Adiabatic elimination in nonlinear dynamical systems

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We reconsider the problem of the adiabatic elimination of selected dynamical variables in the description of nonlinear systems, with a special emphasis on the identification of suitable criteria for the global validity of this procedure. The problem is analyzed in detail using as a guideline the one-mode homogeneously broadened laser model, with an injected signal and an arbitrary population difference for added flexibility. We propose five conditions for the global validity of the adiabatic limit, after consideration not only of the relative size of the time scales involved, but also of the magnitude of all parameters, of the physical variables, and of their fluctuations. The adiabatic elimination is formulated in the context of the dressed-mode description, leading to a generalization of the procedure adopted in earlier studies of multimode absorptive bistability. In addition, we establish a rigorous link between the adiabatic-elimination procedure and the multiple-time-scale approach to dynamic systems. The scaling behavior of the variables, of the eigenvalues of the linearized equations, and of the dressed-mode amplitudes and eigenvectors with respect to the smallness parameter of the problem is studied in detail. The relevance of what we shall define as "normal" and "anomalous" scaling in the text to the applicability of the adiabatic-elimination process is clarified. We also develop an alternative adiabatic-elimination scheme for the special case where all the rate constants of the system have the same order of magnitude, but another smallness parameter can be identified. From our analysis, it should be clear that our main conclusions are model independent, and not at all restricted to the specific features of the dynamical system selected as a test case for our discussion.

I. INTRODUCTION

As emphasized by Haken,[1] the so-called *adiabatic-elimination* process plays a fundamental role in understanding the origin of self-organization in open systems far from thermal equilibrium. In addition, the elimination of the fast variables can lead to a considerable reduction in the complexity of a problem, sometimes to the point that analytic solutions become accessible.

The adiabatic-elimination procedure described by Haken in his study of the generalized Ginzburg-Landau equations for phase-transition-like phenomena in open systems[2] is designed to cope with the behavior of a nearly critical system, a situation where the control parameters are adjusted in such a way that a stationary state is about to become unstable. In this case, part of the system, the so-called unstable modes, evolves through a very slow process of damping or amplification (the unstable modes are said to become soft); thus the remaining (stable) modes, which still evolve at their normal rates, can be eliminated adiabatically. The accuracy of this procedure is controlled by the difference $\lambda-\lambda_c$ between a control parameter λ and its critical value λ_c and is, normally, adequate when this difference is sufficiently small. In addition, this type of adiabatic elimination has a "local character" in that it allows the construction of solutions only when these are suitably "close" to the stationary configuration that has become unstable.

To be more precise, if the state of the system is specified by the element $\vec{V}$ of an n-dimensional vector space, and if $\vec{V}_{st}$ corresponds to the steady-state configuration that becomes unstable, the distance $|\vec{V}-\vec{V}_{st}|$ must be of the order of some appropriate positive power of $\lambda-\lambda_c$. This limitation is common also to standard bifurcation theory.

In practice, however, the adiabatic elimination is carried out in a global sense, after assuming that the rate constants which appear in the equations of motion of the system of interest are sufficiently different from one another. Sometimes, this requirement is supplemented by more refined conditions in connection with specific problems, but we are not aware of the existence of a systematic procedure that specifies the conditions under which the adiabatic elimination is applicable in a global sense. This problem is even more acute in the case of fully quantum-statistical treatments.

The aim of this paper is to attempt a definition of such a procedure, and to lay the foundations (hopefully) for a rigorous theory of the adiabatic elimination in a global sense. In the process, we show that the adiabatic-elimination conditions impose requirements not only on the rate constants, but also on all physical parameters of the system. Even more so the validity of the procedure hinges on the order of magnitude of the vector $\vec{V}$, i.e., of the same variables that appear in the time-evolution equations. On the contrary, no conditions are imposed on the size of the difference $\lambda-\lambda_c$ or on the distance $|\vec{V}-\vec{V}_{st}|$. Thus, in this sense, the elimination is global.

We arrive at this result through (i) the identification of a suitable smallness parameter, given by the ratio between two rate constants and (ii) the selection of the appropriate region in control parameter space where the adiabatic-elimination process is possible on a global scale. The latter step will be guided by consideration of the size of the relevant physical parameters with respect to the smallness parameter. An important point in our procedure is a particular type of scaling of the variables ("normal" scaling). This has been used repeatedly in the past, but its role with respect to the adiabatic elimination has not been clarified to our knowledge. From our analysis, it will become apparent that some of the conditions may not be satisfied in specific instances, and that the adiabatic-elimination procedure cannot be carried out, even if the rate constants are sufficiently different from one another. For an important class of problems of this type, a different scaling of the variables ("anomalous" scaling) becomes necessary.

We also extend the usual adiabatic-elimination ideas to cover the little-studied problem where all the rate constants have the same order of magnitude. This situation requires a separate study because our five conditions no longer apply. Nevertheless, we show that an alternative scheme can be set up, in some instances, which leads to an adiabatic elimination of a new type. It is very unlikely that this work will complete the discussion on the global adiabatic elimination. For this reason, and in order to avoid excessive use of general mathematical formalism in our analysis, we focus our attention on a specific and well-known model of a dynamical system: the one-mode laser, which is equivalent[3] to the Lorenz model[4] of hydrodynamics. This system is sufficiently rich to lend itself to a variety of considerations. We make it more flexible with the inclusion of an external field (so that it can simulate also a bistable system, or a laser with injected signal) and of atomic and cavity detunings. It should be clear from our discussion that the considerations of the following sections are not limited to this model.

An interesting feature that emerges from our study concerns the behavior of the instabilities. Usually, when the rate constants in a nonlinear model are allowed to become sufficiently different from one another, several instabilities disappear. This is well known to occur in the laser model. We show, however, that all the hard-mode instabilities that persist in the adiabatic limit are accompanied by anomalous scaling.

This research takes its premises from the work by some of us[5] on the dressed-mode theory of optical bistability. In that case, it was possible to perform a global and rigorous adiabatic elimination of the atomic variables and to obtain a quasianalytical treatment of self-pulsing. Some of the ideas developed in Ref. 5 will be generalized in this work.

Even if consideration of the character and size of the fluctuations of the variables is essential in deriving the adiabatic-elimination conditions, the treatment here is purely semiclassical. An application and extension of the rigorous global adiabatic-elimination limit to a fully quantum-statistical laser model will be worked out in a separate paper.[6]

II. THE ONE-MODE LASER MODEL WITH INJECTED SIGNAL—NORMAL SCALING

This model was introduced and elaborated on in the 1960's by Haken,[7] Lamb,[8] and their respective collaborators. Subsequently, it was generalized to include a coherent external field by Bonifacio and Lugiato.[9] We consider a system of N identical two-level atoms with transition frequency ω_a interacting with a radiation field mode of frequency ω_c. The atoms fill a resonant cavity of length L and volume V with mirrors of transmittivity T. An external cw coherent field of frequency ω_0 is injected into the cavity. We label with a the expectation value of the field annihilation operator for the cavity mode, with P the expectation value of the complex macroscopic atomic polarization, and with D the expectation value of one-half the difference between the population of the lower and of the upper levels. In the semiclassical, dipole, and rotating-wave approximations, these variables obey the time-evolution equations

$$\frac{da}{dt}=-i\omega_c a-gP-\kappa(a-\alpha_0 e^{-i\omega_0 t}) , \tag{1a}$$

$$\frac{dP}{dt}=-i\omega_a P+2gaD-\gamma_\perp P , \tag{1b}$$

$$\frac{dD}{dt}=-g(aP^*+a^*P)-\gamma_{||}\left[D+\frac{\sigma N}{2}\right] . \tag{1c}$$

The variables a^* and P^* satisfy the complex-conjugate equations of (1a) and (1b), respectively; D is real by definition. The meaning of the symbols in Eqs. (1) is as follows: g is the coupling constant

$$g=\left[\frac{2\pi}{\hbar\omega_c V}\right]^{1/2}\omega_a\mu , \tag{2}$$

where μ is the modulus of the atomic dipole moment, κ is the cavity damping constant

$$\kappa=\frac{cT}{2L} , \tag{3}$$

and α_0, which is taken to be real and positive for definiteness, is a c-number proportional to the amplitude of the coherent injected field. To be more precise, α^2 and a^2 represent the average number of photons in the injected and cavity fields, respectively. $\gamma_\perp$ and $\gamma_{||}$ are the transverse and longitudinal atomic decay rates, which obey the

inequality $\gamma_{\parallel} \leq 2\gamma_{\perp}$, σ is the unsaturated inversion per atom created by the pump; the range of σ is $-1 \leq \sigma \leq 1$. The usual laser model corresponds to the selection of parameters $\sigma > 0$ and $\alpha_0 = 0$, while $\sigma > 0$ and $\alpha_0 \neq 0$ correspond to the laser with injected signal. The choice $\sigma = -1$ (no pump action) and $\alpha_0 \neq 0$ leads to optically bistable action.

In order to eliminate the explicit time dependence from the coefficients of Eqs. (1) we set

$$\tilde{a} = a e^{i\omega_0 t}, \quad \tilde{P} = P e^{i\omega_0 t}, \quad \tilde{D} = D \tag{4}$$

with the result

$$\frac{d\tilde{a}}{dt} = -i(\omega_c - \omega_a)\tilde{a} - g\tilde{P} - \kappa(\tilde{a} - \alpha_0) , \tag{5a}$$

$$\frac{d\tilde{P}}{dt} = -i(\omega_a - \omega_0)\tilde{P} + 2g\tilde{a}\tilde{D} - \gamma_{\perp}\tilde{P} , \tag{5b}$$

$$\frac{d\tilde{D}}{dt} = -g(\tilde{a}\tilde{P}^* + \tilde{a}^*\tilde{P}) - \gamma_{\parallel}\left[\tilde{D} + \frac{\sigma N}{2}\right] . \tag{5c}$$

Equations (5) are not yet in a convenient form to attack the problem of the adiabatic elimination because it is not easy to assess the order of magnitude of the various parameters with respect to the natural smallness parameter (the ratio between the small and large rate constants). The situation becomes much more transparent if we introduce the following scaled variables:

$$x = \frac{\tilde{a}}{\sqrt{N_s}}, \quad y = \frac{\alpha_0}{\sqrt{N_s}} , \tag{6a}$$

$$p = \left[-\frac{\sigma N}{2}\left[\frac{\gamma_{\parallel}}{\gamma_{\perp}}\right]^{1/2}\right]^{-1}\tilde{P}, \quad \tilde{d} = \left[-\frac{\sigma N}{2}\right]^{-1}\tilde{D} , \tag{6b}$$

where N_s is the saturation photon number

$$N_s = \frac{\gamma_{\perp}\gamma_{\parallel}}{4g^2} . \tag{7}$$

We call this transformation the normal scaling because it is the one which is suggested naturally by the steady-state analysis of the system of equations (5). If we restrict ourselves to the resonant case, $\omega_c = \omega_a = \omega_0$, for simplicity, the steady-state values of the atomic variables are given by

$$\tilde{P} = -\frac{\sigma N}{2}\left[\frac{\gamma_{\parallel}}{\gamma_{\perp}}\right]^{1/2}\frac{\tilde{a}/\sqrt{N_s}}{1+\tilde{a}^2/N_s} , \tag{8a}$$

$$\tilde{D} = -\frac{\sigma N}{2}\frac{1}{1+\tilde{a}^2/N_s} , \tag{8b}$$

while $\tilde{a}$ is the solution of the following state equation:

$$\alpha_0 = \tilde{a} - \frac{2C\tilde{a}}{1+\tilde{a}^2/N_s} , \tag{9}$$

where C, the cooperation parameter, is defined by

$$C = \frac{g^2 N\sigma}{2\kappa\gamma_{\perp}} . \tag{10}$$

We note, although it is rather obvious, that the specific form of the scaling relations (6) is not tied to the selection of the variable which appears in the state equation [e.g., a in Eq. (9)]. The state equation could have been expressed in terms of D, for example, with the same results:

$$\tilde{D} + \frac{\sigma N}{2} + \frac{(\alpha_0^2/N_s)\tilde{D}}{1 - 2C(-\sigma N/2)^{-1}\tilde{D}} = 0 . \tag{9'}$$

For a general nonlinear set of equations, the normal scaling is the one that yields the simplest-looking steady-state equations and that reduces the number of free parameters to a minimum. Furthermore, the steady-state values of the normalized variables can be expected to be of the order of unity.

