Path-integral approach to the quantum theory of the degenerate parametric amplifier

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(Received 3 June 1983; revised manuscript received 24 October 1983)

The quantum theory of the degenerate parametric amplifier is usually treated in the parametric approximation where the pump field is treated classically. In this paper we present a fully quantized theory of this nonlinear optical device using a path-integral approach. A perturbation series, the first term of which corresponds to the parametric approximation, is employed to evaluate explicitly the coherent-state propagator. The question of the validity of the parametric approximation is considered and the conditions under which this approximation is justified are elucidated. Finally, certain correlation functions for the signal-mode operators are calculated that are needed to study squeezed states. It is shown that the quantum nature of the pump field tends to decrease the squeezing.

I. INTRODUCTION

The quantum statistical properties of the radiation produced by a degenerate parametric amplifier have recently received renewed attention.[1–5] Theoretical predictions indicate that under the proper conditions one should be able to produce light in both squeezed and antibunched states. Both types of states are nonclassical in nature. It has also been shown that squeezed states can be useful in the detection of very weak signals.[6] A device which can produce such states is, therefore, of some interest.

The degenerate parametric amplifier is a device which provides a nonlinear coupling between two modes of the radiation field.[7] The first, the pump mode, has a frequency of 2ω, while the second, the signal mode, has a frequency ω. The quantum theory of this device is usually treated in the so-called parametric approximation. In this approximation the pump mode is treated classically, i.e., replaced by a c-number, so that a single-mode Hamiltonian is obtained which is quadratic in the field operators. The problem can then be solved without further approximation. It should be noted that the parametric approximation neglects two effects. First, it ignores quantum fluctuations in the pump mode. Second, by treating the pump mode as a fixed c-number it also ignores depletion of this mode.

In this paper we will show that the parametric approximation can be derived from the first term of a perturbation series for the propagator of this system. Examination of the next term in the series allows us both to calculate corrections to the parametric approximation and to set bounds on its region of validity. We then use the lowest-order correction to the propagator to calculate corrections to both the intensity and squeezing of the signal mode. The perturbation series itself is derived from a path-integral representation for the propagator of this system. In a previous paper we presented a formalism for applying path integrals[8] to certain problems in nonlinear optics. Here we employ that formalism. The path-integral approach is useful because it allows one to see more clearly than the canonical approach the connection between the classical and quantum dynamics of the system.

II. PERTURBATION SERIES FOR PROPAGATOR

The Hamiltonian for a degenerate parametric amplifier is given by (we use units in which $\hbar=1$)

$$H=\omega a^\dagger a+2\omega b^\dagger b+\kappa(a^{\dagger 2}b+a^2b^\dagger)\ , \tag{2.1}$$

where a ($a^\dagger$) and b ($b^\dagger$) are the annihilation (creation) operators for the signal and pump modes, respectively, and κ is a coupling constant which depends upon the second-order susceptibility tensor of the medium which mediates the interaction. In the parametric approximation the pump mode is treated classically so that b is replaced by $\beta_0 e^{-2i\omega t}$ where β_0 is the amplitude of the pump mode. The resulting Hamiltonian is

$$H_p=\omega a^\dagger a+\kappa(\beta_0 e^{-2i\omega t}a^{\dagger 2}+\beta_0^* e^{2i\omega t}a^2)\ . \tag{2.2}$$

The propagator for this Hamiltonian was calculated in Ref. 8 and is given by (where β_0 is assumed to be real)

$$G(\alpha_f,t_f;\alpha_i,t_i)=\langle \alpha_f \mid U_p(t_f,t_i)\mid \alpha_i\rangle \tag{2.3}$$

$$\begin{aligned}=&\{\text{sech}[2\kappa\beta_0(t_f-t_i)]\}^{1/2}\\ &\times\exp\{-\tfrac{1}{2}(\mid\alpha_i\mid^2+\mid\alpha_f\mid^2)+\alpha_f^*\alpha_i e^{-i\omega(t_f-t_i)}\text{sech}[2\kappa\beta_0(t_f-t_i)]\\ &\quad-\tfrac{1}{2}i(\alpha_f^*)^2e^{-2i\omega t_f}\tanh[2\kappa\beta_0(t_f-t_i)]-\tfrac{1}{2}i\alpha_i^2e^{2i\omega t_i}\tanh[2\kappa\beta_0(t_f-t_i)]\}\ .\end{aligned} \tag{2.4}$$

Here $U_p(t_f,t_i)$ is the time-development transformation corresponding to H_p and $|\alpha\rangle$ is a coherent state with amplitude α.

The propagator for the Hamiltonian given by Eq. (2.1) is given by

$$K(\alpha_f,\beta_f,t_f;\alpha_i,\beta_i,t_i)=\langle\alpha_f,\beta_f\,|\,e^{-iH(t_f-t_i)}\,|\,\alpha_i,\beta_i\rangle\ , \tag{2.5}$$

where $|\alpha,\beta\rangle=|\alpha\rangle\otimes|\beta\rangle$, i.e., the tensor product of a coherent state for the signal mode with amplitude α and a coherent state for the pump mode with amplitude β. It is also possible to express this propagator in terms of a path integral. We have that

$$K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\int\mathscr{D}[\alpha(\tau)]\int\mathscr{D}[\beta(\tau)]e^{iS}\ , \tag{2.6}$$

where

$$iS=\int_0^t d\tau[\tfrac{1}{2}(\dot{\alpha}^*\alpha-\alpha^*\dot{\alpha})+\tfrac{1}{2}(\dot{\beta}^*\beta-\beta^*\dot{\beta})-iH(\alpha,\alpha^*;\beta,\beta^*)]\ , \tag{2.7}$$

$$H(\alpha,\alpha^*;\beta,\beta^*)=\omega|\alpha|^2+2\omega|\beta|^2+\kappa[(\alpha^*)^2\beta+\alpha^2\beta^*] \tag{2.8}$$

and the paths $\alpha(\tau)$ and $\beta(\tau)$ are such that $\alpha(t)=\alpha_f$, $\beta(t)=\beta_f$, $\alpha(0)=\alpha_i$, and $\beta(0)=\beta_i$.

It is not possible to evaluate the expression appearing in Eq. (2.6) exactly and we, therefore, resort to a perturbation expansion.[9] The first term of this expansion gives the contribution to the propagator corresponding to a classical description of the pump field; that is, if we retain only this term and make a further approximation which corresponds to letting the pump mode propagate in time as if there were no interaction, then we obtain the results given by the parametric approximation. We can calculate corrections by calculating the next term in the perturbation series and by refining the freely-propagating-pump-mode approximation.

Because the Hamiltonian given by Eq. (2.1) has only linear terms in b and $b^\dagger$ appearing in the interaction it is possible to perform the integration over the paths $\beta(\tau)$ by using the results in Ref. 8 for an arbitrary quadratic Hamiltonian. We find that

$$K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\int\mathscr{D}[\alpha(\tau)]\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\times e^{iS_0+iS_1}\ , \tag{2.9}$$

where

$$iS_0=\int_0^t d\tau\{\tfrac{1}{2}(\dot{\alpha}^*\alpha-\alpha^*\dot{\alpha})-i\omega|\alpha|^2-i\kappa[(\beta_f^*e^{-2i\omega t})e^{2i\omega\tau}\alpha^2+\beta_i e^{-2i\omega\tau}(\alpha^*)^2]\}\ , \tag{2.10}$$

$$iS_1=-\kappa^2\int_0^t d\tau_2\int_0^{\tau_2}d\tau_1 e^{-2i\omega(\tau_2-\tau_1)}[\alpha^*(\tau_2)\alpha(\tau_1)]^2\ . \tag{2.11}$$

We have split the action into two parts, S_0 containing terms of zeroth and first order in κ, and S_1 containing only terms of second order in κ. We assume that the interaction is weak so that S_1 is small.

We now expand the propagator in Eq. (2.9) in a power series in S_1:

$$\begin{aligned}K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)&=\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\sum_{n=0}^{\infty}\frac{1}{n!}\int\mathscr{D}[\alpha(\tau)]e^{iS_0}(iS_1)^n\\&\simeq K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)+K^{(1)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)\ ,\end{aligned} \tag{2.12}$$

where

$$K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\int\mathscr{D}[\alpha(\tau)]e^{iS_0}\ , \tag{2.13}$$

$$K^{(1)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\int\mathscr{D}[\alpha(\tau)]e^{iS_0}(iS_1)\ . \tag{2.14}$$

Before evaluating $K^{(0)}$ let us note the following. The exponential factor appearing in both $K^{(0)}$ and $K^{(1)}$ has a magnitude given by

$$|\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]|=[\exp(-|\beta_f-\beta_i e^{-2i\omega t}|^2)]^{1/2}\ , \tag{2.15}$$

so that it is peaked about the value $\beta_f=e^{-2i\omega t}\beta_i$. This simply corresponds to free propagation of the pump mode, i.e., if there were no interactions and at $t=0$ the pump mode were in a coherent state with amplitude β_i, then at time t it would be in a coherent state with amplitude $e^{-2i\omega t}\beta_i$. If we replace β_f in iS_0 by $e^{-2i\omega t}\beta_i$ we find that (again assuming that β_0 is real)

$$iS_0\to\int_0^t d\tau\{\tfrac{1}{2}(\dot{\alpha}^*\alpha-\alpha^*\dot{\alpha})-i\omega|\alpha|^2-i\kappa\beta_0[(\alpha^*)^2e^{-2i\omega\tau}+\alpha^2e^{2i\omega\tau}]\}\ . \tag{2.16}$$

This is just the action for the signal mode in the parametric approximation (corresponding to the Hamiltonian H_p). If the path integral appearing in Eq. (2.13) is a slowly varying function of β_f then this replacement is justified and we can approximate $K^{(0)}$ by

$$K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)\simeq\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]\times G(\alpha_f,t;\alpha_i,0)\ , \tag{2.17}$$

where β_0 in the expression for G [Eq. (2.4)] is set equal to β_i. This expression for $K^{(0)}$ will reproduce all of the results of the parametric approximation. We can calculate corrections to this approximation by doing two things. First, we evaluate $K^{(1)}$ where we set $\beta_f = e^{-2i\omega t}\beta_i$ in the path integral appearing in Eq. (2.14). Second, we must calculate corrections to the approximation implied by Eq. (2.17) for $K^{(0)}$. We will discuss the validity of the approximations we have made in Sec. IV.

