Quantum noise of an injection-locked laser oscillator

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The quantum noise of an injection-locked laser oscillator is analyzed by the operator Langevin equation. The problem is also treated by the Fokker-Planck equation and the same results are obtained in the same regimes of applicability. The steady-state solution of the Fokker-Planck equation gives the probability distribution of amplitude and phase, the Langevin equation arrives more directly at the spectrum of amplitude and phase. The phase of the injection-locked oscillator is related to the phase of the injection signal and thus constitutes a measurement of phase. In the limit of complete inversion and zero internal loss of the laser resonator, the associated uncertainty is twice that dictated by the uncertainty principle. This result is interpreted by comparing it with the uncertainty introduced by a linear amplifier which can perform a simultaneous measurement of amplitude and phase.

INTRODUCTION

The quantum noise of a laser oscillator has received a great amount of study, both theoretical and experimental. The first treatment of the laser oscillator with operator noise sources is due to Haken.[1–4] Lax and co-workers, in a series of papers, developed and expanded the theory further.[5–9] A density matrix description of the laser oscillator was pioneered by Scully and Lamb[10] and expanded upon in a book by Sargent, Scully, and Lamb.[11] Mandel and Wolf contributed to the theoretical description.[12]

The experimental verification of the quantum noise emitted by lasers started with the study of frequency noise of an He-Ne laser by Javan *et al.*[13] was followed up by laser amplifier noise studies by Klüver.[14] The frequency noise initially observed was governed by the thermal vibrations of the laser cavity length. The amplitude noise near threshold, however, was found to be attributable to quantum noise.[15,16] Quantum noise could be detected in semiconductor diode lasers,[17] since quantum noise predominates over classical noise generating mechanisms in such lasers because of their small dimensions and fast relaxation times. More recently, quantum noise fluctuations were observed in He-Ne laser gyros,[18] the emergence of quantum noise having been made possible by cancellation of classical noise contributions in the measurement of difference frequencies.

Injection locking of lasers for communication purposes[19] has rekindled the interest in quantum noise limitations on this form of modulation. The classical theory of injection locking is discussed in Stratonovich's book.[20] Haken *et al.*[21] studied the quantum theory of locking of modes in a laser oscillator. Chow *et al.*[22] pointed out the narrowing of the laser spectrum due to injection locking. No complete quantum-mechanical treatment of the noise accompanying injection locking exists in the literature. The present paper presents such an analysis. In the limit when the oscillator runs at a very high power level with complete inversion, the mean-square phase fluctuations of the output waveform are found to be

$$(\langle \Delta\phi^2 \rangle_{\rm av})^{1/2} = \frac{1}{\sqrt{2\langle n_s \rangle_{\rm av}}} ,$$

where $\langle n_s \rangle_{\rm av}$ is the average photon number of the injection signal. If the phase of the output is viewed as a measurement of the phase of the locking signal, then this measurement results in a phase uncertainty twice that dictated by the uncertainty principle. This result is compared with a measurement of a coherent state after amplification by an ideal linear amplifier. Such an amplification makes possible the simultaneous measurement of amplitude and phase fluctuations, doubling the uncertainty in the process, as pointed out by Haus and Townes,[23] Arthurs and Kelly,[24] and Caves.[25]

We start in Sec. I with the operator Langevin equation and obtain the fluctuation spectra of amplitude and phase of the locked oscillator in Sec. II. In the limit of complete inversion we determine the minimum phase uncertainty. This result is compared with that of the linear amplifier in Sec. III. The only difference between the two systems is that the relaxation times of amplitude and phase, different in the case of the locked oscillator, become identical in the case of the linear amplifier. We ascertain the fact that the linear amplifier is capable of reaching the ideal limit of a simultaneous measurement.

In Sec. IV we set up the Fokker-Planck equation for the $P(\alpha)$ function of the laser oscillator, supplemented by an injection-locking term due to a c-number source. Section V treats the case of a coherent state coupled to the oscillator via an optical isolator and finds that the Fokker-Planck equation is identical with that of a c-number source, with the amplitude of the c-number source replaced by the eigenvalue of the coherent state. The steady-state distribution $P(\alpha)$ is approximately a two-dimensional Gaussian in amplitude and phase. The equations of motion for the expectation values of amplitude and phase are then related to the analysis of Sec. II. The mean-square amplitude and phase fluctuations are obtained in Sec. VI and compared with the results of the Langevin approach. The two approaches are shown to lead to the same result in the same ranges of applicability.