With the use of the normal scaling transformation (6), Eqs. (5) take the form

$$\frac{dx}{dt} = -\kappa[x(1+i\theta) - y - 2Cp] , \tag{11a}$$

$$\frac{dp}{dt} = -\gamma_{\perp}[p(1+i\Delta) - x\tilde{d}] , \tag{11b}$$

$$\frac{d\tilde{d}}{dt} = -\gamma_{\parallel}[\tfrac{1}{2}(xp^* + x^*p) + \tilde{d} - 1] , \tag{11c}$$

where θ and Δ are the cavity and atomic detuning parameters, respectively, i.e.,

$$\theta = \frac{\omega_c - \omega_0}{\kappa} , \tag{12a}$$

$$\Delta = \frac{\omega_a - \omega_0}{\gamma_{\perp}} . \tag{12b}$$

In the following discussion, for simplicity, we shall often restrict ourselves to the case of perfect tuning $\theta = \Delta = 0$; under this condition, if the variables x and p are initially real, they remain real throughout the entire time evolution. Thus, in resonance, Eqs. (11) become

$$\frac{dx}{dt} = -\kappa(x - y - 2Cp) , \tag{13a}$$

$$\frac{dp}{dt} = -\gamma_{\perp}(p - x\tilde{d}) , \tag{13b}$$

$$\frac{d\tilde{d}}{dt} = -\gamma_{\parallel}(xp + \tilde{d} - 1) . \tag{13c}$$

In this form, the equivalence of Eqs. (13) with the Lorenz model[3,4] is easy to assess.

III. ADIABATIC-ELIMINATION CONDITIONS

One of the main advantages of Eqs. (11) and (13) is that *all* terms in each equation are proportional to the *same* rate constant. This feature makes the identification of the conditions for the adiabatic elimination more direct and devoid of ambiguities. In this section we state three conditions for the validity of the adiabatic-elimination process; two additional requirements will be given in Sec. VII. The first condition is obvious.

Condition 1. The rate constants can be divided into two groups, such that all the constants within each set have the same order of magnitude. By arbitrary selection of one rate constant from each group, the ratio between the smaller and the larger of the two rates is the smallness pa-

rameter ϵ of the adiabatic-elimination process ($\epsilon \ll 1$).

The second condition is suggested at once by the structure of Eqs. (11) and (13).

Condition 2. The quantities x, y, p, $\tilde{d}$, C, θ, and Δ remain finite in the limit $\epsilon \to 0$. Condition 2 makes the application of the adiabatic-elimination process straightforward. Consider, for example, Eqs. (13) in the case in which the atomic variables are to be eliminated adiabatically, i.e., $\kappa/\gamma_\perp = \epsilon$ and $\gamma_\parallel/\gamma_\perp = O(\epsilon^0)$. The longer-lived process evolves over a time scale of the order of κ^{-1}. By setting $\tau = \kappa t$, Eqs. (13) become

$$\frac{dx}{d\tau} = -x + y + 2Cp \ , \tag{14a}$$

$$\frac{\kappa}{\gamma_\perp}\frac{dp}{d\tau} = -p + x\tilde{d} \ , \tag{14b}$$

$$\frac{\kappa}{\gamma_\parallel}\frac{d\tilde{d}}{d\tau} = -xp - \tilde{d} + 1 \ , \tag{14c}$$

and, in the limit $\epsilon \to 0$, they reduce to the form

$$\frac{dx}{d\tau} = -x + y + 2C\frac{x}{1+x^2} \ , \tag{15a}$$

$$p = \frac{x}{1+x^2}, \quad \tilde{d} = \frac{1}{1+x^2} \ . \tag{15b}$$

A more rigorous procedure to perform the adiabatic elimination will be described in Sec. IX. Now we consider the parameters that appear in the original Eqs. (5) in order to assess the order of magnitude of each quantity with respect to the smallness parameter. Condition 2 is not sufficient to accomplish this task. We must add the following.

Condition 3. In the limit $\epsilon \to 0$, the fluctuations of all the variables must not diverge. In fact, an intrinsic condition for the validity of the semiclassical equations is that the fluctuations be small during the entire evolution; hence, condition 3 is a minimal requirement.

As it turns out, the magnitude of the fluctuations of x, p, and $\tilde{d}$ is set by the same parameters that appear in the scaling relations (6). More precisely, the fluctuations of x are of the order of N_s^{-1}, while those of p and $\tilde{d}$ are of the order of N^{-1}.[10,11] Hence, according to condition 3, we must require that N_s^{-1} and/or N^{-1} either remain finite or vanish in the limit $\epsilon \to 0$.

We now apply conditions 2 and 3 to the different types of adiabatic eliminations that are usually carried out with the model equations (13). If we generally denote by γ_L the "large" rate constant, we can write

$$2C = \frac{(g\sqrt{N}/\gamma_L)^2}{(\kappa/\gamma_L)(\gamma_\perp/\gamma_L)} \ , \tag{16a}$$

$$N_s^{-1} = 4N^{-1}\frac{(g\sqrt{N}/\gamma_L)^2}{(\gamma_\parallel/\gamma_L)(\gamma_\perp/\gamma_L)} \ , \tag{16b}$$

where $(g\sqrt{N})^{-1}$ can be recognized as the cooperation time in cooperative spontaneous emission.[12] By using condition 2, which requires C to be finite, we obtain in each case the order of magnitude of the ratio $g\sqrt{N}/\gamma_L$ relative to the smallness parameter ϵ. Using condition 3 we can finally assess the order of magnitude of N^{-1}.

(a) Elimination of the atomic polarization and inversion. In this case, we set $\gamma_L \equiv \gamma_\perp$, $\kappa/\gamma_\perp = \epsilon$, and $\gamma_\parallel/\gamma_\perp = O(\epsilon^0)$. Conditions 2 and 3 require that $(g\sqrt{N}/\gamma_\perp)^2 = O(\epsilon)$ and $N^{-1} = O(\epsilon^0)$, and this automatically ensures that $N_s^{-1} = O(\epsilon)$.

(b) Elimination of the atomic polarization. Here we select $\gamma_L = \gamma_\perp$, $\kappa/\gamma_\perp = \epsilon$ and $\gamma_\parallel/\gamma_\perp = O(\epsilon)$. In this case, we obtain the requirements $(g\sqrt{N}/\gamma_\perp)^2 = O(\epsilon)$ and $N^{-1} = O(\epsilon)$, and, as a result, $N_s^{-1} = O(\epsilon)$.

(c) Elimination of the field and of the atomic polarization. Here we set $\gamma_L = \gamma_\perp$, $\kappa/\gamma_\perp = O(\epsilon^0)$, and $\gamma_\parallel/\gamma_\perp = \epsilon$. The orders of magnitude of the parameters are $(g\sqrt{N}/\gamma_\perp) = O(\epsilon^0)$, $N^{-1} = O(\epsilon)$, and, under these conditions, $N_s^{-1} = O(\epsilon^0)$. Note that condition 3 is essential in this case.

(d) Elimination of the field. We select $\gamma_L = \kappa$, $\gamma_\perp/\kappa = \epsilon$, and $\gamma_\parallel/\gamma_\perp = O(\epsilon^0)$ and obtain the order of magnitude $(g\sqrt{N}/\kappa)^2 = O(\epsilon)$ and $N^{-1} = O(\epsilon)$. In this case, $N_s^{-1} = O(\epsilon^0)$ and, again, condition 3 is essential.

As we can see from the above examples, condition 3 does not assign the order of magnitude of N^{-1} uniquely. For example, in case (b) above, the choice $N^{-1} = O(\epsilon^0)$ would also be consistent with condition 3. We have selected, instead, $N^{-1} = O(\epsilon)$ because with this choice the fluctuations of the eliminated variables are of order ϵ, in line with the results of cases (a), (c), and (d). Thus, the smallness of the fluctuations of these variables is automatically guaranteed in the adiabatic limit.

IV. REMARKS

At this point we make the following remarks.

(i) The adiabatic limit, defined in Sec. III, does not impose any special restrictions on the control parameters C and y, apart from being finite. The same holds true for the variables x, p, and $\tilde{d}$. Hence, the adiabatic elimination is global.

(ii) Consider for definiteness case (a) of Sec. III. The condition $\kappa/\gamma_\perp = \epsilon \to 0$ can be satisfied equally well by letting κ approach zero, $\gamma_\perp$ approach infinity, or by any other double limit that produces the same result. The conclusions of Sec. III remain valid regardless of the procedure adopted in carrying out the limit $\epsilon \to 0$.

(iii) In Sec. III, we assumed that the rate constants of the problem can be divided into two groups, one containing only "small," and the other "large" rate constants. It is easy to see that our considerations can be immediately generalized to the case in which the rate constants fall naturally into more than two groups according to their orders of magnitude. Thus, for example, if the rate constants fall into three groups, the elimination can be performed in two steps. First, one can eliminate the variables corresponding to the group with the largest damping rates and, next, the variables corresponding to the intermediate group. In the case of our model, we have the following three different possibilities.

Elimination of the polarization, followed by the atomic inversion ($\kappa/\gamma_\parallel = \epsilon_1$, $\gamma_\parallel/\gamma_\perp = \epsilon_2$): here, we obtain $(g\sqrt{N}/\gamma_\perp)^2 = O(\epsilon_1\epsilon_2)$, $N^{-1} = O(\epsilon_2)$, and, consequently, $N_s^{-1} = O(\epsilon_1\epsilon_2)$.

Elimination of the polarization, followed by the field ($\gamma_{\parallel}/\kappa=\epsilon_1$, $\kappa/\gamma_{\perp}=\epsilon_2$): here, we obtain $(g\sqrt{N}/\gamma_{\perp})^2=O(\epsilon_2)$, $N^{-1}=O(\epsilon_1\epsilon_2)$, and, consequently, $N_s^{-1}=O(\epsilon_2)$.

Elimination of the field, followed by the atomic polarization ($\gamma_{\parallel}/\gamma_{\perp}=\epsilon_1$, $\gamma_{\perp}/\kappa=\epsilon_2$): here, we obtain $(g\sqrt{N}/\kappa)^2=O(\epsilon_2)$, $N_s^{-1}=O(\epsilon_1\epsilon_2)$, consequently, $N_s^{-1}=O(\epsilon^0)$.