Let us now evaluate $K^{(0)}$ and $K^{(1)}$. We can find $K^{(0)}$ in the same way in which we found G in Ref. 8. We have that

$$K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0) = \exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}] \times G^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0) , \tag{2.18}$$

where

$$G^{(0)}(\alpha_f,\beta_f,t_2;\alpha_i,\beta_i,t_1) = \{\operatorname{sech}[2\sqrt{\kappa_1\kappa_2}(t_2-t_1)]\}^{1/2}\exp[-\tfrac{1}{2}(|\alpha_f|^2+|\alpha_i|^2)+A_{21}\alpha_i^2+B_{21}(\alpha_f^*)^2+C_{21}\alpha_f^*\alpha_i] , \tag{2.19}$$

$$A_{jl} = -\frac{1}{2}i\left[\frac{\kappa_2}{\kappa_1}\right]^{1/2} e^{2i\omega t_l}\tanh[2\sqrt{\kappa_1\kappa_2}(t_j-t_l)] , \tag{2.20a}$$

$$B_{jl} = -\frac{1}{2}i\left[\frac{\kappa_1}{\kappa_2}\right]^{1/2} e^{-2i\omega t_j}\tanh[2\sqrt{\kappa_1\kappa_2}(t_j-t_l)] , \tag{2.20b}$$

$$C_{jl} = e^{-i\omega(t_j-t_l)}\operatorname{sech}[2\sqrt{\kappa_1\kappa_2}(t_j-t_l)] , \tag{2.20c}$$

and

$$\kappa_1 = \kappa\beta_i, \quad \kappa_2 = \kappa\beta_f^* e^{-2i\omega t} . \tag{2.21}$$

In the above we assume that $t_j \geq t_l$ and we define $t_0 = 0$. We must also specify which branch of the square-root function is to be chosen. It should be chosen so that $\sqrt{\kappa_1/\kappa_2}\sqrt{\kappa_1\kappa_2} = \kappa_1$.

The evaluation of $K^{(1)}$ is complicated and the details of the calculation are given in Appendix A. We will be interested in the case in which the signal mode is initially in the vacuum state. This means that we will be interested in the propagator for $\alpha_i = 0$. The resulting expression is

$$K^{(1)}(\alpha_f,\beta_f,t;0,\beta_i,0) = \exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0) \times[g_4(t)(e^{-i\omega t}\alpha_f^*)^4+g_2(t)(e^{-i\omega t}\alpha_f^*)^2+g_0(t)] , \tag{2.22}$$

where

$$g_4(t) = \frac{-\kappa^2}{\eta_0^2}\frac{1}{8}[(\eta_0 t)^2\operatorname{sech}^4(\eta_0 t)+2(\eta_0 t)\tanh(\eta_0 t)\operatorname{sech}^2(\eta_0 t)-\tanh^2(\eta_0 t)-2\tanh^2(\eta_0 t)\operatorname{sech}^2(\eta_0 t)] , \tag{2.23a}$$

$$g_2(t) = -i\frac{\kappa^2}{\eta_0^2}\frac{3}{4}\{-(\eta_0 t)^2\operatorname{sech}^2(\eta_0 t)\tanh(\eta_0 t)+(\eta_0 t)[\tfrac{5}{3}\operatorname{sech}^2(\eta_0 t)-\tfrac{1}{3}]-\tfrac{4}{3}\tanh(\eta_0 t)+2\tanh^3(\eta_0 t)\} , \tag{2.23b}$$

$$g_0(t) = -\frac{\kappa^2}{\eta_0^2}\frac{1}{8}\{(\eta_0 t)^2[3\operatorname{sech}^2(\eta_0 t)-1]+(\eta_0 t)4\tanh(\eta_0 t)-6\tanh^2(\eta_0 t)\} , \tag{2.23c}$$

$$\eta_0 = 2\kappa\beta_i . \tag{2.23d}$$

It follows from Eq. (2.12) that Eqs. (2.18) and (2.22) give us an explicit expression for the propagator $K(\alpha_f,\beta_f,t;0,\beta_i,0)$ that contains the quantum corrections to the parametric approximation. The correlation functions for the field operator can be evaluated from the propagator. In Sec. III we calculate the correlation functions that are needed to study the intensity and the squeezing of the signal mode.

III. CORRECTIONS TO CORRELATION FUNCTIONS AND "SQUEEZING" OF THE SIGNAL MODE

The propagator K is closely related to the Q representation of the radiation field and, hence, can be used directly to evaluate expectation values of antinormally ordered products of creation and annihilation operators.[8] For the case of interest in which the pump mode is initially in the state $|\beta_i\rangle$ and the signal mode is initially in the vacuum states we have

$$Q(\alpha_f,\beta_f,t) = \frac{1}{\pi^2}|K(\alpha_f,\beta_f,t;0,\beta_i,0)|^2 . \tag{3.1}$$

The two correlation functions which we wish to calculate, $\langle a^\dagger(t)a(t)\rangle$ and $\langle[a(t)]^2\rangle$, can therefore be expressed as

$$\langle a^\dagger(t)a(t)\rangle = \frac{1}{\pi^2}\int d^2\alpha_f \int d^2\beta_f\, |K(\alpha_f,\beta_f,t;0,\beta_i,0)|^2 \times|\alpha_f|^2-1 , \tag{3.2}$$

$$\langle [a(t)]^2\rangle=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f\,|K(\alpha_f,\beta_f,t;0,\beta_i,0)|^2\times(\alpha_f^*)^2\;. \tag{3.3}$$

The correlation function $\langle a^\dagger(t)a(t)\rangle$ is just the intensity of the signal mode and examination of it will allow us to see how this mode grows with time. Calculation of the correlation function $\langle [a(t)]^2\rangle$ allows us to examine the squeezing of the signal mode.

In a squeezed state, the fluctuations in one quadrature are smaller than the standard quantum limit. The fluctuations are increased in the conjugate one so that the uncertainty relation is not violated. Squeezing is a genuinely quantum-mechanical feature of the radiation field. It has been predicted that a number of nonlinear optical systems will generate such states.[10–19]

We define Hermitian dimensionless amplitudes

$$a_1=\tfrac{1}{2}ae^{i(\omega t-\pi/4)}+\text{H.c.}\;, \tag{3.4a}$$

$$a_2=\frac{1}{2i}ae^{i(\omega t-\pi/4)}+\text{H.c.} \tag{3.4b}$$

For initial vacuum state of the pump mode we obtain the following formulas for the variances of the amplitudes a_1 and a_2:

$$\Delta a_1^2=\tfrac{1}{4}+\tfrac{1}{2}\langle a^\dagger(t)a(t)\rangle+\tfrac{1}{2}\text{Im}\{\langle [a(t)]^2\rangle e^{2i\omega t}\}\;, \tag{3.5a}$$

$$\Delta a_2^2=\tfrac{1}{4}+\tfrac{1}{2}\langle a^\dagger(t)a(t)\rangle-\tfrac{1}{2}\text{Im}\{\langle [a(t)]^2\rangle e^{2i\omega t}\}\;. \tag{3.5b}$$

It is clear that we need to evaluate the correlation functions given in Eqs. (3.2) and (3.3) to study the squeezing in the variables a_1 and a_2.

In order to calculate the lowest-order approximation to the correlation functions $\langle a^\dagger(t)a(t)\rangle$ and $\langle a^2(t)\rangle$ we first substitute $K^{(0)}$ for K in Eqs. (3.2) and (3.3) and make use of the freely-propagating-pump approximation to evaluate the β_f integral. This yields

$$\langle a^\dagger(t)a(t)\rangle=\frac{1}{\pi}\int d^2\alpha_f\,|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2\times|\alpha_f|^2-1$$

$$=\sinh^2(\eta_0 t)\;, \tag{3.6}$$

$$\langle [a(t)]^2\rangle=\frac{1}{\pi}\int d^2\alpha_f\,|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2\alpha_f^2$$

$$=-ie^{-2i\omega t}\sinh(\eta_0 t)\cosh(\eta_0 t)\;. \tag{3.7}$$

These are the results which one obtains from the parametric approximation.

