PHYSICAL REVIEW A VOLUME 29, NUMBER 3 MARCH 1984

Gravity in one dimension: A dynamical and statistical study

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(Received 25 July 1983)

Computer experiments performed on a one-dimensional self-gravitating system are discussed from a dynamical and a statistical point of view. Statistically, the time-averaged position and velocity densities are shown to approach their respective microcanonical ensemble averages for four, five, and six particles. The relaxation time is much longer than previously expected. For three particles the system does not relax to the microcanonical distribution. We have explored the surface of sections for the $N=3$ case and have correlated the statistical properties with explicit dynamical features. We show that this dynamical method is consistent with the statistical interpretation. The mechanism which generates the chaotic behavior for the $N=3$ case is multiple encounters.

INTRODUCTION

This paper concerns a dynamical system in one dimension. The particular system which we consider is that of mass points moving on a line in one dimension under the influence of their mutual gravitational attraction. The particles in one dimension do not interact with the usual $1/r^2$ force like mass points in three dimensions. Instead, the particles are equivalent to sheets of mass in three dimensions. These sheets are of infinite extent in the y and z directions and move in the x direction parallel to their normal. The gravitational acceleration $A(x,t)$ is obtained from Poisson's equation,

$$-\frac{\partial A(x,t)}{\partial x}=4\pi G\rho(x,t) \tag{1}$$

$$=4\pi Gm\sum_{i=1}^{N}\delta(x-x_i(t)) , \tag{2}$$

where $\delta(x)$ is the Dirac delta function, ρ is the mass density, and G is the gravitational constant. Then

$$A(x,t)=2\pi Gm\sum_{i=1}^{N}S(x-x_i(t)) , \tag{3}$$

where $S(x)$ is defined as

$$S(x)=\begin{cases}-1, & x>0\\ 0, & x=0\\ 1, & x<0 .\end{cases} \tag{4}$$

The field acting on the ith particle is proportional to the difference between the number of particles on the right and the number on the left:

$$A(x_i,t)=2\pi Gm(N-2i+1) . \tag{5}$$

Each particle experiences a uniform gravitational field and follows a parabolic trajectory until a crossing occurs, where there is a discontinuity in the field.

The Hamiltonian for this system is given by

$$H(x,p)=\sum_{i=1}^{N}p_i^2/2m+2\pi Gm^2\sum_{j>i}^{N}|x_j-x_i| . \tag{6}$$

N is the number of particles. The particles have equal mass m and when they meet they freely pass through each other. x_i, p_i, and v_i are the position, momentum, and velocity of the ith particle, respectively.

Except for properties which explicitly depend on ordering and labeling, this system is indistinguishable from a one-dimensional model in which particles undergo elastic collisions due to a hard-core potential in addition to the gravitational potential. From this perspective the system is seen as an ordered system with the field experienced by each particle a function only of the ordering and not of the position. The word "encounter" will be used rather than "collision" to describe the crossing of two particles because it more accurately conveys the idea that the interaction is not a simple binary event. Every particle continually interacts with every other particle through the long-range force. There is no discontinuity in the velocity, but there is a discontinuity in the acceleration at an encounter.

Although not the focus of this paper, this model has astrophysical relevance. Hohl and Feix[1] have conjectured that the velocities of stars normal to the galactic plane are decoupled from the rotational velocity components in a disk galaxy. Therefore, the stars will oscillate perpendicular to the galactic plane with a period independent of the galactic rotational period. This model describes the motion of stars perpendicular to the galactic plane if edge effects are neglected.

Much of the work done on the one-dimensional self-gravitating system has been motivated by astronomical considerations. Since the 1960s astronomers have carried out numerical experiments with this model to investigate questions about Vlasov Theory and Lynden-Bell's violent relaxation.[1–4]

In this paper we are interested in the physics of the model from both a dynamical and a statistical point of view. The statistical properties were first studied by Hohl and co-workers.[1,5–7] Hohl reports a relaxation time which is $N^2\tau_c$. N is the number of particles and $2\pi\tau$ is the average period of oscillation of a particle in the system. Hohl bases this estimate on two different kinds of computer experiments: Hohl and Broaddus[1] observed that the time required for the residual noise in the kinetic energy time correlation function to be reduced to a small fraction of the correlation function at short lag times was of this order. In another numerical experiment Hohl[5] found that the time necessary for the time-averaged density to reach an apparent equilibrium was also of the order of $N^2\tau_c$.