(iv) The cooperation parameter C can also be written in the form[12]

$$C=\frac{\alpha L\sigma}{2T}\ , \tag{17}$$

where α is the unsaturated absorption coefficient per unit length

$$\alpha=\frac{g^2N}{C\gamma_{\perp}}\ . \tag{18}$$

Hence, the case (a) studied in Sec. III includes, in particular, the limit $\alpha L\to 0$, $T\to 0$ with C constant, which has been discussed in the dressed-mode theory of optical bistability.[5] In this limit, the fields $\tilde{a}$ and α_0 diverge, while the normalized variables x and y remain finite. Note that the incident and transmitted fields $E_I=(\mu/\hbar\sqrt{\gamma_{\perp}\gamma_{\parallel}T})^{-1}y$ and $E_T=(\mu/\hbar\sqrt{\gamma_{\perp}\gamma_{\parallel}T})^{-1}x$ (see Ref. 9) remain finite in this limit.

(v) With reference to the original variables $\tilde{a},\tilde{P},\tilde{D}$, we see that in all the cases studied in Sec. III, the variables that are not eliminated adiabatically diverge in the adiabatic limit. This divergence occurs also in the thermodynamic limit $N\to\infty$, $V\to\infty$, N/V finite. Because, on the other hand, the coupling constant g is proportional to $V^{-1/2}$, it follows at once that $x,p,\tilde{d}$ remain finite in the thermodynamic limit, so that in this limit, also, the normal scaling is the natural one to use.

We stress, however, that the adiabatic is different from the thermodynamic limit, because the eliminated variables (original variables) often remain finite. In fact, in the adiabatic limit, the smallness parameter is the ratio between small and large rate constants, while in the thermodynamic limit, the smallness parameter is N^{-1}. It is true that in many cases N^{-1} is of the order ϵ also in the adiabatic limit, but $g\sqrt{N}/\gamma_L$ is always finite in the thermodynamic limit, while it is small in most cases under the adiabatic-elimination conditions.

Of course, these considerations are not meant to negate the physical relevance of the fact that N^{-1} is indeed very small for macroscopic systems; for example, this fact is at the origin of the smallness of the fluctuations in all macroscopic systems, quite independently of whether the adiabatic process is valid or not. However, we stress that the *mathematical* definition of the adiabatic limit, as given in Sec. III, is different from and independent of the thermodynamic limit.

(vi) In all cases examined in Sec. III, the ratio g/γ_L vanishes in the adiabatic limit. This does not mean, however, that we are dealing with a weak-coupling theory. This is only a consequence of our prescription that the normalized variables $x,p,\tilde{d}$ remain finite. In fact, if we consider, for example, the case of the adiabatic elimination of the atomic variables, the weak-coupling limit in the sense of Van Hove[13] is defined by the conditions $\kappa\to 0$, $g^2\to 0$, and $t\to\infty$, with κ/g^2 finite and $\tau=\kappa t$ finite. This limit must be carried out in Eqs. (5) where the coupling constant g appears explicitly. In the resonant case $\omega_c=\omega_a=\omega_0$, by keeping the variables $\tilde{a},\tilde{P},\tilde{D}$ finite, we obtain the linear equation

$$\frac{d}{d\tau}\tilde{a}=-\tilde{a}+\alpha_0+\frac{g^2N\sigma}{\kappa\gamma_{\perp}}\tilde{a}\ . \tag{19}$$

This coincides with Eq. (15a) if we drop all the nonlinear terms. On the other hand, if we perform the limit while keeping x finite, the coupling constant g remains incorporated within the definition of x itself. Hence, if we analyze Eq. (15a) we easily see that the nonlinear term $(1+x^2)^{-1}$ contains all powers of the coupling constant. Thus, Eq. (15a) is representative of a strong-coupling theory, in spite of the smallness of g/γ_L.

V. THE CASE WITH DETUNINGS

We now drop the resonance assumption $\omega_c=\omega_a=\omega_0$ and consider first the case of a nonzero external field ($y\neq 0$). In this case, all the considerations of Sec. III remain unchanged, with the exception of the additional requirement that Δ and θ remain finite in the adiabatic limit. Thus from Eq. (12) we see that $\omega_c-\omega_0$ must vanish (diverge) when κ vanishes (diverges). A similar correspondence exists between $\omega_a-\omega_0$ and $\gamma_{\perp}$.

The situation is different when the injected field is absent. In this case, the quantities $\omega_a-\omega_0$ and $\omega_c-\omega_0$ are no longer physically meaningful, and they must be replaced by $\omega_a-\omega_c$. The set of equations (1) with $\alpha_0=0$ can be recast in two equivalent ways:

$$\frac{dx_1}{dt}=-\kappa(x_1-2Cp_1)\ , \tag{20a}$$

$$\frac{dp_1}{dt}=-\gamma_{\perp}[p_1(1+i\Delta_{ac})-x_1\tilde{d}]\ , \tag{20b}$$

$$\frac{d\tilde{d}}{dt}=-\gamma_{\parallel}[\tfrac{1}{2}(x_1p_1^*+x_1^*p_1)+\tilde{d}-1] \tag{20c}$$

or

$$\frac{dx_2}{dt}=-\kappa[x_2(1-i\theta_{ca})-2Cp_2]\ , \tag{21a}$$

$$\frac{dp_2}{dt}=-\gamma_{\perp}(p_2-x_2\tilde{d})\ , \tag{21b}$$

$$\frac{d\tilde{d}}{dt}=-\gamma_{\parallel}[\tfrac{1}{2}(x_2p_2^*+x_2^*p_2)+\tilde{d}-1]\ , \tag{21c}$$

where

$$\Delta_{ac}=\frac{\omega_a-\omega_c}{\gamma_{\perp}},\quad \theta_{ca}=\frac{\omega_a-\omega_c}{\kappa}\ , \tag{22}$$

and where x_1,p_1,x_2,p_2 are defined as

$$x_1=\frac{2ga}{\sqrt{\gamma_\perp\gamma_{||}}}e^{i\omega_c t},\quad x_2=\frac{2ga}{\sqrt{\gamma_\perp\gamma_{||}}}e^{i\omega_a t},$$
$$p_1=\left[\frac{N}{2}\left[\frac{\gamma_{||}}{\gamma_\perp}\right]^{1/2}\right]^{-1}Pe^{i\omega_c t},\qquad(23)$$
$$p_2=\left[\frac{N}{2}\left[\frac{\gamma_{||}}{\gamma_\perp}\right]^{1/2}\right]^{-1}Pe^{i\omega_a t}.$$

The two sets of equations (20) and (21) offer two different possibilities for carrying out the adiabatic limit. In the first, we require that Δ_{ac} remain finite, while in the second case θ_{ca} must remain finite. Note that, in all cases, the adiabatic elimination can be carried out by neglecting the time derivatives of the variables to be eliminated, just as one would in the absence of oscillatory terms in the equations. This procedure is different, but equivalent to the methods proposed by Haken,[1] which involves the use of different tactics, depending on the presence, or absence, of oscillatory terms in the equations for the variables that are not to be eliminated. The method for carrying out the adiabatic limit in the system modeled by Eqs. (2) has been used in Ref. 5.

VI. EIGENVALUES OF THE LINEARIZED PROBLEM

The formulation of the remaining adiabatic-elimination conditions requires consideration of the eigenvalues of the linear stability analysis. To this purpose, consider a stationary solution $x_{st},p_{st},\tilde{d}_{st}$ of Eqs. (13), with [see Eqs. (8) and (9)]

$$p_{st}=\frac{x_{st}}{1+x_{st}^2},\quad \tilde{d}=\frac{1}{1+x_{st}^2},$$
$$y=x_{st}\left[1-\frac{2C}{1+x_{st}^2}\right].\qquad(24)$$

In terms of the deviations from the steady state,

$$\delta x=x-x_{st},$$
$$\delta p=p-p_{st},\qquad(25)$$
$$\delta d=\tilde{d}-\tilde{d}_{st},$$

Eqs. (13) can be cast into the following form:

$$\frac{d\vec{q}}{dt}=\mathscr{L}\vec{q}+\vec{\psi}_{NL},\qquad(26)$$

where $\vec{q}$ is the three-component vector

$$\vec{q}=\begin{bmatrix}\delta x\\ \delta p\\ \delta d\end{bmatrix},\qquad(27)$$

$\mathscr{L}$ is the matrix

$$\mathscr{L}=\begin{bmatrix}-\kappa & 2C\kappa & 0\\ \gamma_\perp\tilde{d}_{st} & -\gamma_\perp & \gamma_\perp x_{st}\\ -\gamma_{||}\tilde{p}_{st} & -\gamma_{||}x_{st} & -\gamma_{||}\end{bmatrix},\qquad(28)$$

and where the vector $\vec{\psi}_{NL}$ contains all the nonlinear terms of the equations

$$\vec{\psi}_{NL}=\begin{bmatrix}0\\ \gamma_\perp\ \delta x\ \delta d\\ -\gamma_{||}\ \delta x\ \delta p\end{bmatrix}.\qquad(29)$$

The stability of the stationary state is governed by the eigenvalues λ of the matrix $\mathscr{L}$, which obey the secular equation

$$\tilde{\lambda}^3+\tilde{\lambda}^2(\tilde{\kappa}+\tilde{\gamma}_\perp+\tilde{\gamma}_{||})+\tilde{\lambda}\left[\tilde{\kappa}(\tilde{\gamma}_\perp+\tilde{\gamma}_{||})+\tilde{\gamma}_\perp\tilde{\gamma}_{||}(1+x_{st}^2)-2C\tilde{\kappa}\tilde{\gamma}_\perp\frac{1}{1+x_{st}^2}\right]+\tilde{\kappa}\tilde{\gamma}_\perp\gamma_{||}\left[1+x_{st}^2+2C\frac{x_{st}^2-1}{1+x_{st}^2}\right]=0.\qquad(30)$$

In Eq. (30) we have scaled the eigenvalues and the rate constants to the largest rate γ_L (thus, for example, $\tilde{\lambda}=\lambda/\gamma_L$, etc.). In the adiabatic limit, the eigenvalues can be calculated immediately by perturbative expansions in the smallness parameter; one only needs to set

$$\tilde{\lambda}=\tilde{\lambda}^{(0)}+\epsilon\tilde{\lambda}^{(1)}+\cdots,\qquad(31)$$

substitute Eq. (31) into Eq. (30), and equate powers of ϵ. Thus, for case (a) of Sec. III, where $\gamma_L=\gamma_\perp$ and $\epsilon=\kappa/\gamma_\perp$, one finds, to order ϵ^0,

$$\tilde{\lambda}_1^{(0)}=0,$$
$$\lambda_{2,3}^{(0)}=\tfrac{1}{2}\{-1-\tilde{\gamma}_{||}\pm[(1-\tilde{\gamma}_{||})^2-4\tilde{\gamma}_{||}x_{st}^2]^{1/2}\}\qquad(32)$$

and, to order ϵ,

$$\lambda_1^{(1)}=-\left[1+2C\frac{x^2-1}{(1+x_{st}^2)^2}\right]=-\frac{dy}{dx_{st}},\qquad(33a)$$
$$\lambda_{2,3}^{(1)}=\frac{2C}{1+x_{st}^2}\frac{\gamma_{||}(1-x_{st}^2)+\lambda_{2,3}^{(0)}}{\lambda_{2,3}^{(0)}(2\lambda_{2,3}^{(0)}+1+\gamma_{||})}.\qquad(33b)$$

Note that $\lambda_1^{(1)}$ coincides exactly with the eigenvalue of Eq. (15a) linearized around the stationary value x_{st}. Equation (15a), of course, is just the equation obtained after performing the adiabatic elimination.