I. QUANTUM-MECHANICAL LANGEVIN EQUATION

In this section, and the subsequent two sections we consider the operator Langevin equation for the injection-locked oscillator (see Fig. 1) and the regenerative amplifier. We start with the Langevin equation for the laser in the absence of an injection signal as derived in Ref. 11. The pertinent equation is (49), p. 334:

$$\dot{a}(t)=-\frac{1}{2}\left[\frac{\omega_0}{Q}-\mathscr{A}+\mathscr{B}a^\dagger a\right]a(t)+G(t)\,, \tag{1.1}$$

where $a(t)$ is the slowly varying envelope of the annihilation operator $a(t)e^{-i\omega_0 t}$.

$\mathscr{A}$ is the linear gain parameter, ω_0/Q is the decay rate of the laser resonator in the absence of the gain medium, $\mathscr{B}$ is the saturation parameter expressing the dependence of the gain on the photon number $a^\dagger a$: the total gain parameter is $\mathscr{A}-\mathscr{B}a^\dagger a$. The function $G(t)$ is the operator noise source with the correlation function[4,11]

$$\langle G^\dagger(t)G(t')\rangle+\langle G(t)G^\dagger(t')\rangle =2\left[\frac{1}{2}\frac{\omega_0}{Q}+\frac{g^2}{\gamma}(N_2+N_1)\right]\delta(t-t')\,, \tag{1.2}$$

where the first term $\frac{1}{2}\omega_0/Q$ represents the zero-point fluctuation of the photon field and the second term $(g^2/\gamma)(N_2+N_1)$ represents the atomic dipole moment fluctuation. g is the atom-field interaction matrix element and γ is the phase decay constant of the dipole moment. The Marcovian assumption that the dipole moment and energy decay constants are much larger than the photon decay constant ω_0/Q is used to derive (1.1) and (1.2). The noise contribution of the level operator is neglected because it is of higher order in (ga).

The interaction Hamiltonian between a laser photon operator a and an injection signal operator b is assumed to be

$$V=i\hbar(\kappa b a^\dagger e^{-i(\omega-\omega_0)t}-\kappa a b^\dagger e^{i(\omega-\omega_0)t})\,, \tag{1.3}$$

where b is the annihilation operator of the injection signal, ω is the frequency of the injection signal, and ω_0 that of the oscillator. The interaction Hamiltonian is quadratic in the excitation amplitudes of the two systems, laser and injection signal. The coupling is thus linear. Here, again, b is an envelope operator, the natural time dependence $\exp(i\omega t)$ has been factored out. The decay rate ω_0/Q consists of two contributions, one from the unloaded quality factor Q_0, and the other from the external Q,Q_e:

$$\frac{\omega_0}{Q}=\frac{\omega_0}{Q_e}+\frac{\omega_0}{Q_0}\,. \tag{1.4}$$

The external Q expresses the coupling to the mode of the injection signal, the unloaded Q incorporates coupling to any other modes and to the loss. We shall find it convenient to use time constants τ_e and τ_0 defined by

$$\frac{\omega_0}{Q_e}=\frac{2}{\tau_e}\,, \tag{1.5}$$

$$\frac{\omega_0}{Q_0}=\frac{2}{\tau_0}\,. \tag{1.6}$$

The coupling constant κ has been assumed real with no loss of generality, since this choice disposes of an arbitrary phase reference.

The Heisenberg equation of motion:

$$\dot{a}=\frac{i}{\hbar}[\mathscr{H},a] \tag{1.7}$$

which led to (1.2) is supplemented by the coupling term (1.3) and results in

$$\dot{a}=\frac{1}{2}\left[\mathscr{A}-\frac{\omega_0}{Q}-\mathscr{B}a^\dagger a\right]a+\kappa b\,e^{-i(\omega-\omega_0)t}+G(t)\,. \tag{1.8}$$

This is the quantum-mechanical Langevin equation of injection locking of an oscillator. The linear laser amplifier, operating below its oscillation threshold, is also described by (1.8) if the gain saturation term $\mathscr{B}a^\dagger a$ is dropped in the above equation. The coupling constant κ can be related to the external Q and the bandwidth B of the injection signal as shown in Appendix A

$$\kappa=2(B/\tau_e)^{1/2}=\left[\frac{2B\omega_0}{Q}\right]^{1/2}\left[1+\frac{\tau_e}{\tau_0}\right]^{-1/2}. \tag{1.9}$$

Here B is the Nyquist bandwidth related to the signal sampling time T as follows:

$$B=1/2T\,.$$

II. AMPLITUDE AND PHASE NOISE OF INJECTION-LOCKED OSCILLATOR

The Langevin approach leads directly to the spectra of the amplitude and phase fluctuations, the Fourier transforms of the correlation functions. The Fokker-Planck approach used later on gives the probability densities of the fluctuating quantities. In this section we derive the amplitude and phase spectra of the injection-locked oscillator starting with (1.8).