These two methods represent two different measures of relaxation. That they yield the same relaxation time is both compelling and surprising. It is surprising because there is no *a priori* reason why time scales which are defined in dynamically different ways should be the same. Thus, Hohl and co-workers describe a self-consistent picture of a dynamical system which equilibrates in $N^2\tau_c$. This is the only published estimate of the thermalization time. It has been quoted as recently as 1980.[8]

Although the question of ergodicity is never explicitly considered in Hohl's work, it is an implicit component of his picture. According to the ergodic hypothesis, the time-averaged velocity and position densities approach the predictions of the microcanonical ensemble. Hohl computed time-averaged position and velocity densities. The question of whether or not they agreed with the microcanonical ensemble was not investigated. He did not consider whether these averages represented the long-time limit. No systematic tests for convergence were reported.

Later, the microcanonical position and velocity distributions were obtained analytically by Rybicki[9] as a function of N. Hohl and other investigators never discussed the question of relaxation in the context of the ergodic hypothesis; thus, the need for the development of the microcanonical ensemble was not recognized by them.

The divergence of neighboring trajectories for the one-dimensional self-gravitating system has been studied by Froeschle and Schiedecker[10] and Benettin, Froeschle, and Scheidecker.[11] They developed a clever technique for uniformly sampling the microcanonical ensemble. They observe that trajectories can be grouped into two broad categories, having either a coarse-grained rapid or slow divergence. By invoking K-system criteria they infer that the rapidly diverging trajectories are ergodic.

Their important work is an essential departure from that of Hohl and others because it treats the system in the context of nonlinear dynamics. They conclude that the fraction of phase space which contains ergodic orbits is an increasing function of N. In contrast to the hard spheres of Sinai, which are completely ergodic for small N, it is well known via the Kolmagorov-Arnold-Moser (KAM) theorem[12] that continuous systems with only a few degrees of freedom have regions of stability and, hence, are not ergodic. It is of some concern to statistical physicists that systems with continuous paths in phase space are generally not ergodic. In this context the work of Froeschle and Scheidecker is significant. It is the only system which has quantitatively demonstrated the approach to ergodicity with increasing degrees of freedom.

The conclusions which Froeschle and Scheidecker have reached contradict Hohl's picture. As pointed out earlier, Hohl's results are consistent with a system which is ergodic for $N \geq 3$. This result is similar to Sinai's hard spheres. In contrast, Froeschle and Scheidecker indicate that regions of stability (slow divergence) and chaos (rapid divergence) coexist for $N \geq 3$. It is their interpretation that the chaotic regions produce ergodic behavior and the stable regions do not. They do not give any direct evidence that the chaotic regions represent ergodic behavior.

Each of these investigators have used fundamentally different methods. Froeschle and Scheidecker have studied the dynamical properties of trajectories. Hohl has examined the development of the position and velocity distributions as a function of time. The latter approach appeals directly to the ergodic hypothesis and is statistical in nature. It is the purpose of this paper to resolve the conflicting views of this model. We will systematically extend the statistical methods introduced by Hohl and explore how they can be harmonized with the dynamical methods of Froeschle and Scheidecker.

Some preliminary results have been reported concerning the $N^2\tau_c$ relaxation time hypothesized by Hohl. We[13] found that a 100-particle system did not approach equilibrium in the predicted time. In that paper it was concluded that either the system does not relax (i.e., is not ergodic) or it relaxes on a time scale much larger than $N^2\tau_c$. We show here that the one-dimensional self-gravitating system is ergodic for large N. Froeschle and Scheidecker have demonstrated that it also has the exponential divergence expected of K systems.

NUMERICAL EXPERIMENT

Because long-range forces do not easily lend themselves to traditional statistical techniques, and because ergodicity is a notoriously difficult analytical problem, this model is an ideal candidate for a numerical experiment. The programming approach is conceptually simple. The minimum encounter time between two adjacent particles is computed by solving the quadratic equation in time. All of the calculations are done in double precision, to fourteen and one-half significant figures, on a Xerox Sigma 9 computer. An inevitable error is propagated through any program based upon the calculation of minimum encounter time. This error propagation occurs in the so-called "exact" programs such as those used by Hohl[1] and by Froeschle and Scheidecker.[10]

For errors in the fourth significant figure the dynamics are practically unaffected. However, the system will eventually generate a spurious encounter between two particles. Because of the accumulated error, an encounter will occur which would not occur in a computer with infinite precision. At this point the system is evolving on a different path in phase space. The test for this sort of problem is to time reverse the system and determine how accurately the original conditions are reproduced. If the error in the particle positions has grown so large that the ordering of the encounters is no longer that obtained with infinite precision calculations, the system will not be time reversible. That is, the original ordering of the numbered particles will not be reproduced.