This example suggests the following definition of normal behavior of the eigenvalues: The eigenvalues behave "normally" in the adiabatic limit when, first, they are analytic functions of ϵ, and, second, can be subdivided into two groups, such that the number of elements of the first (second) group is equal to the number of variables to be eliminated (not to be eliminated). For the eigenvalues of

the first group, the term $\tilde{\lambda}^{(0)}$ in Eq. (31) is different from zero. For the eigenvalues of the second group, $\tilde{\lambda}^{(0)}$ vanishes, while $\tilde{\lambda}^{(1)}$ coincides with the corresponding eigenvalue of the linearized problem as obtained *after* performing the adiabatic elimination.

VII. ADIABATIC-ELIMINATION CONDITIONS (CONTINUED)

We are now in a position to formulate two more conditions for the validity of a global adiabatic elimination.

Condition 4. The eigenvalues of the linearized problem must behave normally in the adiabatic-elimination limit, according to the definition in Sec. VI.

Condition 5. For each eigenvalue of the set corresponding to the variables to be eliminated, the real part of $\lambda^{(0)}$ must be nonpositive.

We now comment on these new conditions, and discuss, in particular, some counterexamples to illustrate their significance. With regard to condition 4, consider first the case of the usual laser ($y=0$) with the field and polarization variables to be eliminated adiabatically [$\gamma_L=\gamma_\perp$, $\kappa/\gamma_\perp=O(\epsilon^0)$, $\gamma_\parallel/\gamma_\perp=\epsilon$]. We linearize Eqs. (13) around the nontrivial stationary solution $x_{\rm st}=\sqrt{2C-1}$. Because in this case

$$2C/(1+x_{\rm st}^2)=1$$

the eigenvalue equation (30) becomes

$$\tilde{\lambda}^3+\tilde{\lambda}^2(\tilde{\kappa}+1+\tilde{\gamma}_\parallel)+\tilde{\lambda}\tilde{\gamma}_\parallel(\tilde{\kappa}+1+x_{\rm st}^2)+2\tilde{\kappa}\tilde{\gamma}_\parallel x_{\rm st}^2=0 . \quad (34)$$

If we now look for eigenvalues of the type (31), we find only one, with

$$\tilde{\lambda}_1^{(0)}=-(\tilde{\kappa}+1) . \quad (35)$$

The other two eigenvalues have the form

$$\tilde{\lambda}=\tilde{\lambda}^{(0)}\sqrt{\epsilon}+\tilde{\lambda}^{(1)}\epsilon+\cdots \quad (36)$$

with

$$\tilde{\lambda}^{(0)}=\pm i\left[\frac{2\tilde{\kappa}}{\tilde{\kappa}+1}\right]^{1/2}x_{\rm st} ,$$
$$\tilde{\lambda}^{(1)}=\tfrac{1}{2}\left[\frac{(\tilde{\kappa}-1)x_{\rm st}^2}{(\tilde{\kappa}+1)^2}-1\right] . \quad (37)$$

In this case, the eigenvalues do not behave normally (according to the definition of normal behavior introduced in Sec. VI), and if we attempt to carry out the adiabatic elimination by setting $dx/dt=dp/dt=0$ in Eqs. (13a) and (13b) with $y=0$, we obtain $x=p=0$, which is not correct. Thus, the adiabatic limit cannot be carried out under these conditions, even if $\gamma_\parallel$ is sufficiently smaller than κ and $\gamma_\perp$.

Two other examples, also for $y=0$, are worth discussing in connection with conditions 4 and 5. Consider first the case $\gamma_\perp\gg\kappa\gg\gamma_\parallel$. After eliminating the polarization variable from Eqs. (13) (with $y=0$), we obtain

$$\frac{dx}{dt}=-\kappa x(1-2C\tilde{d}) ,$$
$$\frac{d\tilde{d}}{dt}=-\gamma_\parallel[\tilde{d}(1+x^2)-1] . \quad (38)$$

The linearized form of Eqs. (38) around the steady state leads to the eigenvalue equations

$$\tilde{\lambda}^2+2\epsilon C\tilde{\lambda}+2\epsilon(2C-1)=0 , \quad (39)$$

where we have selected $\gamma_L=\kappa$, $\tilde{\gamma}_\parallel=\epsilon$; the solutions of Eq. (39) are

$$\tilde{\lambda}_{1,2}=-\epsilon C\pm i\sqrt{2\epsilon(2C-1)} \quad (40)$$

which are, again, of the type shown in Eq. (36). In this case, also, the eigenvalues do not behave normally, and, in fact, the adiabatic elimination of x cannot be performed.

The meaning of condition 4 should now be more transparent: by performing the adiabatic elimination, one eliminates as many eigenvalues as the eliminated variables. This condition requires that the eliminated eigenvalues should scale as the rate constants of the eliminated variables.

With regard to condition 5, we observe that, if one of the eliminated eigenvalues happens to have a positive real part (this may happen even with normal eigenvalues), the adiabatic-elimination process wipes out one of the instabilities.

Our next example illustrates this point. We are dealing here with the laser with injected signal ($C>0$, $y\neq 0$) under conditions such that $\kappa\gg\gamma_\perp\gg\gamma_\parallel$, as in the experiments of Brun and collaborators.[14] The steady-state diagram that results from Eq. (24) is given in Fig. 1. After adiabatic elimination of the field variable, Eqs. (13) take the form

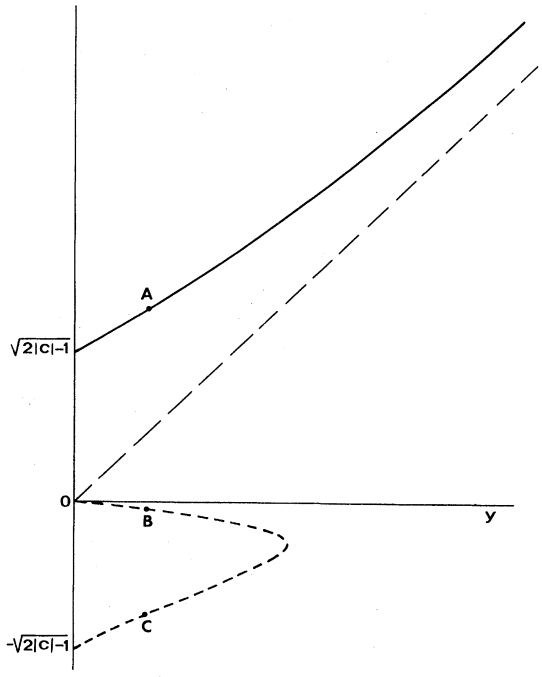

FIG. 1. Steady-state equation linking the input (y) and output (x) field amplitudes. Dashed segment in the lower part of the diagram corresponds to a set of unstable states. Main diagonal in the (y,x) plane refers to the behavior of an empty cavity.

$$\frac{dp}{dt}=-\gamma_\perp[-y\tilde{d}+p(1-2C\tilde{d})]\ ,$$
$$\frac{d\tilde{d}}{dt}=-\gamma_{||}(yp+2Cp^2+\tilde{d}-1)\ . \tag{41}$$

Now, because $\gamma_{||}/\gamma_\perp=\epsilon\to 0$, one would expect that the situation should be well described by the single equation that results from (41) after adiabatic elimination of the polarization, i.e.,

$$\frac{d\tilde{d}}{d\tau}=-\left[\frac{y^2\tilde{d}}{(1-2Cd)^2}+\tilde{d}-1\right],\quad \tau=\gamma_{||}t\ . \tag{42}$$

This is not the case, however. In fact, if we linearize Eqs. (41), the eigenvalue equation takes the form ($\gamma_L=\gamma_\perp$)

$$\tilde{\lambda}^2+\tilde{\lambda}\left[\tilde{\gamma}_{||}+\frac{y}{x_{\rm st}}\right]+\tilde{\gamma}_{||}(1+x_{\rm st}^2)\frac{dy}{dx_{\rm st}}=0\ , \tag{43}$$

where the function $y=y(x_{\rm st})$ is defined by Eq. (24). Upon solving this eigenvalue equation using the expansion ansatz of Eq. (31), we find the two eigenvalues

$$\tilde{\lambda}_1=-\frac{y}{x_{\rm st}}+\epsilon\lambda_1^{(1)}+\cdots\ ,$$
$$\tilde{\lambda}_2=-\epsilon\frac{x_{\rm st}}{y}(1+x_{\rm st}^2)\frac{dy}{dx_{\rm st}}+\lambda_2^{(2)}\epsilon^2+\cdots\ , \tag{44}$$
$$\epsilon=\gamma_{||}/\gamma_\perp\ .$$

On the other hand, if we linearize Eq. (42), the resulting eigenvalue is just the coefficient of the linear term in ϵ in the expansion of $\tilde{\lambda}_2$ [Eq. (44)]. This is a consequence of the elimination of $\tilde{\lambda}_1$ in the adiabatic process. Note that in this case the eigenvalues scale normally. If we now consider Fig. 1, we see that according to Eq. (42), the solution B is stable, while according to Eqs. (41) it is unstable. The stability of the system is determined by the eigenvalue $\tilde{\lambda}_1$ which happens to be positive at B. This example shows that the caveat of condition 5 is indeed necessary.

As an example of the above comments, we compare in Fig. 2 an exact solution of Eqs. (13) for $C=10$, $y=10$, $\kappa=1$, $\gamma_\perp=5$, and $\gamma_{||}=2$, with the approximate solution obtained from Eq. (15a) after elimination of the atomic variables. The agreement is not very good, as expected. A similar comparison for $\kappa=1$, $\gamma_\perp=10$, and $\gamma_{||}=10$ shows instead much closer agreement (Fig. 3).

As an illustration of the contents of condition 5, we compare in Fig. 4 the solutions of the exact equations (13) and of the approximate equations (41) and (42). The parameters chosen for this study are $C=5$, $y=2$, $\kappa=1$, $\gamma_\perp=0.05$, and $\gamma_{||}=0.005$ and the initial state of the system corresponds to a point in the vicinity of B of Fig. 1. The selection of the rate constants is such that the approximate equations (41) are in close agreement with the exact equations of motion. On the other hand, the adiabatic elimination of the polarization from Eq. (41) turns point B from an unstable to a stable steady state. Thus, the solution of Eq. (42) is just a constant in complete disagreement with the solution of the exact equation.