The operator a generally follows the time dependence of the injection signal, if the locking is successful. Thus, it is convenient to write

$$a(t)=(a_0+\Delta a)e^{-i[(\omega-\omega_0)t+\phi_0+\Delta\phi]} \tag{2.1}$$

and

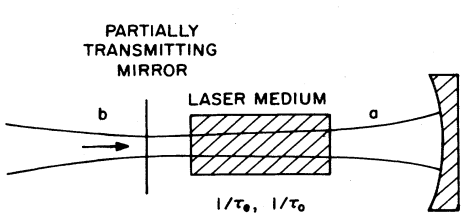

FIG. 1. Schematic of injection-locked laser oscillator.

$$b(t)=(b_0+\Delta b)e^{-i[(\omega-\omega_0)t+\Delta\psi]} \ . \quad (2.2)$$

Here a_0, b_0, and ϕ_0 are real c numbers. The Hermitian operators Δb and $\Delta\psi$ express the excess amplitude and phase noise of the injection signal, and Hermitian operators Δa and $\Delta\phi$ those of the oscillator excitation. The zero-point fluctuations attributable to the input port are already included in the Langevin noise operator $G(t)$. We shall assume that Δb and $\Delta\psi$ commute and are uncorrelated. $G(t)$ commutes with both and is uncorrelated with either because we assume that a and b are operators pertaining to two different subsystems.

The above quantum-mechanical quasilinearization is an extension of the one used for a laser oscillator by Haken[4] and Lax.[7] Use of (2.1) and (2.2) in the quantum-mechanical Langevin equation (1.4) leads, after separation into orders of the perturbation, to an equation for the c-number amplitude a_0 and phase ϕ_0

$$-i(\omega-\omega_0)a_0-\frac{1}{2}\left[\mathscr{A}-\frac{\omega_0}{Q}-\mathscr{B}a_0^2\right]a_0=\kappa b_0 e^{i\phi_0} \ . \quad (2.3)$$

The amplitude a_0 is related to b_0 by

$$a_0=\frac{\kappa}{\left[(\omega-\omega_0)^2+\frac{1}{4}\left[\mathscr{A}-\frac{\omega_0}{Q}-\mathscr{B}a_0^2\right]^2\right]^{1/2}}b_0 \ , \quad (2.4)$$

where the factor multiplying b_0 is the net gain, the enhancement of the injection signal, and

$$\tan\phi_0=\frac{\omega_0-\omega}{\frac{1}{2}\left[\mathscr{B}a_0^2-\mathscr{A}+\frac{\omega_0}{Q}\right]} \ . \quad (2.5)$$

In the absence of an injection signal, the gain is infinite, and from this fact one may evaluate the value of a_0 of the free-running oscillator

$$a_0^2=\frac{\mathscr{A}-\dfrac{\omega_0}{Q}}{\mathscr{B}} \ . \quad (2.6)$$

The injection signal increases a_0 so that

$$\mathscr{A}-\mathscr{B}a_0^2<\omega_0/Q$$

and the gain coefficient is less than the loss coefficient. The increase in amplitude with the injection signal, for small changes Δa_0 from a_0, is given by (2.4) and (2.6)

$$a_0(b_0\neq 0)-a_0(b_0=0)\cong\frac{\kappa\cos\phi_0}{\mathscr{B}a_0^2}b_0 \ . \quad (2.7)$$

At synchronism, $\omega=\omega_0$, the injection signal b_0 and the response $a_0e^{i\phi_0}$ are in phase, $\phi_0=0$. Increased detuning leads to a reduction of the net gain and an increase of the dephasing, provided that the detuning is within the locking bandwidth $\Delta\omega_L$. The locking bandwidth is obtained from the imaginary part of (2.3) and by noting that $|\sin\phi_0|\leq 1$. We then have

$$|\omega_0-\omega|\leq\frac{\kappa b_0}{a_0}\equiv\Delta\omega_L \ .$$

The oscillation frequency ω, the phase shift ϕ_0, and the increase Δa_0 of the amplitude are shown schematically in Fig. 2 as functions of the detuning $\omega_0-\omega_i$.