Small-N systems ($N=3$ to 10) can be followed for several thousand encounters before the inevitable error propagation causes the system to experience a spurious encounter which causes the relative ordering of particles in configuration space to be changed, compared to a calculation of infinite precision. Because of the error propagation the system may be pictured as drifting from orbit to orbit on the energy surface. Thus, there is an averaging effect over the ensemble due to error propagation in the computer. As a result of this extra averaging the system may appear more ergodic than it really is. The time scales which we will demonstrate here are the minimum relaxation time scales. In considering the general dynamical properties, the tendency of the computer to wander from orbit to orbit is not a severe problem since all orbits on the hypersurface are allowable. Only near the boundary of stable and chaotic orbits does this drift become important. However, we have never observed an orbit to cross from a stable region to a chaotic region or vice versa.

DYNAMICAL TESTS OF ERGODICITY

It is necessary to explicitly demonstrate what is meant by stable and chaotic regions, and to know how this behavior might be observed experimentally. We consider the three-particle case, the smallest case that is physically interesting since $N=2$ is trivially integrable. There are six dimensions to this space. From the conservation of energy and momentum, and by fixing the center of mass, we can restrict the phase motion to a three-dimensional manifold. This configuration is equivalent to a conservative dynamical system with two degrees of freedom.

Such a system can in turn be reduced to a plane area-preserving mapping by the introduction of the widely used Poincare surface of section.[14] The area-preserving property of a plane mapped into itself is the counterpart of a Hamiltonian dynamical system. The surface of section which we consider is the cut defined by the surface $(X_2-X_3)=0$, the crossing of particles 2 and 3. X_1 and V_1 then are the coordinates of the surface of section. Hereafter, we will abbreviate surface of section as SS.

The trajectory of the system on the three-dimensional phase surface will repeatedly intersect the cut made by the two-dimensional SS as the system evolves. For a chaotic orbit we expect the pattern of intersections to fill the area defined by the three-dimensional energy hypersurface on the SS in a seemingly random way. However, if there is another integral of the motion, the dimensionality of the available region in phase space is reduced to a two-dimensional surface embedded in the three-dimensional space.

Froeschle and Scheidecker,[10] referred to hereafter as FS, have made a surface of section for the $N=3$ case. They were the first to note that chaotic and integrable regions coexist for this model. They also make the general comment that the $N=3$ case "... exhibits the usual features of dynamical systems with two degrees of freedom." We will explore these features explicitly and discuss their implications. FS randomly sample the microcanonical ensemble in order to determine the fraction of chaotic orbits. In addition, we will describe the dynamical characteristics of the chaotic orbits.

In a mapping, an important role is played by stable fixed points, or points which are mapped back into themselves. Such an orbit will have a symmetrical oscillating behavior in the following sense: Since $X_2=X_3$ at the SS and the center of momentum is constrained to zero, the greatest symmetry is obtained whenever $V_2=-V_3$ and, hence, $V_1=0$. Therefore, we expect the fixed point to lie somewhere on the line $V_1=0$ on the SS. We have found a fixed point by iterating the ratio of kinetic to potential energy along this line. The SS which results from perturbing the initial conditions of the stable point slightly (and then renormalizing to the energy surface) is shown as the oval in the center of Fig. 1. The elliptical point is marked at the center. Larger perturbations yield the increasingly distorted ellipses (the cardioid shaped curves) shown in Fig. 1. The time sequence of the intersections at the SS reveals that these orbits are quasiperiodic. The point of intersection falls on the cardioid in a regular manner. After several hundred intersections the initial point has completed a cycle around the closed curve.

The surface of sections generated by most randomly selected initial conditions are similar in shape. They all look like this basic cardioid shape but they have different sizes. The initial conditions which generate the cardioid have the slow (linear) divergence that comprise 96% of the $N=3$ orbits reported by FS. Thus, nondivergent orbits lie on smooth surfaces in phase space as revealed by the closed regular one-dimensional curves in the SS. These integrable orbits are stable and quasiperiodic.