In order to calculate corrections to the above expressions we need (i) to improve the freely-propagating-pump approximation and (ii) to include the effects of $K^{(1)}$. It is clear how to do the latter as $K^{(1)}$ has been calculated in Sec. II. The idea behind the former is as follows. In making the freely-propagating-pump approximation we assumed that $G^{(0)}$ was a constant as a function of β_f in a neighborhood of $\beta_f=e^{-2i\omega t}\beta_i$. We can correct this by taking into account some of the variation of $G^{(0)}$ as a function of β_f in this region. This can be done by expanding in a power series in $\delta\beta_f=\beta_f-\beta_i e^{-2i\omega t}$. It turns out that the convenient quantity to expand is $\int d^2\alpha_f G^{(0)}$ multiplied by either α_f^2 or $|\alpha_f|^2$ (where we choose α_f^2 if we are evaluating $\langle a^2\rangle$ and $|\alpha_f|^2$ if we are evaluating $\langle a^\dagger a\rangle$) because we can do the α_f integration exactly. We then expand these quantities up to second order in $\delta\beta_f$ and then perform the β_f integration. The linear and quadratic terms in $\delta\beta_f$ give corrections to the freely-propagating-pump approximation. The details of these calculations are given in Appendix B. We obtain

$$\langle a^\dagger(t)a(t)\rangle=\sinh^2(\eta_0 t)+\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[2\sinh^2(\eta_0 t)+1]+\eta_0 t[2\sinh(\eta_0 t)\cosh(\eta_0 t)]-3\sinh^4(\eta_0 t)-3\sinh^2(\eta_0 t)\}\;, \tag{3.8}$$

$$\langle [a(t)]^2\rangle=-ie^{-2i\omega t}\sinh(\eta_0 t)\cosh(\eta_0 t)-ie^{-2i\omega t}\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[2\sinh(\eta_0 t)\cosh(\eta_0 t)]+\eta_0 t[2\sinh^2(\eta_0 t)+2]-3\sinh^3(\eta_0 t)\cosh(\eta_0 t)-2\sinh(\eta_0 t)\cosh(\eta_0 t)\}\;. \tag{3.9}$$

The fluctuations in the conjugate variables $a_1(t)$ and $a_2(t)$ are obtained on substituting from Eqs. (3.8) and (3.9) in Eqs. (3.5):

$$\Delta a_1^2=\tfrac{1}{4}e^{-2\eta_0 t}+\frac{\kappa^2}{2\eta_0^2}\{(\eta_0 t)^2e^{-2\eta_0 t}-\eta_0 t(e^{-2\eta_0 t}+1)+[3\sinh^2(\eta_0 t)+2]\sinh(\eta_0 t)e^{-\eta_0 t}-\sinh^2(\eta_0 t)\}\;, \tag{3.10}$$

$$\Delta a_2^2=\tfrac{1}{4}e^{2\eta_0 t}+\frac{\kappa^2}{2\eta_0^2}\{(\eta_0 t)^2e^{2\eta_0 t}+\eta_0 t(e^{2\eta_0 t}+1)-[3\sinh^2(\eta_0 t)+2]\sinh(\eta_0 t)e^{\eta_0 t}-\sinh^2(\eta_0 t)\}\;. \tag{3.11}$$

Equations (3.8), (3.10), and (3.11) give us the lowest-order quantum corrections to the parametric approximation for the quantities $\langle a^\dagger(t)a(t)\rangle$, Δa_1^2, and Δa_2^2. In Table I, we have calculated Δa_1^2 as a function of $\eta_0 t$ for different values of β_i. It is clear that the quantum fluctuations in the pump mode tend to decrease the squeezing in the signal mode.

As we will see in Sec. IV our values for Δa_1^2 will be

TABLE I. Calculated values of Δa_1^2 as a function of $\eta_0 t$ for different values of β_i.

	Δa_1^2 (10^4)			
$\eta_0 t$	Parametric approx.	$\beta_i=1000$	$\beta_i=100$	$\beta_i=10$
0.0	2500.00	2500.00	2500.00	2500.00
0.2	1675.80	1675.80	1675.80	1675.82
0.4	1123.32	1123.32	1123.32	1123.49
0.6	752.986	752.986	752.991	753.561
0.8	504.741	504.741	504.756	
1.0	338.338	338.339	338.373	
1.2	226.795	226.796	226.865	
1.4	152.025	152.026	152.157	
1.6	101.906	101.908	102.140	
1.8	68.309 3	68.313 3	68.7075	
2.0	45.789 1	45.795 6		
2.2	30.693 3	30.703 8		
2.4	20.574 4	20.590 9		
2.6	13.791 4	13.817 0		
2.8	9.244 66	9.283 90		
3.0	6.196 88	6.256 65		

good approximations to the actual values as long as $\eta_0 t$ is of order one or less and $\exp(2\eta_0 t) \ll \beta_i$. For values of $\eta_0 t$ which satisfy these conditions we find that the corrections to the parametric approximation are of the order of 1%. If one considers values of $\eta_0 t$ beyond the range specified by these conditions one finds that Δa_1^2 reaches a minimum and then starts increasing. This type of behavior is not unexpected because as the pump becomes depleted and loses its coherent-state character its phase becomes less well defined. This results in a decrease in the squeezing of the signal mode. An analysis with a classical pump with phase noise shows this explicitly.[5] For the case of a quantum-mechanical pump mode our results provide, at best, an indication of this type of behavior as we are extrapolating our results beyond their range of validity. Finally, we note that the minimum uncertainty relation $\Delta a_1 \Delta a_2 = \frac{1}{4}$ which holds for the signal mode in the parametric approximation is now no longer satisfied. The quantization of the pump mode removes the minimum uncertainty characteristic of the signal mode.

IV. DISCUSSION OF APPROXIMATIONS

In this section we would like to consider a number of the approximations which were made in Secs. II and III. First we will examine some limitations on the validity of the perturbation expansion itself. We will then consider the conditions under which the approximation implied by Eq. (2.17) is reasonable. Finally, we will examine under what conditions $K^{(0)}$ can be used to give an accurate evaluation of correlation functions.

An examination of the expressions we have obtained for $K^{(0)}$ and $K^{(1)}$ shows that they cannot be valid for all values of β_i and β_f. Both of these variables occur in the arguments of the functions sech and tanh. Both of these functions have singularities on the imaginary axis so that for certain values of β_i and β_f, $K^{(0)}$ and $K^{(1)}$ have essential singularities. This implies that for these values the perturbation expansion given in Eq. (2.12) does not make sense. There is, however, a more stringent requirement on β_i and β_f: The integrals which must be performed to compute the terms of the series, e.g., those in Eq. (A4), must converge. This restricts the range of values which β_i and β_f can assume.

In the determination of these restrictions we will work with the variables κ_1 and κ_2 [see Eq. (2.21)] rather than with β_i and β_f directly. Let us assume that κ_1 is real and positive. We then find a range of values of κ_2 for which the above-mentioned integrals converge. In Appendix C we show that the following region satisfies this requirement. Let $\kappa_2 = |\kappa_2| e^{i\theta}$ and define $\sigma(\theta)$ as

$$\sigma(\theta) = 4\left\{\frac{\cosh[\pi/2s(\theta)]-1}{\cosh[\pi/2s(\theta)]+1}\right\}, \tag{4.1}$$

where $s(\theta)=\tan(\theta/2)$. We define the region R as

$$R = \{\kappa_2 \mid \; -\theta_{\max} \le \theta \le \theta_{\max} \text{ and } \tfrac{1}{2}\{\sigma(\theta)-2-[\sigma^2(\theta)-4]^{1/2}\} \le |\kappa_2/\kappa_1| - 1 \le \tfrac{1}{2}\{\sigma(\theta)-2+[\sigma^2(\theta)-4]^{1/2}\}\} \tag{4.2}$$

and picture it in Fig. 1. The angle $\theta_{\max}$ is the angle for which $\sigma(\theta_{\max})=2$, i.e., the angle for which the inequality in Eq. (4.2) gives $0 \le |(\kappa_2/\kappa_1)| - 1 \le 0$. We find from Eq. (4.1) that $\theta_{\max}=0.46\pi$. If $\kappa_2 \in R$ then the necessary integrals will converge. Unless this is true our perturbation series will not be justified.