The equations for the amplitude and phase perturbations are

$$\Delta\dot a=-\frac{\Delta a}{\tau_a}-(\omega_0-\omega)a_0\Delta\phi+\kappa\cos\phi_0\Delta b+\kappa\sin\phi_0 b_0\Delta\psi$$
$$+\tfrac{1}{2}(G(t)\exp\{i[(\omega-\omega_0)t+\phi_0]\}$$
$$+G^\dagger(t)\exp\{-i[(\omega-\omega_0)t+\phi_0]\}) \ , \quad (2.8)$$

where we have used (2.4) and (2.5). Further,

$$\Delta\dot\phi=-\frac{\kappa b_0}{a_0}\cos\phi_0(\Delta\phi-\Delta\psi)+(\omega_0-\omega)\frac{\Delta a}{a_0}-\frac{\kappa}{a_0}\sin\phi_0\Delta b$$
$$+\frac{i}{2a_0}(G(t)\exp\{i[(\omega-\omega_0)t+\phi_0]\}$$
$$-G^\dagger(t)\exp\{-i[(\omega-\omega_0)t+\phi_0]\}) \ , \quad (2.9)$$

where

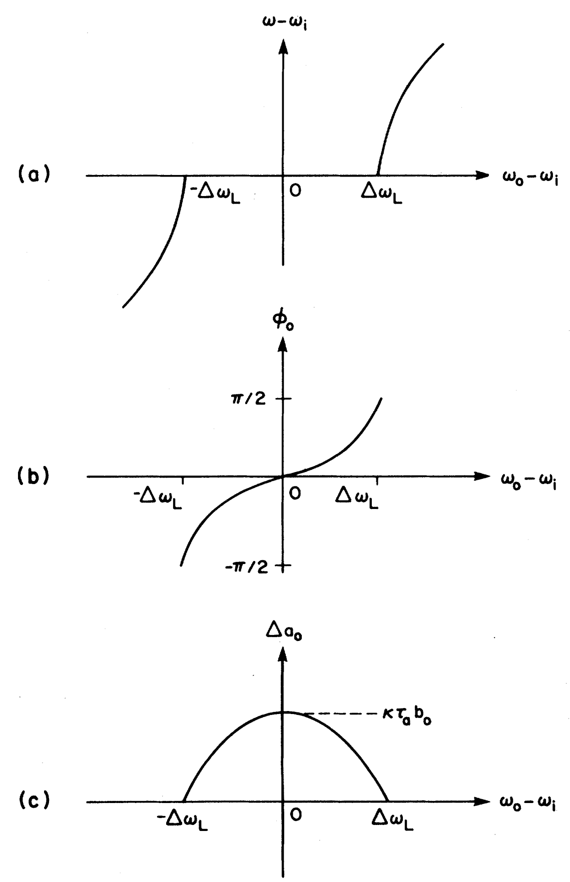

FIG. 2. Oscillation frequency ω, phase shift ϕ_0 and increase in amplitude Δa_0 vs frequency detuning $\omega_i-\omega_0$. ω_i is the input signal frequency, ω_0 is the oscillator frequency without input signal and $\Delta\omega_L$ is the locking bandwidth.

$$\frac{1}{\tau_a}=\mathscr{B}a_0^2 \tag{2.10}$$

is the decay rate of the amplitude perturbation, i.e.,

$$-\frac{1}{2}\left[\mathscr{A}-\frac{\omega_0}{Q}-\mathscr{B}a^2\right]a$$

expanded to first order in Δa, giving $\mathscr{B}a_0^2\Delta a$. The decay rate of the phase is zero in the absence of an injection signal. In the presence of an injection signal it is

$$\frac{1}{\tau_p}=\frac{\kappa b_0}{a_0}\cos\phi_0=\frac{1}{2}\left[\mathscr{B}a_0^2-\mathscr{A}+\frac{\omega_0}{Q}\right] \tag{2.11}$$

and is larger, the larger the injection signal. The phase relaxation time is directly related to the gain at resonance, $a_0=\kappa\tau_p b_0$ at resonance, as can be seen from (2.4) and (2.11).