VIII. ADIABATIC ELIMINATION IN THE DRESSED-MODE APPROACH

An alternative procedure for the global adiabatic elimination is one that exploits the mode variables introduced in Ref. 2, and generalizes the method described in Ref. 5. A valuable feature of this approach is that the eigenvalues of the linearized problem appear explicitly in the equations. Following Haken, we introduce the eigenstates of the linear operator $\mathscr{L}$ defined in Eq. (26):

$$\mathscr{L}\vec{\mathscr{O}}_j=\lambda_j\vec{\mathscr{O}}_j\ . \tag{45}$$

These are given explicitly by

$$\vec{\mathscr{O}}_j=\begin{bmatrix}\mathscr{O}_j^{(1)}\\ \mathscr{O}_j^{(2)}\\ \mathscr{O}_j^{(3)}\end{bmatrix}=\mathscr{N}_j\begin{bmatrix}1\\ \tilde{d}_{\rm st}\dfrac{\lambda_j/\gamma_{||}+1-x_{\rm st}^2}{G(\lambda_j)}\\ -p_{\rm st}\dfrac{2+\lambda_j/\gamma_\perp}{G(\lambda_j)}\end{bmatrix}, \tag{46}$$

where

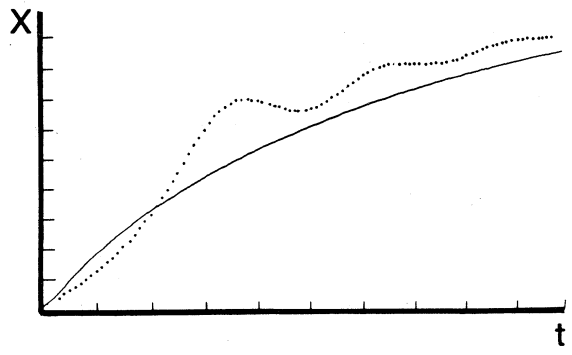

FIG. 2. Output field (dotted line) according to Eqs. (13) and corresponding to the parameters $C=10$, $y=10$, $\kappa=1$, $\gamma_\perp=5$, and $\gamma_{||}=2$. Approximate solution obtained after adiabatic elimination of the atomic variables is shown as the solid line.

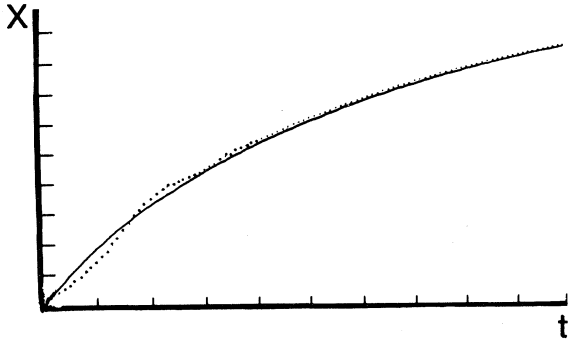

FIG. 3. Output field (dotted line) according to Eqs. (13) and corresponding to the parameters $C=10$, $y=10$, $\kappa=1$, $\gamma_\perp=10$, and $\gamma_{||}=10$. The agreement with the approximate solution (solid line) obtained after adiabatic elimination of the atomic variables is much better than in Fig. 2.

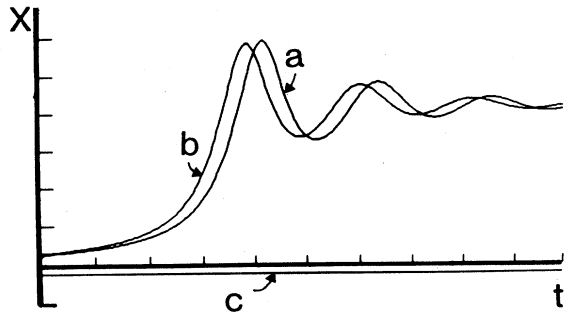

FIG. 4. Output field (curve a) according to Eqs. (13) is compared with the solutions of Eq. (41) (curve b), and of Eq. (42) (curve c). Parameters used in this figure are $C=5$, $y=2$, $\kappa=1$, $\gamma_\perp=0.05$, and $\gamma_\parallel=0.005$. A large initial spike of the two unstable solutions has been left out of the picture for clarity.

$$G(\lambda_j)=(1+\lambda/\gamma_\perp)(1+\lambda/\gamma_\parallel)+x_{\rm st}^2 \tag{47}$$

and $\mathcal{N}_j$ denotes a set of three arbitrary constants. Consider also the adjoint operator $\mathcal{L}^\dagger$

$$\mathcal{L}^\dagger=\begin{bmatrix} -\kappa & \gamma_\perp \tilde{d}_{\rm st} & -\gamma_\parallel \tilde{p}_{\rm st} \\ 2C\kappa & -\gamma_\perp & -\gamma_\parallel x_{\rm st} \\ 0 & \gamma_\perp x_{\rm st} & -\gamma_\parallel \end{bmatrix} \tag{48}$$

and its eigenvalue equation

$$\mathcal{L}^\dagger \tilde{\vec{\mathcal{O}}}_j=\lambda_j^* \tilde{\vec{\mathcal{O}}}_j \ . \tag{49}$$

The explicit form of the eigenvectors $\tilde{\vec{\mathcal{O}}}_j$ of the adjoint operator is

$$\tilde{\vec{\mathcal{O}}}_j=\begin{bmatrix} \tilde{\mathcal{O}}_j^{(1)} \\ \tilde{\mathcal{O}}_j^{(2)} \\ \tilde{\mathcal{O}}_j^{(3)} \end{bmatrix}=\tilde{\mathcal{N}}_j \begin{bmatrix} 1 \\ -2C\dfrac{\kappa}{\gamma_\perp}\dfrac{(1+\lambda_j^*/\gamma_\parallel)}{G(\lambda_j^*)} \\ -2C\dfrac{\kappa}{\gamma_\parallel}\dfrac{x}{G(\lambda_j^*)} \end{bmatrix} . \tag{50}$$

Note that all the elements of $\vec{\mathcal{O}}_j$ and $\tilde{\vec{\mathcal{O}}}_j$ are written in terms of dimensionless quantities. The following orthogonality relation can be easily proved:

$$(\tilde{\vec{\mathcal{O}}}_j,\vec{\mathcal{O}}_{j'})=\sum_{l=1}^{3}\tilde{\mathcal{O}}_j^{(l)*}\mathcal{O}_{j'}^{(l)}=0, \quad j\neq j' \ . \tag{51}$$

The normalization constant $\tilde{\mathcal{N}}_j$ is chosen in such a way that

$$(\tilde{\vec{\mathcal{O}}}_j,\vec{\mathcal{O}}_{j'})=\delta_{jj'} \ . \tag{52}$$

Next, and still following the method of Ref. 2, we introduce the following expansion for the fluctuation variables:

$$\vec{q}(t)=\sum_j \xi_j(t)\vec{\mathcal{O}}_j \ , \tag{53}$$

where the coefficients $\xi_j(t)$, the so-called "dressed-mode" variables are defined by

$$\xi_j(t)=(\tilde{\vec{\mathcal{O}}}_j,\vec{q}(t)) \ . \tag{54}$$

Obviously, because $\vec{\mathcal{O}}_j$ and $\vec{q}(t)$ are dimensionless vectors, the dressed-mode amplitudes are also dimension-free. On substituting Eq. (53) into Eq. (26) we obtain the following coupled nonlinear equations for the dressed-mode variables:

$$\frac{d\xi_j}{dt}=\lambda_j\xi_j+(\tilde{\vec{\mathcal{O}}}_j,\vec{\psi}_{\rm NL}) \ . \tag{55}$$

From the explicit form, Eq. (29), of the nonlinear vector $\vec{\psi}_{\rm NL}$, and the definition (53), we can easily derive the explicit form of the scalar product in Eq. (55):

$$(\tilde{\vec{\mathcal{O}}}_j,\vec{\psi}_{\rm NL})=\sum_{j',j''}\mathcal{O}_{j''}^{(1)}(\gamma_\perp\tilde{\mathcal{O}}_j^{(2)*}\mathcal{O}_{j'}^{(3)*}-\gamma_\parallel\tilde{\mathcal{O}}_j^{(3)*}\mathcal{O}_{j'}^{(2)}) \ . \tag{56}$$

Now we can turn our attention to the adiabatic-elimination process, under the assumption that the eigenvalues behave normally. The constants $\mathcal{N}_j$, which may very well be explicit functions of ϵ, must be chosen in such a way that none of the components $\mathcal{O}_j^{(l)}$ $(j,l=1,2,3)$ diverge in the limit $\epsilon\to 0$, but also in such a way that for each eigenvector $\vec{\mathcal{O}}_j$ $(j=1,2,3)$ the three components $\mathcal{O}_j^{(l)}$ do not vanish simultaneously in the limit $\epsilon\to 0$. Because we assume that the eigenvalues behave normally, there will be $n^{(e)}$ eigenvalues that scale as the large decay rate γ_L and $n^{(ne)}$ eigenvalues that scale as the small decay rate γ_s $(\gamma_s/\gamma_L\equiv\epsilon)$.[15] It will be convenient to refer to the elements of the first set as the "eliminated eigenvalues," and similarly to the elements of the second set as the "noneliminated eigenvalues."

As shown in Appendix A for a general nonlinear system, the expansion of the elements $\mathcal{O}_j^{(l)}$ and $\tilde{\mathcal{O}}_j^{(l)}$ in powers of ϵ

$$\begin{aligned} \mathcal{O}_j^{(l)}&=\sum_{n=0}^{\infty}\mathcal{O}_{jn}^{(l)}\epsilon^n \ , \\ \tilde{\mathcal{O}}_j^{(l)}&=\sum_{n=0}^{\infty}\tilde{\mathcal{O}}_{jn}^{(l)}\epsilon^n \end{aligned} \tag{57}$$

is characterized by the following properties. In the case of eigenstates $\vec{\mathcal{O}}_j$ corresponding to noneliminated eigenvalues, the components $\mathcal{O}_{j,0}^{(l)}$ are different from zero for all values of l. In the case of eigenstates $\vec{\mathcal{O}}_j$ corresponding to eliminated eigenvalues, we have, instead, $\mathcal{O}_{j,0}^{(l)}=0$ for all the components that correspond to noneliminated variables. Likewise, the eigenstates $\tilde{\vec{\mathcal{O}}}_j$ corresponding to eliminated eigenvalues have components $\tilde{\mathcal{O}}_{j,0}^{(l)}$ which are different from zero for all values of l, while the eigenstates $\tilde{\vec{\mathcal{O}}}_j$ corresponding to noneliminated eigenvalues are such that $\tilde{\mathcal{O}}_{j,0}^{(l)}=0$ for all the components that correspond to eliminated variables. This behavior of the eigenstates is typical of all situations where the eigenvalues are normal.

As shown in Appendix A, the above properties of the eigenstates lead to dressed-mode equations of the form

$$\frac{d\xi_j}{dt}=\gamma_s[\tilde{\lambda}_j^{(1)}\xi_j+\Phi_j(\xi_1,\xi_2,\xi_3)+\mathcal{O}(\epsilon)] \tag{58a}$$

for the mode variables that correspond to noneliminated eigenvalues, and

$$\frac{d\xi_j}{dt}=\lambda_L[\tilde{\lambda}_j^{(0)}\xi_j+\Phi_j(\xi_1,\xi_2,\xi_3)+\mathcal{O}(\epsilon)] \tag{58b}$$

for the modes ξ_j, corresponding to the eliminated variables. The expressions $\tilde{\lambda}^{(0)}$ and $\tilde{\lambda}^{(1)}$ have been defined by Eq. (31); the nonlinear terms Φ_j are of order ϵ^0. If we now introduce the scaled time variable $\tau=\gamma_s t$, and carry out the limit $\epsilon\to 0$, while insisting that the mode variables ξ_j remain finite, the relevant dressed-mode equations for the noneliminated eigenvalues take the form

$$\frac{d\xi_j}{d\tau}=\tilde{\lambda}_j^{(1)}+\Phi_j(\xi_1,\xi_2,\xi_3)\ . \tag{59a}$$

The equations for the eliminated variables, instead, reduce to the algebraic system

$$0=\tilde{\lambda}_j^{(0)}\xi_j+\Phi_j(\xi_1,\xi_2,\xi_3)\ . \tag{59b}$$

The required closed set of equations for the noneliminated mode variables is obtained by solving Eq. (59b) for the eliminated modes and substituting the result into Eq. (59a).

