^{1/2},\quad F_{3/2}(\eta)\cong\tfrac{2}{5}D^{5/2}\,. \quad \text{(A6)}$$

(ii) For $\theta\gg1$,

$$F_{-1/2}(\eta)\cong\tfrac{4}{3}D^{3/2},\quad F_{3/2}(\eta)\cong D^{3/2}\,. \quad \text{(A6′)}$$

Rather obvious interpolations between these results are given by the simple functions

$$F_{-1/2}(\eta)\cong\frac{4}{(4\theta+9\theta^3)^{1/2}}\,, \quad \text{(A7)}$$

$$F_{3/2}(\eta)\cong\frac{1}{\theta^{5/2}}\left(\tfrac{4}{25}+\theta^2\right)^{1/2}\,, \quad \text{(A8)}$$

and for completeness we recall the exact result,

$$F_{1/2}(\eta)=\tfrac{2}{3}D^{3/2}\,. \quad \text{(A9)}$$

Equations (A7) and (A8) agree with the exact numerical results for $F_{-1/2}(\eta)$ and $F_{3/2}(\eta)$, in Table III, with precisions better than 5% and 6%, respectively (maximum errors occur at $\eta\cong0$ and $\eta\cong1$).

*Deceased.

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