The next thing which we would like to consider is the freely-propagating-pump approximation. We noted before that Eq. (2.17) would be a good approximation for $K^{(0)}$, at least in the region of interest where the Gaussian factor is

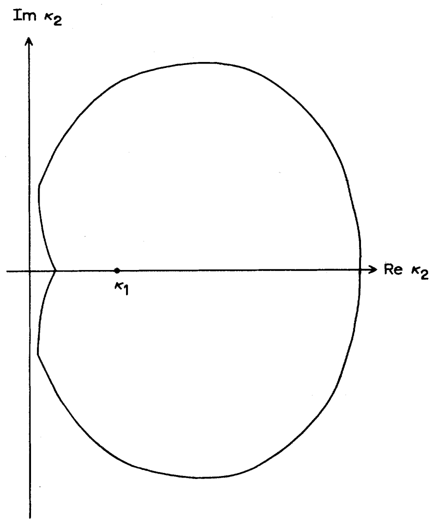

FIG. 1. Region R in complex κ_2 plane.

not small, if $G^{(0)}$ is a slowly varying function of β_f. We now want to determine when this is the case. Let us first define

$$\delta\beta_f=\beta_f-e^{-2i\omega t}\beta_i \tag{4.3}$$

and note that the exponential factor in Eq. (2.17) starts to drop off rapidly for $\delta\beta_f\sim 1$. Examining $G^{(0)}$ now, we see that if $\delta\beta_f\sim 1$ then the deviations in $\tanh(2\sqrt{\kappa_1\kappa_2}t)$ and $\operatorname{sech}(2\sqrt{\kappa_1\kappa_2}t)$ will be small if $\kappa t\ll 1$, and the deviation in $\sqrt{\kappa_1/\kappa_2}$ and $\sqrt{\kappa_2/\kappa_1}$ is of order $|\delta\beta_f/\beta_i|$ and so will be small if $|\beta_i|\gg 1$. These factors are, however, multiplied by $(\alpha_f^*)^2$ and α_i^2. Therefore, $G^{(0)}$ will be a slowly varying function of β_f if

$$|\alpha_i|^2\kappa t\ll 1\ , \tag{4.4a}$$

$$|\alpha_i|^2/|\beta_i|\ll 1\ , \tag{4.4b}$$

$$|\alpha_f|^2\kappa t\ll 1\ , \tag{4.4c}$$

$$|\alpha_f|^2/|\beta_i|\ll 1\ . \tag{4.4d}$$

We now want to discuss the calculation of correlation functions. Let us again assume that β_i is real and take it to be fixed. We define the function $f(\alpha_f,\beta_f,t;\alpha_i,\beta_i)$ by

$$K^{(1)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)=K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)f(\alpha_f,\beta_f,t;\alpha_i,\beta_i)\ . \tag{4.5}$$

The region $S(t)$ in which we would expect the approximate propagator $K^{(0)}$ to be close to the actual propagator K is just

$$S(t)=\{(\alpha_i,\alpha_f,\beta_f)\mid\ \beta_f\in R \text{ and } |f(\alpha_f,\beta_f,t;\alpha_i,\beta_i)|\ll 1\}\ , \tag{4.6}$$

i.e., a point $(\alpha_i,\alpha_f,\beta_f)$ is in $S(t)$ if β_f is in R and α_i, α_f, and β_f are such that $|f|$ is small. If a point $(\alpha_i,\alpha_f,\beta_f)$ is in $S(t)$ we are justified in neglecting $K^{(1)}$ in comparison to $K^{(0)}$ but for the parametric approximation to hold we need also that the freely-propagating-pump approximation be valid. That is, we require that conditions (4.4) be satisfied. Therefore, we are interested in a region $S'(t)$ where

$$S'(t)=\{(\alpha_i,\alpha_f,\beta_f)\mid\ (\alpha_i,\alpha_f,\beta_f)\in S(t) \text{ and Eqs. (4.4) are satisfied}\}\ . \tag{4.7}$$

When calculating the correlation functions for the signal mode one encounters an expression of the form[8]

$$\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\alpha_i\int d^2\beta_f P(\alpha_i)|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2\times(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}\ , \tag{4.8}$$

where the initial state of the system is given by

$$\rho=\int d^2\alpha_i P(\alpha_i)|\alpha_i,\beta_i\rangle\langle\alpha_i,\beta_i| \tag{4.9}$$

and $P(\alpha_i)$ is the P representation for the signal mode at $t=0$. If the "function" $P(\alpha_i)|K|^2$ is small outside of $S'(t)$ and falls off rapidly enough then we can accurately approximate Eq. (4.8) by confining the integration to $S'(t)$ and replacing K by the expression on the right-hand side of Eq. (2.17), at least for sufficiently small values of n_j and m_j where $j=1,2$. This replacement of K by the approximate expression given in Eq. (2.17) is nothing but the parametric approximation. We need to find, then, some sort of measure of the extent to which $P(\alpha_i)|K|^2$ is concentrated in $S'(t)$ and some information on the falloff properties of $|K|^2$.

Let us now consider a measure of the extent to which $P(\alpha_i)|K|^2$ is concentrated on $S'(t)$. The propagator K obeys the identity

$$1=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f\int d^2\alpha_i|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2\times P(\alpha_i)\ . \tag{4.10}$$

If we assume that $P(\alpha_i)$ is positive semidefinite (or the limit of positive semidefinite functions) then the quantity

$$\mu(t)=\frac{1}{\pi^2}\int\int_{S'(t)}\int d^2\alpha_f d^2\beta_f d^2\alpha_i|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2\times P(\alpha_i) \tag{4.11}$$

will provide a good indication of the extent to which the region in which $P(\alpha_i)|K|^2$ is concentrated is contained in $S'(t)$. If $\mu(t)$ is close to 1 then $P(\alpha_i)|K|^2$ can be considered to be well concentrated in $S'(t)$.

It is possible to simplify the expression appearing on the right-hand side of Eq. (4.11). First, because of the definition of $S'(t)$ we have that

$$\mu(t)\simeq\frac{1}{\pi^2}\int\int_{S'(t)}\int d^2\alpha_f d^2\beta_f d^2\alpha_i\times|K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)|^2P(\alpha_i)\ . \tag{4.12}$$

We can go still further because the conditions for the freely-propagating-pump approximation hold. We can perform the β_f integration with the result that

$$\mu(t)\simeq\frac{1}{\pi}\int_{M(t)}\int d^2\alpha_f d^2\alpha_i \,| \times G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)\,|^2 P(\alpha_i)\,, \tag{4.13}$$

where

$$M(t)=\{(\alpha_f,\alpha_i)\mid\ (\alpha_i,\alpha_f,e^{-2i\omega t}\beta_i)\in S'(t)\}\ .$$

If $1-\mu(t)\ll 1$, then it is possible to simplify the expression (4.8). We have that

$$\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\alpha_i\int d^2\beta_f P(\alpha_i)\,|K(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)\,|^2(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}$$
$$\simeq\frac{1}{\pi^2}\int\int_{S'(t)}\int d^2\alpha_f d^2\alpha_i d^2\beta_f P(\alpha_i)\,|K^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)\,|^2(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}$$
$$\simeq\frac{1}{\pi^2}\int_{M(t)}\int d^2\alpha_f d^2\alpha_i P(\alpha_i)\,|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i;\alpha_i,\beta_i,0)\,|^2(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}\ . \tag{4.14}$$

Because $\mu(t)$ is close to 1 we have that $P(\alpha_i)\,|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)\,|^2$ is concentrated in $M(t)$ and is, therefore, small outside this region. If it also falls off rapidly enough outside of $M(t)$ then we can extend the α_i and α_f integrations over the entire complex plane without much error. Our final approximation to expression (4.8) is then

$$\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\alpha_i P(\alpha_i)\,|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)\,|^2(\alpha_f^*)^{n_1}\alpha_f^{n_2}(\alpha_i^*)^{m_1}\alpha_i^{m_2}\ . \tag{4.15}$$

If one substitutes expression (4.15) in the calculation of correlation functions one will obtain the results given by the parametric approximation. This is because $G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)$ is just the propagator for the Hamiltonian given in Eq. (2.2).

In the preceding discussion we had to assume that $P(\alpha_i)\,|K\,|^2$ fell off rapidly outside of $S'(t)$ in order for the parametric approximation to be valid. Proving this is difficult, but it is possible to provide some much weaker results which at least give some idea of the behavior of $P(\alpha_i)\,|K\,|^2$. For simplicity let us consider the case $P(\alpha_i)=\delta^2(\alpha_i)$. We are then interested in the properties of $K(\alpha_f,\beta_f,t;0,\beta_i,0)$. One can then show that for any integer $n\geq 1$ there exist constants $c_n(\beta_i)$ and $d_n(\beta_i)$ such that

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)\,|\leq\frac{c_n(\beta_i)}{|\alpha_f\,|^n}\ , \tag{4.16a}$$

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)\,|\leq\frac{d_n(\beta_i)}{|\beta_f\,|^n} \tag{4.16b}$$

so that $|K\,|$ falls off faster than any power of $|\beta_f\,|$ or $|\alpha_f\,|$. This is demonstrated in Appendix D. Because

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)\,|\leq 1\ , \tag{4.17}$$

inequalities (4.16) only really start providing useful information when $|\alpha_f\,|$ and $|\beta_f\,|$ are sufficiently large to make the right-hand sides less than 1. In general this will happen when α_f and β_f are far outside of $S'(t)$. Therefore, while inequalities (4.16) do tell us that $|K\,|$ falls off rapidly they do not really provide us with as much information as we would like. Therefore, the assumption that if $P(\alpha_i)\,|K\,|^2$ is well concentrated in $S'(t)$ [$\mu(t)$ close to 1], then the contribution to the integral in expression (4.8) from outside $S'(t)$ is small, must remain an assumption. The behavior of $|K\,|$ indicated by inequality (4.16) indicates, however, that it is a plausible one.