Thus, we have generalized the global adiabatic-elimination procedure in the dressed-mode formalism, which in Ref. 5 was restricted to the case $\kappa\ll\gamma_\perp,\gamma_{||}$.

IX. MULTIPLE-TIME-SCALE APPROACH

In this section, we describe an adiabatic-elimination procedure which is more rigorous and complete than the one developed in Sec. II [Eqs. (14) and (15)]. Again, we direct our discussion to the case of normal eigenvalues, and for definiteness, consider the specific case of $\kappa/\gamma_\perp=\epsilon$ and $\gamma_{||}/\gamma_\perp\equiv\tilde{\gamma}_{||}=\mathcal{O}(\epsilon^0)$. On the basis of the existence of two well-separated time scales, we introduce the two scaled-time variables

$$t_1\equiv\gamma_\perp t,\quad \tau=\kappa t=\epsilon t_1\ . \tag{60}$$

Equations (31) can now be written as follows:

$$\frac{dx}{dt_1}=-\epsilon(x-y-2Cp)\ , \tag{61a}$$

$$\frac{dp}{dt_1}=-p+x\tilde{d}\ , \tag{61b}$$

$$\frac{d\tilde{d}}{dt_1}=-\tilde{\gamma}_{||}(xp+\tilde{d}-1)\ . \tag{61c}$$

Now, if we let

$$x=x(t_1,\tau),\quad p=p(t_1,\tau),\quad \tilde{d}=\tilde{d}(t_1,\tau)$$

we can use a multiple-time-scale approach which is well known in the literature[16] starting from the pioneering work by Sandri.[17] First of all, we observe that

$$\frac{d}{dt_1}=\frac{\partial}{\partial t_1}+\epsilon\frac{\partial}{\partial\tau}\ .$$

Next, we expand the system variables in powers of ϵ as shown below, for example, for the field variable:

$$x(t_1,\tau)=x_0(t_1,\tau)+\epsilon x_1(t_1,\tau)+\cdots\ . \tag{62}$$

After inserting these expansions into Eqs. (61), and equating the coefficients of equal powers of ϵ, we obtain the following equations to order ϵ^0:

$$\frac{\partial x_0}{\partial t_1}=0\ , \tag{63a}$$

$$\frac{\partial p_0}{\partial t_1}=-p_0+x_0\tilde{d}_0\ , \tag{63b}$$

$$\frac{\partial\tilde{d}_0}{\partial t_1}=-\tilde{\gamma}_{||}(\tilde{d}_0-1+x_0p_0)\ . \tag{63c}$$

The first immediate conclusion is that x_0 does not depend explictly on t_1, so that Eqs. (63b) and (63c) can be readily solved. The result is

$$p_0=\frac{x_0(\tau)}{1+x_0^2(\tau)}+a_+(\tau)\exp[-\lambda_+(\tau)t_1]+a_-(\tau)\exp[-\lambda_-(\tau)t_1]\ , \tag{64a}$$

$$\tilde{d}_0=\frac{1}{1+x_0^2(\tau)}+b_+(\tau)\exp[-\lambda_+(t)t_1]+b_-(\tau)\exp[-\lambda_-(\tau)t_1]\ , \tag{64b}$$

where

$$\lambda_\pm(\tau)=\tfrac{1}{2}\{-1-\tilde{\gamma}_{||}\pm[(1-\tilde{\gamma}_{||})^2-4x_0^2(\tau)]^{1/2}\}\ . \tag{65}$$

Now, we consider a range of values of t such that $t_1\gg 1$ and $\tilde{\gamma}_{||}t_1\gg 1$. Accordingly, we keep only the first-order terms in Eqs. (64) and drop from Eqs. (61) all terms involving $\partial/\partial t_1$, because over the chosen time range the variables depend only on τ. To first order in ϵ, Eqs. (60) produce the following system:

$$\frac{dx_0}{d\tau}=-(x_0-y-2Cp_0)\ , \tag{66}$$

$$\frac{dp_0}{d\tau}=-(p_1+x_0\tilde{d}_1+\tilde{d}_0x_1)\ , \tag{67a}$$

$$\frac{d\tilde{d}_0}{d\tau}=-\tilde{\gamma}_{||}(\tilde{d}_1+x_0p_1+p_0x_1)\ , \tag{67b}$$

where in Eqs. (67) we have replaced $\partial/\partial\tau$ with $d/d\tau$. Equation (66) coincides with Eq. (15a), *which is thus recovered.* On the other hand, Eqs. (67) allow us to calculate x_1, p_1, and $\tilde{d}_1$ by solving them together with Eq. (61a) expanded to order ϵ^2. This has the form

$$\frac{dx_1}{d\tau}=-(x_1-2Cp_1)\ . \tag{67c}$$

In principle, this analysis can be continued to arbitrary orders in ϵ. It is important to observe that the initial condition to be used for Eq. (66) is just $x(t=0)$, because, as we have shown, x_0 does not depend on the short-time-scale variable t_1. This feature is related also to the fact that the eigenstates corresponding to eliminated eigenvalues are

characterized by $\mathcal{O}_{j,0}^{(l)}=0$ for all the components corresponding to the noneliminated variables (see Sec. VIII).

X. ANOMALOUS SCALING

We now turn our attention to the case in which the eigenvalues do not behave normally, so that the adiabatic elimination cannot be carried out along the lines described above. As illustrated by the first example of Sec. VII where we selected $\gamma_L=\gamma_\perp$, $\kappa/\gamma_\perp=O(\epsilon)$, and $\gamma_{||}/\gamma_\perp=\epsilon$ [see Eqs. (34)—(37)], the eigenvalues need not be analytic functions of ϵ, although they may be analytic in $\sqrt{\epsilon}$. The eigenvalues λ_2 and λ_3 corresponding to the "long" time scale have real and imaginary parts that scale very differently with respect to ϵ because $\mathrm{Re}\lambda_{2,3}=O(\epsilon)$ while $\mathrm{Im}\lambda_{2,3}=O(\sqrt{\epsilon})$. Clearly, the oscillations imposed by these eigenvalues are much faster than the damping mechanism.

We can verify at once that, in this case, the eigenstates also do not behave normally. In fact, only the eigenstate corresponding to the eigenvalue $\tilde{\lambda}_1$ behaves in a normal way, because from Eqs. (31), (35a), and (46), and on selecting $\mathcal{N}_1=O(\epsilon^0)$, its components have the following orders of magnitude: $\mathcal{O}_1^{(1)},\mathcal{O}_1^{(2)}=O(\epsilon^0)$ and $\mathcal{O}_1^{(3)}=O(\epsilon)$, as expected of normal eigenvectors. On the other hand, in the case of the eigenstates $\vec{\mathcal{O}}_2$ and $\vec{\mathcal{O}}_3$, Eq. (36) and the selection $\mathcal{N}_2=O(\epsilon^0)$ lead to $\mathcal{O}_2^{(1)},\mathcal{O}_2^{(2)}=O(\epsilon^0)$, as it must be, but $\mathcal{O}_2^{(3)}=O(\sqrt{\epsilon})$ instead of $\mathcal{O}_2^{(3)}=O(\epsilon^0)$. The same holds true for the eigenvector $\vec{\mathcal{O}}_3$. Consequently, in the limit $\epsilon\to 0$ the component $\vec{\mathcal{O}}^{(3)}$ vanishes for all three eigenstates. This result suggests that the fluctuation variable should be rescaled in such a way that the anomaly disappears. Hence we define

$$\overline{\delta d}=\delta d/\sqrt{\epsilon} \tag{68}$$

so that $\mathcal{O}_2^{(3)}$ and $\mathcal{O}_3^{(3)}$ become $O(\epsilon^0)$, while $\mathcal{O}_1^{(3)}$ becomes $O(\sqrt{\epsilon})$. In this case we require that in the adiabatic limit $\epsilon\to 0$ the new scaled population fluctuation $\overline{\delta d}$, and not δd, remains finite. This implies that, in the adiabatic limit, the fluctuation variable δd, which is of the order $\sqrt{\epsilon}$, scales differently from the stationary value $\tilde{d}_{\mathrm{st}}$ of the same variable, which is instead of order ϵ^0. For this reason, whenever the eigenvalues do not behave normally, we call the scaling anomalous.

We now turn our attention to the matter of hard-mode instabilities. Often, when carrying out the adiabatic limit in nonlinear models, several instabilities are seen to disappear. For the case of the model selected in this study, we have invariably found that the hard-mode instabilities that survive the adiabatic limit are accompanied by anomalous scaling.

Consider again, for example, Eqs. (13) with $y=0$, i.e., the single-mode model of the laser without injected signal. From our analysis of the secular equation (34), and as shown for the first time in Ref. 18, the stationary state becomes unstable when $\kappa>\gamma_\perp+\gamma_{||}$ and

$$x_{\mathrm{st}}^2>\frac{(\kappa+\gamma_\perp+\gamma_{||})(\kappa+\gamma_\perp)}{(\kappa-\gamma_{||}-\gamma_\perp)\gamma_\perp}\ . \tag{69}$$

In fact, when Eq. (69) is satisfied, two complex-conjugate eigenvalues have a positive real part. Now we consider the case in which $\gamma_{||}/\gamma_\perp=\epsilon$ and $\kappa/\gamma_\perp\equiv\tilde{\kappa}=O(\epsilon^0)$. In this limit the instability survives and the inequality (69) becomes

$$x_{\mathrm{st}}^2>\frac{(\tilde{\kappa}+1)^2}{\tilde{\kappa}-1}\ . \tag{70}$$

As we have shown above, this case corresponds to anomalous scaling, and the conditions $\mathrm{Re}\tilde{\lambda}_{2,3}>0$ (which reduce to $\mathrm{Re}\tilde{\lambda}^{(1)}>0$) coincide with the inequality (70) as we can see from Eq. (37). The use of the anomalous scaling (68) is essential if one wants to calculate the form of the limit cycle that bifurcates from the stationary solution when this becomes unstable.