In contrast, orbits which are chaotic should be area-filling on the SS since there is no integral of the motion to restrict the dimensionality of the region in which the phase point may be found. The SS which is obtained for nearly all of the randomly chosen chaotic orbits is the "batwing" shown in Fig. 1. We say "nearly all" because we have found another, separate, chaotic region for one of the divergent orbits. We will return to this point shortly. At first glance it appears that the batwing is not a two-dimensional surface, but a simple one-dimensional curve. However, if we magnify a portion of the central tip of the

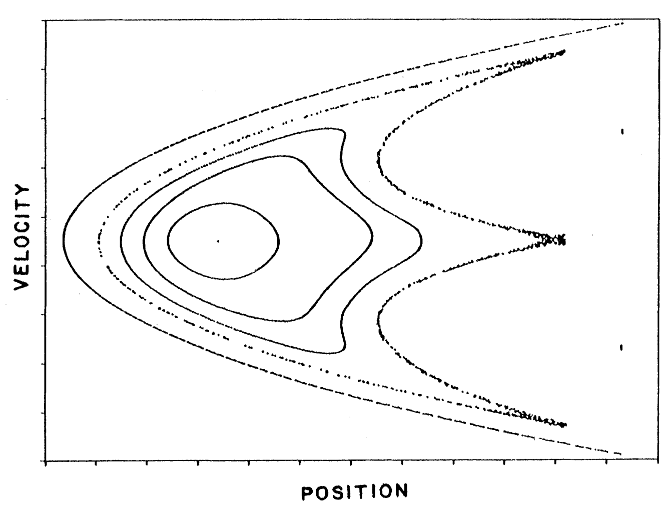

FIG. 1. Surface of section (position and velocity of particle 1 when particles 2 and 3 are colliding) for a 3-particle system. The stable point, three stable orbits, a chaotic orbit (batwing) and the stable outer boundary are shown.

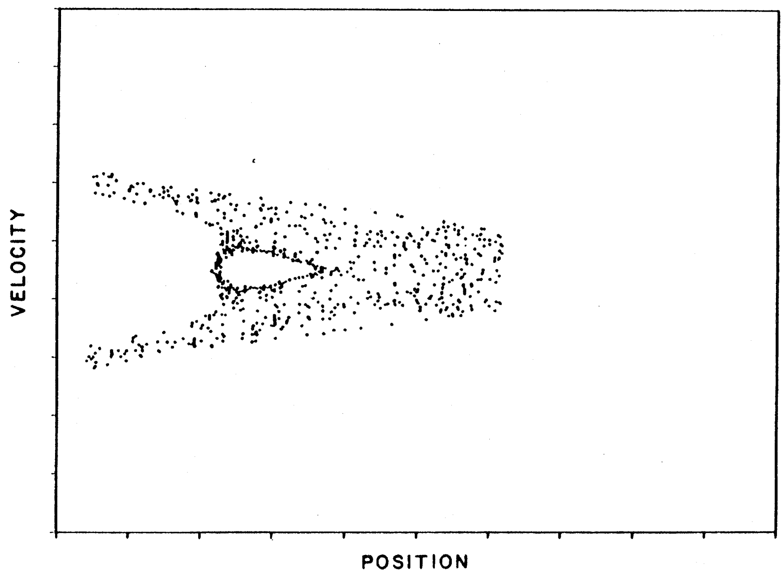

FIG. 2. Magnification of the central tip of the batwing in Fig. 1, showing the random filling of the region by one orbit.

right edge of the batwing, we see (Fig. 2) that it is indeed a two-dimensional surface. Any other portion of the batwing, when magnified, will show a narrow band which fills in a stochastic manner. Note that all of the points in Fig. 2 are made by one orbit. Upon magnification, the stable curves do not show such characteristics but are restricted to one-dimensional closed curves.

We have produced chaotic orbits in the following way: The initial conditions are set such that there is a near triple encounter, $x_1 \simeq x_2 \simeq x_3$. An exact triple encounter is an extremely divergent orbit which fills the whole phase space in less than 100 encounters. Orbits generated with a near triple encounter produce the batwing. If we make increasingly larger perturbations from the exact triple encounter, we still obtain chaotic orbits, but with progressively weaker divergence. Such orbits require several hundred to several thousand encounters to diverge and fill the available phase space. All of these orbits have a surface of section like the batwing. Eventually, the perturbation will be made so large that the orbit is no longer divergent. In this instance the stable SS is again a one-dimensional line which closely borders either the inner or outer boundary of the stochastic boundary.