Finally, let us give some general conditions under which $\mu(t)$ is close to 1. We will consider the case β_i real and positive and the signal mode initially in the vacuum state, i.e., $P(\alpha_i)=\delta^{(2)}(\alpha_i)$. An examination of the expression for $f(\alpha_f,\beta_f,t;0,\beta_i)$ for the case $\kappa_2\in R$ (see Appendix E) shows that $|f(\alpha_f,\beta_f,t;0,\beta_i)\,|\ll 1$ if

$$1/\beta_i\ll 1,\quad |\alpha_f\,|^2/\beta_i\ll 1\ ,$$
$$\kappa t\ll 1,\quad (\kappa t/\beta_i)\,|\alpha_f\,|^4\ll 1\ . \tag{4.18}$$

These conditions determine $S(t)$. If we now impose the requirement that Eqs. (4.4) must also be satisfied we find that a point $(\alpha_i,\alpha_f,\beta_f)$ is in $S'(t)$ if $\beta_f\in R$, $1/|\beta_i\,|\ll 1$, and

$$\kappa t\ll 1,\quad |\alpha_f\,|^2/|\beta_i\,|\ll 1,\quad |\alpha_f\,|^2\kappa t\ll 1\ . \tag{4.19}$$

We now use these results in Eq. (4.13) to obtain

$$\mu(t)\simeq\frac{1}{\pi}\int_L d^2\alpha_f\,|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)\,|^2\ , \tag{4.20}$$

where

$$L=\{\alpha_f\mid\ |\alpha_f\,|^2\ll|\beta_i\,|\ \text{and}\ |\alpha_f\,|^2\kappa t\ll 1\}\ ,$$

and we have assumed that $\kappa t\ll 1$ and $1/|\beta_i\,|\ll 1$.

It is possible to derive a more convenient condition than Eq. (4.20) if we note that

$$|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;\alpha_i,\beta_i,0)\,|^2$$
$$=\exp[-|x_f+y_f\tanh(\eta_0 t)-x_i\operatorname{sech}(\eta_0 t)\,|^2 - |y_f\operatorname{sech}(\eta_0 t)+x_i\tanh(\eta_0 t)-y_i\,|^2]\ , \tag{4.21}$$

where $\alpha_i=x_i+iy_i$ and $\alpha_f=e^{-i\omega t}(x_f+iy_f)$. From this expression we see that $|G^{(0)}|$ is peaked at

$$\alpha_f=e^{-i\omega t}[\alpha_i\cosh(\eta_0 t)-i\alpha_i^*\sinh(\eta_0 t)] \tag{4.22}$$

and that this peak has a width given roughly by $\cosh(\eta_0 t)$. If $\alpha_i=0$ this peak will lie within the disc-shaped region in the α_f plane given by

$$D=\{\alpha_f \mid \ |\alpha_f| \lesssim \cosh(\eta_0 t) \sim e^{\eta_0 t}\} . \tag{4.23}$$

If $D \subseteq L$ then $\mu(t)$ will be approximately 1. This will be the case when

$$e^{2\eta_0 t} \ll |\beta_i| , \quad e^{2\eta_0 t}\kappa t \ll 1 . \tag{4.24}$$

Let us summarize our conclusions. In order for the parametric approximation to give accurate values for correlation functions it must be the case that $\mu(t)$ be close to one. In the case in which the signal mode is initially in the vacuum state this condition will be satisfied if

$$1/|\beta_i| \ll 1 , \tag{4.25a}$$

$$\kappa t \ll 1 , \tag{4.25b}$$

$$\kappa t e^{4\beta_i \kappa t} \ll 1 , \tag{4.25c}$$

$$e^{4\beta_i \kappa t} \ll \beta_i . \tag{4.25d}$$

There is a certain amount of redundancy in these conditions. For example, if Eqs. (4.25a) and (4.25d) are satisfied then Eq. (4.25b) follows as a consequence. We also note that if Eqs. (4.25a) and (4.25d) are satisfied and the condition that $\kappa t \beta_i$ be of order one or less is also satisfied then Eqs. (4.25b) and (4.25c) follow as consequences. This is in contrast to ordinary perturbation theory which is valid only for times such that $\kappa t \beta_i \ll 1$ so that the parametric approximation represents a definite improvement over the perturbative result.

V. CONCLUDING REMARKS

We have presented a fully quantum-mechanical theory of the degenerate parametric amplifier using a path-integral representation of the coherent-state propagator. We have developed a perturbation series for this propagator, the first term of which, under certain conditions, corresponds to the parametric approximation. We studied the effect of the quantum fluctuations of the pump mode on the squeezing of the signal mode and showed that these fluctuations not only reduce the squeezing but also that the minimum uncertainty relation does not hold. Finally we examined the conditions under which the parametric approximation will be valid.

ACKNOWLEDGMENTS

We would like to thank C. Chandler for useful conversations. This research is supported in part by the U.S. Navy Office of Naval Research.

APPENDIX A

In order to calculate $K^{(1)}$ we must first evaluate the path integral appearing in Eq. (2.14):

$$\int \mathscr{D}[\alpha(\tau)] e^{iS_0}(iS_1) = -\kappa^2 \int_0^t dt_2 \int_0^{t_2} dt_1 e^{-2i\omega(t_2-t_1)} F(t_1,t_2) , \tag{A1}$$

where

$$F(t_1,t_2) = \int \mathscr{D}[\alpha(\tau)] e^{iS_0} [\alpha^*(t_2)\alpha(t_1)]^2 . \tag{A2}$$

The path integral in the above equation can be evaluated by making use of the following rule: If $t_2 > t' > t_1$ and $f(\alpha(t'))$ is a function of the path $\alpha(\tau)$ at the time t', then[9]

$$\int \mathscr{D}[\alpha(\tau)] e^{iS_0} f(\alpha(t')) = \frac{1}{\pi} \int d^2\alpha' G^{(0)}(\alpha_f,\beta_f,t_2;\alpha',\beta_i,t') \times f(\alpha') G^{(0)}(\alpha',\beta_f,t';\alpha_i,\beta_i,t_1) , \tag{A3}$$

where $G^{(0)}$ is the propagator corresponding to S_0 and is given by Eq. (2.19). Application of this rule twice gives us that

$$F(t_1,t_2) = \frac{1}{\pi^2} \int d^2\alpha_1 \int d^2\alpha_2 G^{(0)}(\alpha_f,\beta_f,t;\alpha_2,\beta_i,t_2) G^{(0)}(\alpha_2,\beta_f,t_2;\alpha_1,\beta_i,t_1) G^{(0)}(\alpha_1,\beta_f,t_1;\alpha_i,\beta_i,0)(\alpha_2^*)^2 \alpha_1^2 . \tag{A4}$$

It should be noted, though we have not explicitly indicated it, that $F(t_1,t_2)$ depends upon α_f, β_f, α_i, β_i, and t as well as on t_1 and t_2. Evaluation of the integrals in Eq. (A4) is lengthy but straightforward. Upon performing them we find that

$$\begin{aligned} F(t_1,t_2) = {} & G^{(0)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0) \\ & \times \left\{ \frac{1}{D_1^2 D_2^2}(2B_{10}C_{21})^2 \left[\left[\frac{1}{B_{20}}(B_{30}-B_{32})(\alpha_f^*)^2 + 4C_{30}A_{32}\alpha_f^*\alpha_i + 4A_{32}(A_{30}-A_{20})\alpha_i^2 \right]^2 \right.\right. \\ & \left. + 12A_{32} \left[\frac{1}{B_{20}}(B_{30}-B_{32})(\alpha_f^*)^2 + 4C_{30}A_{32}\alpha_f^*\alpha_i + 4A_{32}(A_{30}-A_{20})\alpha_i^2 \right] + 12A_{32}^2 \right] \\ & + \frac{4}{D_1 D_2^2} B_{10}C_{20}\alpha_i (C_{32}\alpha_f^* + 2A_{32}C_{20}\alpha_i) \\ & \times \left[\frac{1}{B_{20}}(B_{30}-B_{32})(\alpha_f^*)^2 + 4C_{30}A_{32}\alpha_f^*\alpha_i + 4A_{32}(A_{30}-A_{20})\alpha_i^2 + 6A_{32} \right] \\ & + \frac{1}{D_2} \left[\frac{1}{D_1^2} C_{10}^2\alpha_i^2 + \frac{2}{D_1} B_{10} \right] \\ & \left. \times \left[\frac{1}{B_{20}}(B_{30}-B_{32})(\alpha_f^*)^2 + 4C_{30}A_{32}\alpha_f^*\alpha_i + 4A_{32}(A_{30}-A_{20})\alpha_i^2 + 2A_{32} \right] \right\} , \end{aligned} \tag{A5}$$

where

$$D_1=1-4A_{21}B_{10}\ , \tag{A6}$$

$$D_2=1-4A_{32}B_{20}\ , \tag{A7}$$

and $t_3=t$ and $t_0=0$. We will be interested in the case in which the signal mode is initially in the vacuum state. This means that we need only consider the propagator of the system when $\alpha_i=0$ which results in a considerable simplification of Eq. (A5). We also note that the same exponential factor which appears in $K^{(0)}$, i.e., $\exp[-\frac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]$, also appears in $K^{(1)}$. Therefore, we can approximate $K^{(1)}$ by