XI. ANOMALOUS SCALING IN THE LASER WITH SATURABLE ABSORBER

In arriving at Eq. (69) we have assumed $\kappa>\gamma_\perp$. In this section, instead, we focus our attention on the situation in which $\gamma_\perp\gg\kappa\gg\gamma_{||}$. This corresponds to the case of the usual laser model [Eq. (38)] where, as we have seen, we cannot eliminate the field variable despite the fact that $\kappa\gg\gamma_{||}$. The model described by Eq. (38) does not possess any hard-mode instabilities, but this is no longer true if we add a saturable absorber inside the cavity.[19] This new system is described by a straightforward extension of Eqs. (38):

$$\frac{dx}{dt}=-\kappa x(1-2C\tilde{d}-2C_{\mathrm{abs}}\tilde{d}_{\mathrm{abs}})\ , \tag{71a}$$

$$\frac{d\tilde{d}}{dt}=-\gamma_{||}[\tilde{d}(1+x^2)-1]\ , \tag{71b}$$

$$\frac{d\tilde{d}_{\mathrm{abs}}}{dt}=-\gamma_{||}^{\mathrm{abs}}[\tilde{d}_{\mathrm{abs}}(1+ax^2)-1]\ , \tag{71c}$$

where the symbols C_{abs}, $\tilde{d}_{\mathrm{abs}}$, and γ^{abs} have the same meaning as C, $\tilde{d}$, and $\gamma_{||}$, but refer to the absorber instead of the amplifier. For this reason C_{abs} is negative. The parameter a is defined as the ratio between the saturation photon numbers of the amplifier and of the absorber, i.e.,

$$a=N_s/N_s^{\mathrm{abs}}\ . \tag{72}$$

We are using Eqs. (71) only to provide an example of hard-mode instability. It should be noted that these equations are, in fact, rate equations obtained from the semiclassical theory via an adiabatic-elimination procedure which has forced the disappearance of a class of soft-mode instabilities. This class of instabilities has been studied previously,[20] and a further analysis is to be published soon.[21]

In the steady state we have

$$\tilde{d}_{\mathrm{st}}=\frac{1}{1+x_{\mathrm{st}}^2},\quad \tilde{d}_{\mathrm{abs,st}}=\frac{1}{1+ax_{\mathrm{st}}^2}\ , \tag{73a}$$

where x_{st} obeys the state equation

$$x_{\mathrm{st}}\left[1-\frac{2C}{1+x_{\mathrm{st}}^2}-\frac{2C_{\mathrm{abs}}}{1+ax_{\mathrm{st}}^2}\right]=0\ . \tag{73b}$$

If we linearize Eqs. (71) around a nonzero stationary solution we obtain the eigenvalue equation

$$\tilde{\lambda}^3+[\tilde{\gamma}_{||}(1+x_{st}^2)+\tilde{\gamma}_{||}^{abs}(1+ax_{st}^2)]\tilde{\lambda}^2+\left[\tilde{\gamma}_{||}\tilde{\gamma}_{||}^{abs}(1+x_{st}^2)(1+ax_{st}^2)+4x_{st}^2\left[\tilde{\gamma}_{||}^{abs}aC_{abs}\frac{1}{1+ax_{st}^2}+\tilde{\gamma}_{||}C\frac{1}{1+x_{st}^2}\right]\right]\tilde{\lambda}+4\tilde{\gamma}_{||}\tilde{\gamma}_{||}^{abs}x_{st}^2\left[aC_{abs}\frac{1+x_{st}^2}{1+ax_{st}^2}+C\frac{1+ax_{st}^2}{1+x_{st}^2}\right]=0\ , \quad (74)$$

where $\tilde{\lambda}=\lambda/\kappa$, $\tilde{\gamma}_{||}=\gamma_{||}/\kappa$, and $\tilde{\gamma}_{||}^{abs}=\gamma_{||}^{abs}/\kappa$. If we analyze Eq. (74) using the Routh-Hurwitz method,[1] we find two conditions for the stability of the chosen stationary solution. The first requires that the steady state lie along a segment with positive slope of the function $x_{st}=x_{st}(C)$. The second takes the form

$$\frac{4x_{st}^2}{(1+x_{st}^2)(1+ax_{st}^2)}\left[\frac{\tilde{\gamma}_{||}}{\tilde{\gamma}_{||}^{abs}}C+\frac{\tilde{\gamma}_{||}^{abs}}{\tilde{\gamma}_{||}}aC_{abs}\right]+\tilde{\gamma}_{||}(1+x_{st}^2)+\tilde{\gamma}_{||}^{abs}(1+ax_{st}^2)>0\ . \quad (75)$$

When Eq. (75) is not satisfied, two complex-conjugate eigenvalues of the secular equation acquire a positive real part (hard-mode instability). Consider now the adiabatic limit $\epsilon\to 0$, with $\epsilon=\gamma_{||}/\kappa$ and $\gamma_{||}^{abs}/\gamma_{||}=\mathcal{O}(\epsilon^0)$. Equation (74) yields two solutions with the general form displayed by Eq. (36) and with

$$\tilde{\lambda}_{2,3}^{(0)}=\pm 2ix_{st}F^{1/2}\ , \quad (76a)$$

$$\tilde{\lambda}_{2,3}^{(1)}=-\tfrac{1}{2}\left[C+\left[\frac{\gamma_{||}^{abs}}{\gamma_{||}}\right]^2 aC_{abs}\right]F^{-1}\ , \quad (76b)$$

and where

$$F=\frac{\gamma_{||}^{abs}}{\gamma_{||}}aC_{abs}\frac{1}{1+ax_{st}^2}+C\frac{1}{1+x_{st}^2} \quad (77)$$

is positive at the instability threshold. Here again the eigenvalues do not behave normally and anomalous scaling arises. The stability condition $\mathrm{Re}\tilde{\lambda}^{(1)}<0$, as obtained from Eq. (76b), implies

$$C>-\left[\frac{\gamma_{||}^{abs}}{\gamma_{||}}\right]^2 aC_{abs} \quad (78)$$

which coincides with Eq. (75) in the limit $\tilde{\gamma}_{||}=\epsilon\to 0$. The simple expression given by Eq. (76a) for the oscillation frequency at the instability threshold has never, to our knowledge, been derived before.

XII. AN ALTERNATIVE ADIABATIC-ELIMINATION SCHEME

Up to this point we have analyzed adiabatic-elimination schemes which are tailored to situations such that conditions 1 through 5 are satisfied. We now consider a case in which condition 1 fails to hold, i.e., when all the decay rates have the same order of magnitude [in the example of Eqs. (13) we assume that $\kappa\sim\gamma_\perp\sim\gamma_{||}$]. Clearly, this situation requires a different treatment because no ratio of the decay rates can be used as the smallness parameter. This problem has been studied already in the context of absorptive optical bistability,[22,23] and will be applied here to the case of the single-mode laser without injected signal. In order to focus on the essential features of the argument, we shall also assume perfect tuning. Thus we consider Eqs. (13) with the restriction $y=0$.

The procedure to be described is appropriate when C is much larger than unity because, in this case, a convenient smallness parameter is $\epsilon=(2C)^{-1}$. In the limit $\epsilon\to 0$, the stationary solution of Eqs. (13) becomes

$$x^2=\epsilon^{-1}-1\ , \quad (79a)$$

$$p=\epsilon^{1/2}+O(\epsilon^{3/2})\ , \quad (79b)$$

$$\tilde{d}=\epsilon\ . \quad (79c)$$

In linear stability analysis around the steady state leads to Eq. (30) which, in the limit $\epsilon\to 0$ takes the form

$$\lambda^3+\lambda^2(\kappa+\gamma_{||}+\gamma_\perp)+\lambda\gamma_{||}(\kappa+\gamma_\perp\epsilon^{-1})+2\kappa\gamma_{||}\gamma_\perp(\epsilon^{-1}-1)=0\ . \quad (80)$$

The roots of Eq. (8) to the dominant order in ϵ are given by

$$\lambda_1=-2\kappa+O(\epsilon)\ , \quad (81a)$$

$$\lambda_{2,3}=-\tfrac{1}{2}(\gamma_{||}+\gamma_\perp-\kappa)\pm i(\gamma_{||}\gamma_\perp/\epsilon)^{1/2}+O(\epsilon^{1/2})\ . \quad (81b)$$

The imaginary part of the complex-conjugate roots represents the Rabi frequency of the problem; their real part shows that the stationary solutions (79) will be stable if and only if $\kappa<\gamma_{||}+\gamma_\perp$. On the basis of the scaling properties of the stationary solutions (79) we seek time-dependent solutions which scale in the same way with respect to ϵ, i.e.,

$$x(t)=\epsilon^{-1/2}X(t)\ , \quad (82a)$$

$$p(t)=\epsilon^{1/2}P(t)\ , \quad (82b)$$

$$\tilde{d}(t)=\epsilon D(t)\ , \quad (82c)$$

where $X(t)$, $P(t)$, and $D(t)$ are of order ϵ^0 for all time, and which satisfy the coupled equations

$$\dot{X}=-\kappa(X-P)\ , \quad (83a)$$

$$\dot{P}=-\gamma_\perp(P-XD)\ , \quad (83b)$$

$$\epsilon\dot{D}=-\gamma_{||}(\epsilon D-1+XP)\ . \quad (83c)$$

A simpleminded, but incorrect way to solve Eqs. (83) to dominant order in ϵ is to set $\epsilon=0$ with the result

$$\dot{X}=\kappa(X^{-1}-X)\ , \tag{84a}$$

$$P=X^{-1}\ , \tag{84b}$$

$$D=X^{-2}\left[1+\frac{\kappa}{\gamma_\perp}\right]-\frac{\kappa}{\gamma_\perp}X^{-4}\ . \tag{84c}$$

This turns out to be wrong, as shown below by a more careful analysis. In Sec. IX we have already discussed how one can handle problems with two well-separated time scales by means of a multiple-time-scale approach. The same solution occurs here because from Eqs. (81) we have

$$\mathrm{Re}\lambda=0(\epsilon^0),\quad \mathrm{Im}\lambda=O(\epsilon^{-1/2})\ . \tag{85}$$

We then introduce the two new time variables suggested by the scaling of the eigenvalues

$$\begin{aligned} t_1&=t\ ,\\ dt_2&=\epsilon^{-1/2}\omega(t_1)dt_1 \end{aligned} \tag{86}$$

and apply a multiple-time-scale perturbation expansion of the form

$$Z(\epsilon,t)=Z(\epsilon,t_1,t_2)=\sum_{n=0}^{\infty}\epsilon^{n/2}Z(n,t_1,t_2)\ , \tag{87a}$$

$$\omega(\epsilon,t)=\omega(\epsilon,t_1)=\sum_{n=0}^{\infty}\epsilon^{n/2}\omega(n,t_1)\ , \tag{87b}$$

where Z denotes either X or D. The technical details of the analysis, which rests on the solvability condition, are given in Appendix B. The resulting equations to dominant order in ϵ take the form

$$X(0,t_1,t_2)=X(0,t_1)\ , \tag{88a}$$

$$\frac{d}{dt_1}X(0,t_1)=-\kappa[X(0,t_1)-X^{-1}(0,t_1)]\ , \tag{88b}$$

$$\begin{aligned} D(0,t_1,t_2)=&X^{-2}(0,t_1)(1+\kappa/\gamma_\perp)-\frac{\kappa}{\gamma_\perp}X^{-4}(0,t_1)\\ &+\alpha(t_1)e^{it_2}+\alpha^*(t_1)e^{-it_2}\ , \end{aligned} \tag{88c}$$

$$P(0,t_1,t_2)=P(0,t_1)=X^{-1}(0,t_1)\ , \tag{88d}$$

where

$$\frac{d\alpha}{dt_1}=-\frac{\alpha}{2}[\gamma_{||}+\gamma_\perp-\kappa X^{-2}(0,t_1)] \tag{89}$$

and

$$\omega^2(0,t_1)=\gamma_{||}\gamma_\perp X^2(0,t_1)\ . \tag{90}$$

As in the previous discussion of the multiple-time-scale method, we are left with a single nonlinear differential equation which determines, in closed form, the long-time evolution of the field. The atomic polarization follows the field adiabatically as indicated by Eq. (88d). The two equations (88b) and (88d) are identical to Eqs. (84a) and (84b), respectively. However, the equation for the atomic inversion D contains two distinct contributions: the first corresponds to an adiabatic following of the field and was already produced by the naive approach of Eq. (84c), the new feature is provided by the second term describing high-frequency oscillations with an amplitude of order ϵ^0 and frequency $\epsilon^{-1/2}\omega(0,t_1)$ (a slowly varying Rabi frequency). The amplitude of these oscillations is given by $\alpha(t_1)$ which decays over the long time scale t_1. The occurrence of a slow damping for the fast oscillations seems to be a characteristic of the limit considered in this section and was also observed to appear in the study of optical bistability reported in Ref. 22. This is in contrast with the more common situation where our condition 1 holds and where fast oscillations are quickly damped out.