The chaotic orbits for the $N=3$ case are those orbits which experience a near triple encounter, $x_1 \simeq x_2 \simeq x_3$, while stable orbits have no such occurrences. We have observed similar conditions for encounters between the particles in divergent orbits for $N=4$, 5, and 6. We conjecture that a general class of close encounters are responsible for the chaotic properties of this model for all N. Such encounters are not just localized in space. This was the case for $N=3$. For systems larger than $N=3$ pairs of encounters which are separated by a small time interval are associated with the chaotic behavior as well. In practice, we define these encounters as any for which the time between encounters is small,

$$\left| \frac{2\,\Delta A\,\Delta X}{\Delta V} \right| < 0.1 \;, \tag{7}$$

where ΔA, ΔV, and ΔX is the difference in the acceleration, velocity, and position of the adjacent particles used to determine the encounter. As N increases there is an increase in the percentage of the encounters which are close encounters. 9% of the $N=4$, 17% of the $N=5$, and 41% of the $N=6$ encounters were close encounters in the orbits used for the relaxation experiment. The dependence on N is made more obvious by recalling that the acceleration depends only on the ordering [Eq. (5)], so ΔA is always $2m$. The total mass is scaled to unity, so $m=1/N$. The close encounters are then

$$\left| \frac{\Delta X}{N\,\Delta V} \right| < 0.025 \;. \tag{8}$$

As FS have correctly noted there are chaotic and stable regions which are intermingled. An elliptic point is surrounded by a stable region which is surrounded by a chaotic region which, in turn, is bounded by a stable region. This picture is reminiscent of that described by Kolmogorov-Arnol'd-Moser (KAM) systems. KAM systems also have small stable regions embedded within the chaotic region, with the whole stable-chaotic hierarchy again reproduced in miniature.

This structure is reproduced in the 3-particle system. Note the small oval in the center of the chaotic region of Fig. 2. Inside the oval is a stable region. Most initial conditions which represent points inside the oval are stable orbits. We thus see a stable orbit lying in the middle of the chaotic region.

With one exception, all of the chaotic orbits which we have constructed using a near triple encounter or which have been found by picking orbits at random have produced the batwing surface of section. Approximately 15 chaotic orbits have been studied. The one exception was found randomly. By this we mean that the initial conditions were picked randomly, the energy was normalized, and then the divergence characteristics were checked to find the chaotic orbits. The SS of this exception is shown in Fig. 3. Again, all of the points were made by one orbit.

It has been conjectured "... that there is only one stochastic region...we cannot exclude the possibility that there exist very small stochastic regions which are separated from the large one."[11] However, for systems with two degrees of freedom, like our $N=3$ case, the chaotic regions are isolated from each other by stable surfaces. In contrast, chaotic regions in higher dimensions are linked by Arnol'd diffusion and, hence, are not isolated. Therefore, Fig. 3 represents a truly separate chaotic region from the usual batwing shown in Fig. 1. We have not been able to characterize the dynamical properties which distinguish these two chaotic regions from each other at this time.

The outer boundary of the SS is defined by generating initial conditions such that nearly all of the system energy is in the potential energy of particle 1. It is a stable (slowly diverging) orbit. This is the outer boundary shown in Fig. 1.

We have explored the SS in a systematic manner, from the central elliptic point out to the boundary which is defined by conservation of energy. This has been accomplished by relating various parts of the SS to particular dynamical features of the initial conditions.

The one-dimensional self-gravitating system does not satisfy the requirements for a KAM system because the force is not sufficiently smooth. Unlike the KAM systems, the energy cannot be varied to change the size of the stochastic regions.[15] For this model all of the dynamical features scale with the energy.[10] Nevertheless, this system has many of the characteristics of a KAM system. This feature may indicate tht the KAM theorem is applicable to a more general class of Hamiltonians than those for which it was proven.

The results which were discussed in Ref. 13 on the $N=100$ system raise questions about whether this system is ergodic. In order to understand why the $N=100$ system does not seem to relax to the state predicted by microcanonical ensemble theory, we examine small-N systems. Since the relaxation time is thought to be an increasing function of N, by looking at smaller-sized systems it is feasible to run for many relaxation times. Of course this assumes that the system is ergodic and that the time average of a dynamical variable will eventually converge to the microcanonical ensemble average.

STATISTICAL TESTS OF ERGODICITY

A series of experiments was designed to answer the question, "Does the system time average ever approach the ensemble average? If so, on what time scale?" If one does see an approach to the ensemble average, the time scale of this relaxation would represent the minimum time for a physical system to relax. It is a minimum because the effects of machine averaging would be added to the natural dynamical averaging. Using the criterion that the time average equals the ensemble average, we can obtain a lower bound on the thermalization time if the system is ergodic.