$$K^{(1)}(\alpha_f,\beta_f,t;\alpha_i,\beta_i,0)\cong -\kappa^2\exp[-\tfrac{1}{2}(|\beta_f|^2+|\beta_i|^2)+\beta_f^*\beta_i e^{-2i\omega t}]$$
$$\times\int_0^t dt_2\int_0^{t_2}dt_1 e^{-2i\omega(t_2-t_1)}F(t_1,t_2)|_{\beta_f=e^{-2i\omega t}\beta_i}\ , \tag{A8}$$

where we have assumed that $F(t_1,t_2)$ is a slowly varying function of β_f. If we now calculate $F(t_1,t_2)$ under these two restrictions, i.e., $\alpha_i=0$ and $\beta_f=e^{-2i\omega t}\beta_i$, we find that

$$F(t_1,t_2)=G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)e^{2i\omega(t_2-t_1)}$$
$$\times(-\operatorname{sech}^2(\eta_0 t)\cosh^2(\eta_0 t_2)\sinh^2(\eta_0 t_1)(e^{-i\omega t}\alpha_f^*)^4$$
$$+i\operatorname{sech}^2(\eta_0 t)\cosh(\eta_0 t_2)\sinh(\eta_0 t_1)$$
$$\times\{6\operatorname{sech}(\eta_0 t)\sinh(\eta_0 t_1)\sinh[\eta_0(t-t_2)]-\cosh[\eta_0(t_2-t_1)]\}(e^{-i\omega t}\alpha_f^*)^2$$
$$+\operatorname{sech}(\eta_0 t)\sinh(\eta_0 t_1)\sinh[\eta_0(t-t_2)]\{3\operatorname{sech}(\eta_0 t)+\sinh(\eta_0 t_1)\sinh[\eta_0(t-t_2)]-\cosh[\eta_0(t_2-t_1)]\})\ , \tag{A9}$$

where $\eta_0=2\kappa\beta_i$ (again β_i is assumed real). Before proceeding we note the identities

$$D_1=\operatorname{sech}[\eta_0(t_2-t_1)]\cosh(\eta_0 t_2)\operatorname{sech}(\eta_0 t_1)\ , \tag{A10a}$$

$$D_2=\operatorname{sech}[\eta_0(t-t_2)]\cosh(\eta_0 t)\operatorname{sech}(\eta_0 t_2) \tag{A10b}$$

which were of use in deriving Eq. (A9) from Eq. (A5).

In order to complete our calculation of $K^{(1)}$ we must perform the time integrations appearing in Eq. (A8). On doing so we obtain Eqs. (2.22) and (2.23) of the text.

APPENDIX B

We first consider the improvement of the freely-propagating-pump approximation. The contribution of $K^{(0)}$ to $\langle a^\dagger(t)a(t)\rangle$ is

$$\langle a^\dagger(t)a(t)\rangle\cong\frac{1}{\pi^2}\int d^2\beta_f\int d^2\alpha_f\exp(-|\beta_f-e^{-2i\omega t}\beta_i|^2)|G^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)|^2|\alpha_f|^2-1\ . \tag{B1}$$

We evaluate the α_f integral first; this can be done exactly. We then expand the result in terms $\delta\beta_f=\beta_f-e^{-2i\omega t}\beta_i$, i.e.,

$$\frac{1}{\pi}\int d^2\alpha_f|G^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)|^2|\alpha_f|^2=c_1^{(0)}+c_1^{(1)}\delta\beta_f+c_1^{(1)*}\delta\beta_f^*+c_1^{(2)}\delta\beta_f^2+c_2^{(2)}|\delta\beta_f|^2+c_1^{(2)*}(\delta\beta_f^*)^2\ , \tag{B2}$$

where $c_i^{(j)}$ is a function of β_i and t. We recall that if inequalities (4.4) are satisfied, then $|G^{(0)}|$ is a slowly varying function of β_i. Therefore, we expect $c_1^{(1)}$ and $c_{1,2}^{(2)}$ to be small if these inequalities are satisfied. We can now evaluate the integral. The terms linear in $\delta\beta_f$ give no contribution, and the terms proportional to $c_1^{(2)}$ integrate to zero as well. The $c_1^{(0)}$ term is just given by Eq. (3.6) while the term proportional to $c_2^{(2)}$ represents a correction to this. It is this term which is the lowest-order correction to the freely-propagating-pump approximation. We find that

$$\frac{1}{\pi}\int d^2\beta_f c_2^{(2)}(\beta_i,t)e^{-|\delta\beta_f|^2}|\delta\beta_f|^2=\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[\tfrac{5}{2}\sinh^2(\eta_0 t)+\tfrac{3}{2}]-(\eta_0 t)[6\sinh^3(\eta_0 t)\cosh(\eta_0 t)+3\cosh(\eta_0 t)\sinh(\eta_0 t)]$$
$$+\tfrac{15}{4}\sinh^6(\eta_0 t)+\tfrac{21}{4}\sinh^4(\eta_0 t)+\tfrac{3}{2}\sinh^2(\eta_0 t)\}\ . \tag{B3}$$

In the case of $\langle[a(t)]^2\rangle$ the calculation is carried out in the same way. Now one has

$$\frac{1}{\pi}\int d^2\alpha_f|G^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)|^2\alpha_f^2=d_1^{(0)}+d_1^{(1)}\delta\beta_f+d_1^{(1)*}\delta\beta_f^*+d_1^{(2)}(\delta\beta_f)^2+d_2^{(2)}|\delta\beta_f|^2+d_1^{(2)*}(\delta\beta_f^*)^2\ . \tag{B4}$$

Upon performing the β_f integration we find that only the terms proportional to $d_1^{(0)}$ and $d_2^{(2)}$ contribute. The $d_1^{(0)}$ term yields the result in Eq. (3.7) while the $d_2^{(2)}$ term yields

$$\frac{1}{\pi}\int d^2\beta_f d_2^{(2)}(\beta_i,t)e^{-|\delta\beta_f|^2}|\delta\beta_f|^2$$
$$=-ie^{-2i\omega t}\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[\tfrac{5}{2}\sinh^2(\eta_0 t)\tanh(\eta_0 t)+\tfrac{5}{2}\tanh(\eta_0 t)]$$
$$-\eta_0 t[6\sinh^4(\eta_0 t)+4\sinh^2(\eta_0 t)+\tfrac{3}{2}\sinh^2(\eta_0 t)\tanh^2(\eta_0 t)+\tfrac{3}{2}\tanh^2(\eta_0 t)]$$
$$+[\tfrac{15}{4}\sinh^6(\eta_0 t)+\tfrac{27}{4}\sinh^4(\eta_0 t)+3\sinh^2(\eta_0 t)]\tanh(\eta_0 t)\}\ . \tag{B5}$$

We now want to briefly discuss some of the assumptions underlying this approximation. We are assuming that inequalities (4.24) are satisfied so that the range of integration in Eq. (4.6) should be restricted to $S'(t)$. Now, because these inequalities are satisfied the freely-propagating-pump approximation is valid so that only the integration region in which $|\delta\beta_f|\sim 1$ and $\alpha_f\in L$ is important. If inequalities (4.24) are satisfied, then for $|\delta\beta_f|\sim 1$, $|G^{(0)}|^2|\alpha_f|^2$ decreases rapidly outside of L and its integral over the entire complex plane converges. Therefore, we can, with little error, extend the α_f integration to the entire complex plane. We expand the resulting expression about $\delta\beta_f=0$ in order to take into account the variation of $\int d^2\alpha_f|G^{(0)}|^2|\alpha_f|^2$ with $\delta\beta_f$ in the neighborhood $|\delta\beta_f|\sim 1$. This expansion, when multiplied by $\exp(-|\delta\beta_f|^2)$, decreases rapidly away from the region in which $|\delta\beta_f|\sim 1$. We can, therefore, again extend the integration to the entire complex plane. The results of this procedure are exhibited in the preceding paragraph.