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APPENDIX A

Consider a nonlinear system of differential equations for n variables $a_1(t),\ldots,a_n(t)$:

$$\frac{d}{dt}a_i=\gamma_i f_i(\{a\}),\quad i=1,\ldots,n \tag{A1}$$

where γ_i are rate constants and f_i are, typically, nonlinear functions of the variables $\{a\}\equiv a_1,\ldots,a_n$. We assume that the variables a_i are linked to the original variables by a normal scaling. Let a_i^{st} $(i=1,\ldots,n)$ represent a stationary solution of Eq. (A1). We consider the n-component vector

$$\vec{\mathrm{q}}=\begin{bmatrix}\delta a_1\\ \delta a_2\\ \vdots\\ \delta a_n\end{bmatrix}\ , \tag{A2}$$

where $\delta a_i=a_i-a_i^{\mathrm{st}}$ $(i=1,\ldots,n)$. Equation (A1) can be put into the general form of Eq. (26), where $\mathscr{L}\vec{\mathrm{q}}$ contains all the linear and $\vec{\psi}_{\mathrm{NL}}$ all the nonlinear terms. We further assume that the linear system $d\vec{\mathrm{q}}/dt=\mathscr{L}\vec{\mathrm{q}}$ does not decompose into separate (independent) subsystems. Consider now the adiabatic limit $\epsilon=\gamma_s/\gamma_L\to 0$ in which, for definiteness, we eliminate adiabatically the variables $a_{m+1},\ldots,a_n$ $(1\le m\le n-1)$. The matrix elements $\mathscr{L}_{il}(i,l=1,\ldots,n)$ of the operator $\mathscr{L}/\gamma_L$ are such that $\mathscr{L}_{il}=O(\epsilon)$ for $1\le i\le m$ and $\mathscr{L}_{il}=O(\epsilon^0)$ for $m+1\le i\le n$.

If we now consider the eigenvalue equation

$$\sum_l \mathscr{L}_{il}\mathscr{O}_j^{(l)}=\tilde{\lambda}_j\mathscr{O}_j^{(i)} \tag{A3}$$

for the case of the eliminated eigenvalues [in this case we have $\tilde{\lambda}_j=O(\epsilon^0)$] we see that Eq. (A3) for all values of $i=1,\ldots,m$ requires that $\mathscr{O}_j^{(i)}=O(\epsilon)$ $(i=1,\ldots,m)$.

On the other hand, consider the matrix elements $\mathscr{L}^\dagger_{il}(i,l=1,\ldots,n)$ of the operator $\mathscr{L}^\dagger/\gamma_L$. These are such that $\mathscr{L}^\dagger_{il}=O(\epsilon)$ for $1\le l\le m$, and $\mathscr{L}_{il}=O(\epsilon^0)$ for

$m+1 \le l \le n$. Hence, if we consider the eigenvalue equation

$$\sum_{l=1}^{n} \mathscr{L}_{il}^{\dagger} \mathscr{O}_j^{(l)} = \tilde{\lambda}_j^* \mathscr{O}_j^{(i)} \tag{A4}$$

for the case of a noneliminated eigenvalue, so that $\tilde{\lambda}_j = O(\epsilon)$, to order ϵ^0 we obtain the set of n linear homogeneous equations

$$\sum_{l=m+1}^{n} \mathscr{L}_i^{\dagger} \mathscr{O}_{j,0}^{(l)} = 0 \tag{A5}$$

for the $n-m$ variables $\mathscr{O}_{j,0}^{(l)}$ with $l = m+1, \ldots, n$. It follows that, barring exceptional situations, the only possible solution is $\mathscr{O}_{j0}^{(l)} = 0$ for $l = m+1, \ldots, n$.

Next, we consider the order of magnitude of the nonlinear term

$$(\tilde{\vec{\mathscr{O}}}_j, \vec{\psi}_{\mathrm{NL}}) = \sum_l \tilde{\mathscr{O}}_j^{(l)*} \psi_{\mathrm{NL}}^{(l)}$$

which appears in Eq. (55). From Eq. (A1) we see that $\psi_{\mathrm{NL}}^{(l)}$ scales as γ_L. Consider first the case of the modes ξ_j that correspond to the nonelimination eigenvalues ($j = 1, \ldots, m$). For $1 \le l \le m$, we have $\tilde{\mathscr{O}}_j^{(l)} = O(\epsilon^0)$ and $\psi_{\mathrm{NL}}^{(l)} = O(\gamma_s)$, so that their product scales as γ_s. For $(m+1) \le l \le n$ we have $\tilde{\mathscr{O}}_j^{(l)} = O(\epsilon)$ and $\psi_{\mathrm{NL}}^{(l)} = O(\gamma_L)$, and their product scales again as $\gamma_s = \epsilon\gamma_L$. In conclusion, we find the structure of Eq. (58a).

Consider now the case, $j = m+1, \ldots, n$, in which $\tilde{\mathscr{O}}_j^{(l)} = O(\epsilon^0)$ for every l. For $1 \le l \le m$, we have $\psi_{\mathrm{NL}}^{(l)} = O(\gamma_s)$, so that this term scales as $\gamma_s = \epsilon\gamma_L$ and contributes to order $\gamma_L \epsilon$ in Eq. (58b). For $m+1 \le l \le n$, we have $\psi_{\mathrm{NL}}^{(l)} = O(\gamma_L)$, so that these terms contribute to $\gamma_L \Phi_J$ in Eq. (58b).

APPENDIX B

In this appendix, we discuss the derivation of Eqs. (86)–(88). Since from the structures of the eigenvalues we expect the appearance of oscillatory behavior, a convenient strategy is to express P as a function of X and D, with the help of Eq. (83c), and to eliminate the polarization from the remaining equations (83a) and (83b):

$$P = X^{-1} - \frac{\epsilon}{\gamma_{||} X}(\dot{D} + \gamma_{||} D) , \tag{B1}$$

$$\dot{X} + \kappa X - \kappa X^{-1} + \frac{\epsilon\kappa}{\gamma_{||} X}(\dot{D} + \gamma_{||} D) = 0 , \tag{B2}$$

$$\epsilon X \ddot{D} + \dot{D}[-\epsilon\dot{X} + \epsilon X(\gamma_{||} + \gamma_{\perp})] + D(-\epsilon\gamma_{||}\dot{X} + \epsilon\gamma_{||}\gamma_{\perp} X + \gamma_{||}\gamma_{\perp} X^3) + \gamma_{||}\dot{X} - \gamma_{||}\gamma_{\perp} X = 0 . \tag{B3}$$

Using the expansion (87) and the differentation rule

$$\frac{df}{dt} = \frac{\partial f}{\partial t_1} + \epsilon^{-1/2}\omega(t_1)\frac{\partial f}{\partial t_2} \equiv \dot{f} + \epsilon^{-1/2}\omega(t_1) f'$$

we obtain the following equations.

(i) To order $\epsilon^{-1/2}$,

$$X'(0) = 0 , \tag{B4}$$

i.e.,

$$X(0) = X(0, t_1) .$$

(ii) To order ϵ^0,

$$\omega(0) X(0) X'(1) = -\kappa X(0)\dot{X}(0) - \kappa X^2(0) + \kappa , \tag{B5}$$

$$X(0)\omega^2(0) D''(0) + \gamma_{||}\gamma_{\perp} X^3(0) D(0) + \gamma_{||}\dot{X}(0) - \gamma_{||}\gamma_{\perp} X(0) = 0 . \tag{B6}$$

Since $X(0)$ is a function of t_1 only, we see from Eq. (B5) that $X(1)$ will diverge linearly with t_2 unless the right-hand side of that equation vanishes identically. Hence, in order for $X(1)$ to remain bounded, we must impose that

$$X(0)\dot{X}(0) + \kappa X^2(0) - \kappa = 0$$

which is precisely Eq. (88b). In order to solve Eq. (B6), we first consider the homogeneous part of the equation

$$\omega^2(0) D''(0) + \gamma_{||}\gamma_{\perp} X^2(0) D(0) = 0 . \tag{B7}$$

Up to this point, $\omega^2(0)$ is yet undetermined; we fix its value by requiring that

$$\omega^2(0) = \gamma_{||}\gamma_{\perp} X^2(0) \tag{B8}$$

so that the general solution of (B7) becomes

$$D(0) = \alpha(t_1) e^{it_2} + \alpha^*(t_1) e^{-it_2} \tag{B9}$$

in view of the reality of $D(0)$. The selection of $\omega(0)$ corresponding to (B8) represents the only choice that will converge to the results of the linear stability analysis in the long-time limit, when $\omega(0)$ becomes equal to the imaginary part of the root λ given by Eq. (81b). A particular solution of the inhomogeneous equation (B6) is

$$\gamma_{\perp} X^3(0) D(0) = -\dot{X}(0) + \gamma_{\perp} X(0) . \tag{B10}$$

The sum of Eqs. (B9) and (B10) yields the general solution of Eq. (B6) and this is the result contained in Eq. (88c).

(iii) To order $\epsilon^{1/2}$,

$$\omega(0) X(0) X'(2) + X(1)\dot{X}(0) + X(0)\dot{X}(1) + 2\kappa X(0) X(1) + \frac{\kappa}{\gamma_{||}}\omega(0) D'(0) = 0 , \tag{B11}$$

$$-\gamma_{||}\gamma_{\perp} X^3(0)[D''(1) + D(1)] = F(t_1) + A(t_1) e^{it_2} + A^*(t_1) e^{-it_2} , \tag{B12}$$

where F and A are functions of α, $X(0)$, and $X(1)$. Again, we require that $X(2)$ be a bounded function, and this leads to the result

$$\dot{X}(1) = -X(1)[X^{-1}(0)\dot{X}(0) + 2\kappa] , \tag{B13}$$

$$\gamma_{||}\omega(0) X'(2) = \kappa\sqrt{\gamma_{||}\gamma_{\perp}} D'(0) . \tag{B14}$$

At this point, it is easy to verify that $A(t_1)$ is a function of α and $X(0)$ only. Since the homogeneous part of Eq. (B12) leads to oscillations of the form $\exp(\pm it_2)$ as in the

homogeneous terms proportional to A and A^*, secular terms will appear in the general solution $D(1)$. These secular terms will diverge in time (as t_2^n) and, therefore, must vanish identically. The condition $A=0$ yields the differential equation (89) for α. This completes the full characterization of $X(0)$ and $D(0)$.

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