The positions and velocities were sampled and time-averaged density histograms for 4-, 5-, and 6-particle systems were constructed. In each case an arbitrary initial configuration was normalized to the chosen energy, and the system was allowed to evolve under the system dynamics. In constructing the time average, the position and velocities were sampled at a mean frequency of ten times per encounter. The time average for the density was made into a discrete 50-point histogram. The logarithm of the variance of this time average from the ensemble

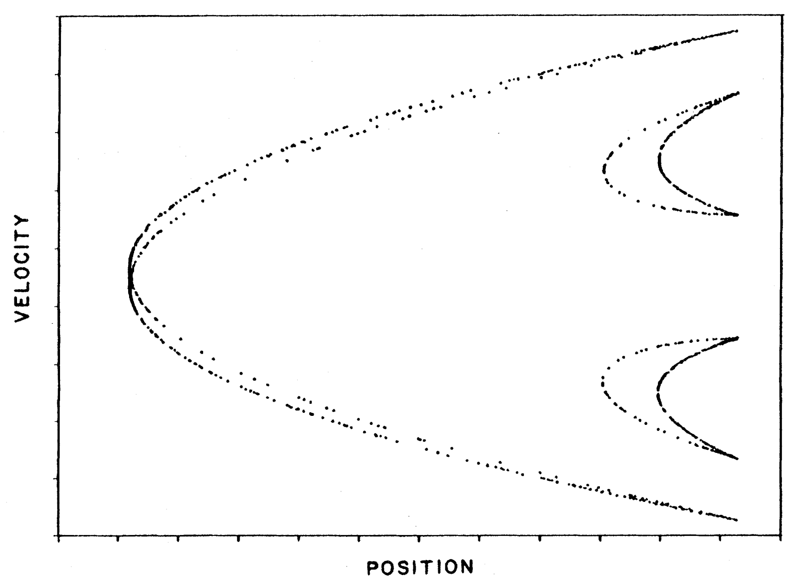

FIG. 3. $N=3$ surface of section showing a different chaotic region than the batwing of Fig. 1.

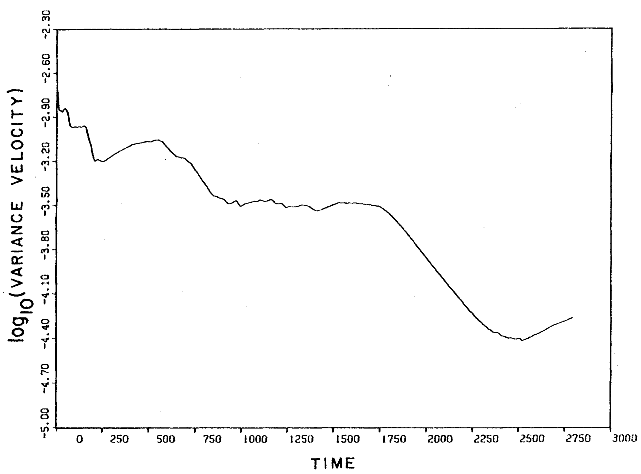

FIG. 4. Log_{10}(variance) [Eq. (9)] from the microcanonical ensemble average velocity density for an $N=4$ chaotic orbit. The time unit is $47.5N^2\tau_c$.

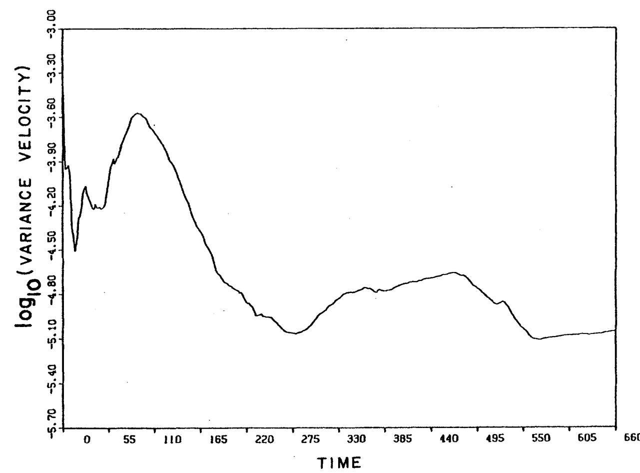

FIG. 6. Log_{10}(variance) [Eq. (9)] from the microcanonical ensemble average velocity density for an $N=6$ chaotic orbit. The time unit is $21.1N^2\tau_c$.

average is shown as a function of time in Figs. 4—6. This variance is computed by first numerically integrating the Rybicki microcanonical density over the same grid pattern used for the histogram. Then

$$\log_{10}(\text{variance})=\log_{10}\sum_{i=1}^{50}(I_{Ei}-I_{Ri})^2\ , \qquad (9)$$

where I_{Ri} is the Rybicki density integrated over the ith interval, and I_{Ei} is the normalized experimental density integrated over the same interval.