Next we consider the effects of $K^{(1)}$ on $\langle a^\dagger(t)a(t)\rangle$ and $\langle[a(t)]^2\rangle$. This is done by evaluating the following integrals:

$$\langle a^\dagger(t)a(t)\rangle|_{K^{(1)}}=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f K^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)[K^{(1)}(\alpha_f,\beta_f,t;0,\beta_i,0)]^*|\alpha_f|^2+\text{c.c.}$$
$$=\int d^2\alpha_f|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2[g_4(t)(e^{-i\omega t}\alpha_f^*)^4+g_2(t)(e^{-i\omega t}\alpha_f^*)^2+g_0(t)]|\alpha_f|^2+\text{c.c.}\ , \tag{B6}$$

$$\langle[a(t)]^2\rangle|_{K^{(1)}}=\frac{1}{\pi^2}\int d^2\alpha_f\int d^2\beta_f K^{(0)}(\alpha_f,\beta_f,t;0,\beta_i,0)[K^{(1)}(\alpha_f,\beta_f,t;0,\beta_i,0)]^*\alpha_f^2+\text{c.c.}$$
$$=\int d^2\alpha_f|G^{(0)}(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i,0)|^2[g_4(t)(e^{-i\omega t}\alpha_f^*)^4+g_2(t)(e^{-i\omega t}\alpha_f^*)^2+g_0(t)]\alpha_f^2+\text{c.c.} \tag{B7}$$

In writing Eqs. (B6) and (B7), we have substituted for $K^{(0)}$ and $K^{(1)}$ from Eqs. (2.18) and (2.22), respectively. Furthermore we have made the substitution $\beta_f=\exp(-2i\omega t)\beta_i$ in the integrands following our earlier discussion. The integrals in Eqs. (B6) and (B7) are rather lengthy but straightforward. On carrying them out, we obtain

$$\langle a^\dagger(t)a(t)\rangle|_{K^{(1)}}=\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[-\tfrac{1}{2}\cosh^2(\eta_0 t)]+\eta_0 t[5\sinh(\eta_0 t)\cosh(\eta_0 t)+6\sinh^3(\eta_0 t)\cosh(\eta_0 t)]$$
$$-\tfrac{15}{4}\sinh^6(\eta_0 t)-\tfrac{33}{4}\sinh^4(\eta_0 t)-\tfrac{9}{2}\sinh^2(\eta_0 t)\}\ , \tag{B8}$$

$$\langle[a(t)]^2\rangle|_{K^{(1)}}=-ie^{-2i\omega t}\frac{\kappa^2}{\eta_0^2}\{(\eta_0 t)^2[2\sinh(\eta_0 t)\cosh(\eta_0 t)-\tfrac{5}{2}\sinh^2(\eta_0 t)\tanh(\eta_0 t)-\tfrac{5}{2}\tanh(\eta_0 t)]$$
$$+\eta_0 t[6\sinh^4(\eta_0 t)+6\sinh^2(\eta_0 t)+\tfrac{3}{2}\sinh^2(\eta_0 t)\tanh^2(\eta_0 t)+\tfrac{3}{2}\tanh^2(\eta_0 t)+2]$$
$$-[\tfrac{15}{4}\sinh^6(\eta_0 t)+\tfrac{39}{4}\sinh^4(\eta_0 t)+8\sinh^2(\eta_0 t)+2]\tanh(\eta_0 t)\}\ . \tag{B9}$$

We now add the contributions to $\langle a^\dagger(t)a(t)\rangle$ and $\langle[a(t)]^2\rangle$ due to the freely-propagating-pump approximation, and the corrections to it (Eqs. (3.6), (B2), (B3), and (B8) for $\langle a^\dagger(t)a(t)\rangle$ and Eqs. (3.7), (B4), (B5), and (B9) for $\langle[a(t)]^2\rangle$). We then obtain Eqs. (3.8) and (3.9) of the text.

APPENDIX C

In this appendix we would like to examine the convergence of the integrals which occur in the perturbation series, i.e., integrals of the type which occur in Eq. (A4). All of these integrals are of the form

$$I_0=\int dx_2 \int dy_2 \exp[a_2x_2^2+b_2y_2^2+a_1x_2+b_1y_2+c_1x_2y_2]x_2^n y_2^m \ , \quad \text{(C1)}$$

where

$$\begin{aligned} a_2&=-1+B_{21}+A_{32} \ ,\\ b_2&=-1-B_{21}-A_{32} \ ,\\ c_1&=2i(A_{32}-B_{21}) \ , \end{aligned} \quad \text{(C2)}$$

and A_{ij}, B_{ij}, and C_{ij} are defined by Eqs. (2.20). The coefficients a_1 and b_1 can also be expressed in terms of A_{ij}, B_{ij}, and C_{ij} but are not relevant to convergence considerations. The integral I_0 will converge if

$$\mathrm{Re}\left[(x_2 y_2)\begin{bmatrix} a_2 & \frac{1}{2}c_1 \\ \frac{1}{2}c_1 & b_2\end{bmatrix}\begin{bmatrix} x_2\\ y_2\end{bmatrix}\right]<0 \quad \text{(C3)}$$

for all values of x_2 and y_2. This will be the case if both of the eigenvalues of the real, symmetric matrix $\underline{A}$ given by

$$\underline{A}=\mathrm{Re}\begin{bmatrix} a_2 & \frac{1}{2}c_1 \\ \frac{1}{2}c_1 & b_2\end{bmatrix}$$

are negative. This is equivalent to the two conditions

$$\mathrm{Tr}\underline{A}<0, \quad \det\underline{A}>0 \ . \quad \text{(C4)}$$

Substituting from Eq. (C2) we obtain

$$\mathrm{Tr}\underline{A}=-2, \quad \det\underline{A}=1-|B_{21}+A_{32}^*|^2 \ . \quad \text{(C5)}$$

As can be seen the trace condition is satisfied automatically so that we are left with the condition

$$1>|B_{21}+A_{32}^*|^2 \ . \quad \text{(C6)}$$

Let us define

$$\xi_1=\tanh(2\sqrt{\kappa_1\kappa_2}\tau_1), \quad \xi_2=\tanh(2\sqrt{\kappa_1\kappa_2}\tau_2) \ . \quad \text{(C7)}$$

We then have that inequality (C6) will be satisfied for all values of κ_1 and κ_2 such that the inequality

$$1>\tfrac{1}{4}|\kappa_1/\kappa_2|\,|\xi_1|^2+\tfrac{1}{4}|\kappa_2/\kappa_1|\,|\xi_2|^2-\tfrac{1}{2}\mathrm{Re}(\xi_1\xi_2) \quad \text{(C8)}$$

is satisfied for all values of τ_1 and τ_2 greater than zero. That is, if inequality (C8) is satisfied for some specific values of κ_1 and κ_2, and all values of $\tau_1>0$ and $\tau_2>0$, then these values of κ_1 and κ_2 will be such that inequality (C6) is also satisfied. Therefore we want to examine inequality (C8).

Before doing so, however, we need to place a bound on $|\tanh z|$ for z on the line

$$L=\{z \mid z=re^{i\theta},\ \theta \text{ fixed and } |\theta|<\pi/2,\ r\geq 0\} \ .$$

We have that

$$|\tanh z|=\left[\frac{\cosh(2x)-\cos(2y)}{\cosh(2x)+\cos(2y)}\right]^{1/2} \ , \quad \text{(C9)}$$

where $z=x+iy$. This expression achieves a maximum on L when $\pi/4<y<\pi/2$, i.e., $\pi/4u<x<\pi/2u$, where $u=\tan\theta$. Therefore, on L we have

$$|\tanh z|\leq\left[\frac{\cosh(\pi/2u)+1}{\cosh(\pi/2u)-1}\right]^{1/2}\equiv m(\theta) \ . \quad \text{(C10)}$$

Let us now assume that κ_1 is real and positive, $\arg\kappa_2=\theta_0$, and that $|\theta_0|<\pi$. Then we have that $\arg\sqrt{\kappa_1\kappa_2}=\frac{1}{2}\theta_0$. This then implies that

$$\begin{aligned} &0\leq|\xi_1|\leq m(\tfrac{1}{2}\theta_0), \quad \mathrm{Re}\xi_1\geq 0\\ &0\leq|\xi_2|\leq m(\tfrac{1}{2}\theta_0), \quad \mathrm{Re}\xi_2\geq 0 \ . \end{aligned} \quad \text{(C11)}$$

If we let $x=|\kappa_2/\kappa_1|$ we see that

$$\frac{1}{4}\left[x+\frac{1}{x}\right]m(\tfrac{1}{2}\theta_0)^2>\tfrac{1}{4}x|\xi_1|^2+\frac{1}{4}\left[\frac{1}{x}\right]|\xi_2|^2-\tfrac{1}{2}\mathrm{Re}(\xi_1\xi_2) \quad \text{(C12)}$$

so that inequality (C8) is satisfied if

$$x+\frac{1}{x}<\frac{4}{m(\frac{1}{2}\theta_0)^2}\equiv\sigma(\theta_0) \ . \quad \text{(C13)}$$

The function $(1/x)+x$ has a minimum of 2 for $x>0$ so that we must have $\sigma(\theta_0)\geq 2$. The angle for which $\sigma(\theta_0)=2$ is $\theta_{\max}=0.46\pi$. For all angles $|\theta_0|\leq\theta_{\max}$ we have $\sigma(\theta_0)\geq 2$. Inequality (C13) is satisfied, then, if $|\theta_0|\leq\theta_{\max}$ and

$$\tfrac{1}{2}[\sigma-2-(\sigma^2-4)^{1/2}]\leq x-1\leq\tfrac{1}{2}[\sigma-2+(\sigma^2-4)^{1/2}] \quad \text{(C14)}$$

which is the condition given in the text.

APPENDIX D

Here we would like to show that the propagator falls off more rapidly than any power of $|\alpha_f|$ or $|\beta_f|$. In order to do this we first note that the operator

$$M=2b^\dagger b+a^\dagger a \quad \text{(D1)}$$

commutes with the Hamiltonian. Therefore, the Hilbert space for the problem splits into the direct sum of the Hilbert spaces $\mathscr{H}_m$ on which M has the eigenvalue m. If $\psi\in\mathscr{H}_m$, then $\exp(-itH)\psi\in\mathscr{H}_m$. Let P_m be the projection operator onto $\mathscr{H}_m$. We then have that

$$e^{-itH}|\alpha_i,\beta_i\rangle=\sum_{m=0}^{\infty}\psi_m(t) \ , \quad \text{(D2)}$$

where

$$\psi_m(t)=P_m e^{-itH}|\alpha_i,\beta_i\rangle \ . \quad \text{(D3)}$$

Because $[M,H]=0$ the norm of $\psi_m(t)$ is independent of time.

The power bounds are obtained from the inequality

$$\begin{aligned}|\alpha_f^r\beta_f^s\langle\alpha_f,\beta_f\,|\,e^{-itH}\,|\,\alpha_i,\beta_i\rangle\,| &= |\,\langle\alpha_f,\beta_f\,|\,(a^\dagger)^r(b^\dagger)^s e^{-itH}\,|\,\alpha_i\beta_i\rangle\,| \\ &\le ||(a^\dagger)^r(b^\dagger)^s e^{-iHt}\,|\,\alpha_i,\beta_i\rangle||\ . \end{aligned} \tag{D4}$$

This is really all that is necessary to obtain bounds of the form given in Eq. (4.16) as the right-hand side of inequality (D4) is independent of α_f and β_f. It is useful, however, to examine this expression a little more closely. We see that

$$(a^\dagger)^r(b^\dagger)^s\psi_m(t)\in\mathscr{H}_{m+r+2s} \tag{D5}$$

so that

$$\langle(a^\dagger)^r(b^\dagger)^s\psi_{m'}(t)\,|\,(a^\dagger)^r(b^\dagger)^s\psi_{m''}(t)\rangle=0 \tag{D6}$$

if $m''\neq m'$. Therefore

$$||(a^\dagger)^r(b^\dagger)^s e^{-iHt}\,|\,\alpha_i,\beta_i\rangle|| = \left[\sum_{m=0}^{\infty}||(a^\dagger)^r(b^\dagger)^s\psi_m(t)||^2\right]^{1/2} . \tag{D7}$$

Let us now consider the case $\alpha_i=0$ and $s=0$. We then have that

$$||\psi_m(0)||=e^{-|\beta_i|^2/2}\frac{|\beta_i|^{m/2}}{\sqrt{(m/2)!}}\ . \tag{D8}$$

Because there can be at most m signal-mode photons in $\mathscr{H}_m$ we have

$$||(a^\dagger)^r\psi_m(t)||\le\left[\frac{(r+m)!}{m!}\right]^{1/2}||\psi_m(0)||\ . \tag{D9}$$

This then, provides the bound

$$\begin{aligned}||(a^\dagger)^r e^{-itH}\,|\,0,\beta_i\rangle||&\le e^{-|\beta_i|^2/2}\left[\sum_{l=0}^{\infty}\frac{|\beta_i|^{2l}}{l!}\frac{(2l+r)!}{(2l)!}\right]^{1/2}\\ &\le e^{-|\beta_i|^2/2}\left[\frac{d^r}{d\,|\beta_i|^r}(\,|\beta_i|^r e^{|\beta_i|^2})\right]^{1/2}\\ &\le\left[\sum_{l=0}^{r}\binom{r}{l}\frac{r!}{(r-l)!}|\beta_i|^{r-l}(-i)^{r-l}H_{r-l}(i\,|\beta_i|)\right]^{1/2} ,\end{aligned} \tag{D10}$$

where $H_n(x)$ is the nth Hermite polynomial. Combining inequalities (D4) and (D9) gives

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)|\le\frac{1}{|\alpha_f|^r}\left[\sum_{l=0}^{r}\binom{r}{l}\frac{r!}{(r-l)!}|\beta_i|^{r-l}(-i)^{r-l}H_{r-l}(i\,|\beta_i|)\right]^{1/2} . \tag{D11}$$

A similar derivation for the case $\alpha_i=0$ and $r=0$ gives

$$|K(\alpha_f,\beta_f,t;0,\beta_i,0)|\le\frac{1}{|\beta_f|^s}[s!L_s(-|\beta_i|^2)]^{1/2} , \tag{D12}$$

where L_s is the sth Laguerre polynomial. We note that for large $|\beta_i|$

$$\left[\sum_{l=0}^{r}\binom{r}{l}\frac{r!}{(r-l)!}|\beta_i|^{r-l}(-i)^{r-l}\times H_{r-l}(i\,|\beta_i|)\right]^{1/2}\sim(\sqrt{2}\,|\beta_i|)^r , \tag{D13}$$

$$[s!L_s(-|\beta_i|^2)]^{1/2}\sim|\beta_i|^s \tag{D14}$$

so that the bounds (D11) and (D12) start being useful (i.e., the right-hand sides become less than 1) for $|\alpha_f|\sim|\beta_i|$ and $|\beta_f|\sim|\beta_i|$.

APPENDIX E

In this appendix we want to find the conditions on α_f, β_i, and t so that $|f(\alpha_f,\beta_f,t;0,\beta_i)|\ll 1$ for $\kappa_2=\kappa\beta_f^*e^{-2i\omega t}\in R$. In Eqs. (2.22) and (2.23) we have given an expression for $K^{(1)}$ from which $f(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i)$ can be immediately derived, i.e.,

$$\begin{aligned}&f(\alpha_f,e^{-2i\omega t}\beta_i,t;0,\beta_i)\\ &\quad=g_4(t)(e^{-i\omega t}\alpha_f^*)^4+g_2(t)(e^{-2i\omega t}\alpha_f^*)^2+g_0(t) ,\end{aligned} \tag{E1}$$

where $g_4(t)$, $g_2(t)$, and $g_0(t)$ are given by Eq. (2.23). By going back through the derivation of $K^{(1)}$ we can find an expression for $f(\alpha_f,\beta_f,t;0,\beta_i)$, i.e., for the case in which β_f is not equal to $\exp(-2i\omega t)\beta_i$. The result is

$$\begin{aligned}f(\alpha_f,\beta_f,t;0,\beta_i)=&\left[\frac{\kappa_1}{\kappa_2}\right]\tilde{g}_4(t)(e^{-i\omega t}\alpha_f^*)^4\\ &+\left[\frac{\kappa_1}{\kappa_2}\right]^{1/2}\tilde{g}_2(t)(e^{-i\omega t}\alpha_f^*)^2+\tilde{g}_0(t) ,\end{aligned} \tag{E2}$$

where $\tilde{g}_j(t)$ is just $g_j(t)$ with η_0 replaced by $2\sqrt{\kappa_1\kappa_2}$.

We now note that for $\kappa_2\in R$, κ_1/κ_2 is of order 1 and

$$\begin{aligned}
&|\tanh(2\sqrt{\kappa_1\kappa_2}t)| \le \sqrt{2}\ ,\\
&|\text{sech}(2\sqrt{\kappa_1\kappa_2})| \le 1\ ,\\
&|(2\sqrt{\kappa_1\kappa_2}t)^2\text{sech}^4(2\sqrt{\kappa_1\kappa_2}t)| \le 3\ ,\\
&|(2\sqrt{\kappa_1\kappa_2}t)^2\text{sech}^2(2\sqrt{\kappa_1\kappa_2}t)| \le 3\ .
\end{aligned} \tag{E3}$$

It can then be seen that

$$\begin{aligned}
&|f(\alpha_f,\beta_f,t;0,\beta_i)|\\
&\quad\le \left|\frac{\kappa_1}{\kappa_2}\right| |\tilde{g}_4(t)|\,|\alpha_f|^4\\
&\quad\quad+\left|\left[\frac{\kappa_1}{\kappa_2}\right]^{1/2}\right| |\tilde{g}_2(t)|\,|\alpha_f|^2+|\tilde{g}_0(t)|\ ,
\end{aligned} \tag{E4}$$

where

$$\begin{aligned}
&\left|\frac{\kappa_1}{\kappa_2}\right| |\tilde{g}_4(t)| \sim \frac{\kappa^2}{\eta_0^2}[\eta_0 t+O(1)]\ ,\\
&\left|\left[\frac{\kappa_1}{\kappa_2}\right]^{1/2}\right| |\tilde{g}_2(t)| \sim \frac{\kappa^2}{\eta_0^2}[\eta_0 t+O(1)]\ ,\\
&|\tilde{g}_0(t)| \sim \frac{\kappa^2}{\eta_0^2}[(\eta_0 t)^2+\eta_0 t+O(1)]\ ,
\end{aligned} \tag{E5}$$

where we have used Eqs. (E3) and the fact that for $\kappa_2 \in R$, $(\eta_0/2\sqrt{\kappa_1\kappa_2}) \sim 1$. Putting Eqs. (E4) and (E5) together we find that $|f|$ is small if

$$\begin{aligned}
&1/\beta_i \ll 1,\quad |\alpha_f|^2/\beta_i \ll 1\ ,\\
&\kappa t \ll 1,\quad \kappa t/\beta_i\,|\alpha_f|^4 \ll 1\ .
\end{aligned} \tag{E6}$$

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