In this way the approach to equilibrium as a function of time in Figs. 4—6, for $N=4$, 5, and 6 is demonstrated. The time unit shown is 1000 of the natural time units obtained for the normalization $2\pi G=1$ and $M=1$. The time unit shown is related to τ_c, the characteristic time, as follows: It equals $960\tau_c$ for $N=6$, $860\tau_c$ for $N=5$, and $760\tau_c$ for $N=4$. It required approximately 50 hours of CPU time for each of these experiments.

Two immediate conclusions can be drawn from these data: the time-averaged density is approaching the ensemble average, but the relaxation time is many orders of magnitudes longer than the $N^2\tau_c$ relaxation time hypothesized by Hohl. For comparison purposes a fit of the time-averaged velocity density histogram with the Rybicki density is shown in Figs. 7 and 8. These averages were made over the time interval shown in Fig. 6 (but they were actually terminated near the minimum in the variance to show the best fit which was obtained). Only the velocity data are shown because similar results are obtained from the position data.

Each of the systems exhibit the same initial short period oscillations in the variance. As the time average acquires statistical significance, it takes a longer time for new configurations to add weight to the average, and the oscilla-

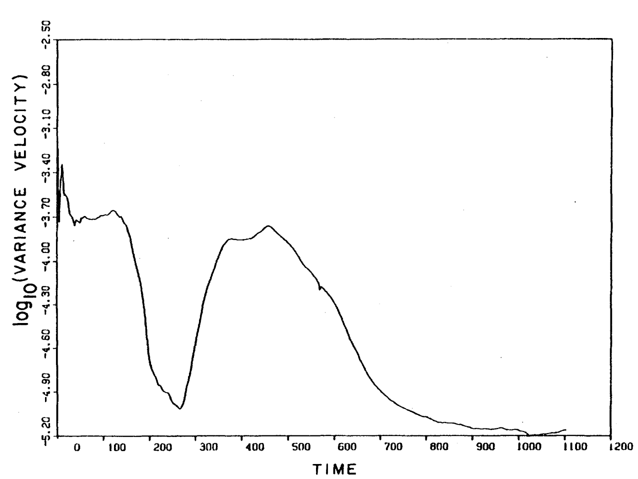

FIG. 5. Log_{10}(variance) [Eq. (9)] from the microcanonical ensemble average velocity density for an $N=5$ chaotic orbit. The time unit is $34.4N^2\tau_c$.

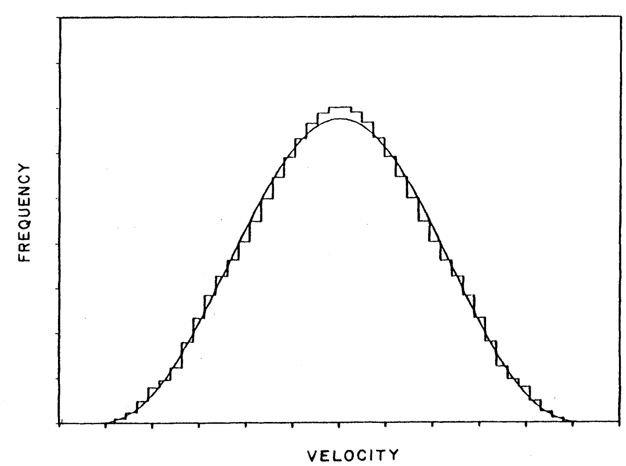

FIG. 7. $N=4$ velocity density histogram obtained from a time average over the interval in Fig. 4, compared to the microcanonical ensemble average.

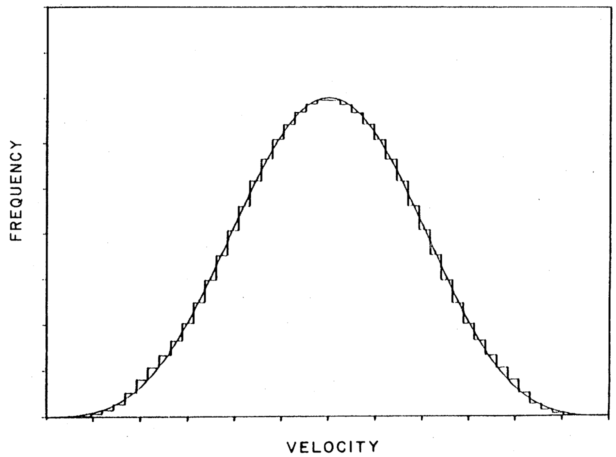

FIG. 8. $N=6$ velocity density histogram obtained from a time average over the interval in Fig. 6, compared to the microcanonical ensemble average.

tions in the variance acquire a longer period. It is somewhat surprising that all of the systems exhibit such large long-term oscillations. In each case the ensemble average appears to have significance as the distribution towards which the system time average is converging.

It is worth noting that the $N=4$ case did not achieve the degree of convergence that the $N=6$ case did, even though there were more encounters. The $N=4$ data of Fig. 4 represents 1.67 million encounters, $N=5$ in Fig. 5 has 1.15 million encounters, and $N=6$ has 1.05 million encounters. If the relaxation time increases with N, one would then expect the $N=4$ to relax sooner. The fact that it took longer and that it did not converge to the microcanonical ensemble as well as $N=6$ is evidence to support the claim[10] that there are finite stochastic and integrable regions in the $N=4$ phase space. To this extent the Froeschle and Scheidecker dynamical results and our statistical results are consistent.

Rybicki[9] has noted that Hohl's[16] results for the time-averaged density histogram for $N=3$ are quite different than the microcanonical ensemble average. That would imply an integral of the motion. He says "it would be interesting to repeat the $N=3$ experiment with other initial conditions to see how strict these integrals may be." We have performed many such experiments and find that the microcanonical ensemble average is not obtained for any of the orbits, even for divergent orbits. Furthermore, the time averages quickly converge to some function not characterized by the microcanonical average. This is to be expected since the chaotic portion of the $N=3$ phase space is very small.[10] The divergent orbits may sample the small stochastic bands, but they are excluded from the larger stable regions. Even though there is no integral of the motion for a divergent orbit, integrals for the stable orbits which bound the chaotic region prevent the phase point from sampling the whole phase space. Thus, even the chaotic orbits cannot have time-averaged densities which approach the ensemble average in any time limit for $N=3$.

CONCLUSION

A direct measure of relaxation, the time-averaged position and velocity density, has been used to attempt to determine the time required for the one-dimensional self-gravitating model to relax. We first used Hohl and Broaddus's[6] estimate of $N^2\tau_c$ for a numerical experiment for 100 particles. Not only did the system not relax, there was no evidence that it was moving toward the equilibrium state defined by the microcanonical ensemble on that time scale. It would be expected to relax to the microcanonical ensemble-averaged density only if the system were ergodic and there were no other isolating integrals, except for the energy. Because of the limitations on computer time our efforts were redirected towards the small-N cases in order to study relaxation and the question of ergodicity.

Hohl[16] reported time-averaged position and velocity densities for small-N cases that roughly approximate the isothermal distribution within a few $N^2\tau_c$ time units. However, he did not demonstrate any long-term stability in the density, which would be expected of equilibrium. We find that the small-N time-averaged densities oscillate about the exact microcanonical densities. Only after thousands of $N^2\tau_c$ time units do the cumulatively averaged densities stabilize around the microcanonical densities.

We have explored the surface of section for the $N=3$ case and have correlated the statistical properties of the orbits with dynamical features. A second stochastic region has been found which is distinct from the first. The peculiar condition which produces chaotic behavior is a near-multiple encounter.

There is consistency between the statistical and dynamical interpretations of this model. The time-averaged densities for $N=5$ and 6 converged better to the densities predicted by the ensemble average than the $N=4$ case did. The $N=3$ case did not converge. To this extent, our results are consistent with the Froeschle and Scheidecker[10] observation that as the size of the chaotic regions increase (with increasing N), the system appears more nearly ergodic. The $N=4$ case may be regarded as somewhat of an anomaly because it did approach the ensemble average even though only 86% of the phase space is chaotic. This might be explained by the fortuitous distribution of the stable zones throughout the phase space in such a way that the omission of the stable regions did not strongly affect the average.

We have shown that the divergent orbits have explicit ergodic properties: The time-averaged density approaches the ensemble averaged density. These divergent orbits lie in the chaotic region of phase space. The stable regions are associated with the nondivergent orbits. The intermediate class of orbits reported by FS which have first slow, then rapid divergence are points which lie in the chaotic region near a stable boundary. Thus, there is a correspondence between the exponential divergence, the statistical time average, and the chaotic regions of phase space.

ACKNOWLEDGMENTS

The authors wish to thank Dr. William E. Stein of Texas A&M University for valuable discussion concerning their research. They also acknowledge the assistance provided by Dr. Ivan Taylor and the Texas Christian University Computer Center.

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