A useful relation is obtained by combining (2.11) and (2.5):

$$\omega_0-\omega=\frac{1}{\tau_p}\tan\phi_0=\frac{\kappa b_0}{a_0}\sin\phi_0\ . \tag{2.12}$$

The determinantal equation for the homogeneous equation, for an assumed $\exp(-i\Omega t)$ dependence, is

$$\left[-i\Omega+\frac{1}{\tau_a}\right]\left[-i\Omega+\frac{1}{\tau_p}\right]+(\omega_0-\omega)^2=0\ , \tag{2.13}$$

where we have used (2.4) and (2.5). The solution is

$$\Omega=-\frac{i}{2}\left[\frac{1}{\tau_a}+\frac{1}{\tau_p}\right]\pm i\left[\frac{1}{4}\left[\frac{1}{\tau_a}-\frac{1}{\tau_p}\right]^2-(\omega_0-\omega)^2\right]^{1/2}.$$

When the injection signal is not detuned, $\omega_0=\omega$, the eigenfrequencies are imaginary and equal in magnitude to the decay rates $1/\tau_a$ and $1/\tau_p$ of amplitude and phase. The amplitude and phase fluctuations are decoupled. When $\omega_0\neq\omega$, the two fluctuations couple and the decay rates of the resulting solutions are affected by the individual decay rates and degree of detuning.

The Fourier transform of the amplitude fluctuation operator, treated as a periodic function of period T_p is

$$\Delta a(n\,\Delta\Omega)\equiv\lim_{T_p\to\infty}\frac{1}{T_p}\int_{-T_p/2}^{T_p/2}\Delta a(t)e^{in\,\Delta\Omega\,t}dt\ . \tag{2.14}$$

From here on, we treat

$$\Omega=\lim_{\substack{\Delta\Omega\to 0\\ n\to\infty}} n\,\Delta\Omega$$

as a continuous variable Ω. From (2.8) and (2.9)

$$\Delta a(\Omega)=\frac{\left[-i\Omega+\frac{1}{\tau_p}\right]N_c(\Omega)-(\omega_0-\omega)N_s(\Omega)}{\left[-i\Omega+\frac{1}{\tau_a}\right]\left[-i\Omega+\frac{1}{\tau_p}\right]+(\omega_0-\omega)^2}\ , \tag{2.15}$$

where the noise sources $N_c(\Omega)$ and $N_s(\Omega)$ are defined, with $n\,\Delta\Omega\equiv\Omega$:

$$N_c(\Omega)=\lim_{T_p\to\infty}\frac{1}{T_p}\int_{-T_p/2}^{T_p/2}dt\,e^{in\,\Delta\Omega\,t}\{(\cos\phi_0)\kappa\,\Delta b(t)+(\sin\phi_0)\kappa b_0\Delta\psi(t)$$

$$+\tfrac{1}{2}[e^{i\phi_0}G(t)e^{i(\omega-\omega_0)t}+e^{-i\phi_0}G^\dagger(t)e^{-i(\omega-\omega_0)t}]\}\ , \tag{2.16}$$

$$N_s(\Omega)=\lim_{T_p\to\infty}\frac{1}{T_p}\int_{-T_p/2}^{T_p/2}dt\,e^{in\,\Delta\Omega\,t}\{(\cos\phi_0)\kappa b_0\Delta\psi(t)-(\sin\phi_0)\kappa\,\Delta b(t)$$

$$+\frac{i}{2}[e^{i\phi_0}G(t)e^{i(\omega-\omega_0)t}-e^{-i\phi_0}G^\dagger(t)e^{-i(\omega-\omega_0)t}]\}\ . \tag{2.17}$$

The noise sources consist of the inphase (cosine, subscript c) and quadrature (sine, subscript s) contributions of the excess noise of the signal and the noise source $G(t)$. The former are due to the amplitude Δb and phase $\Delta\psi$ and are weighted by $\cos\phi_0$ and $\pm\sin\phi_0$, respectively; the weighting is interchanged between that for amplitude and phase. Further, the spectrum of the noise source $G(t)$ is shifted from that centered around ω_0, as implied by the noise envelope $G(t)$ of (1.8), to that centered around ω.

The Fourier transform of $\Delta\phi(t)$,

$$\Delta\phi(\Omega)=\lim_{T_p\to\infty}\frac{1}{T_p}\int_{-T_p/2}^{T_p/2}\Delta\phi(t)e^{in\,\Delta\Omega\,t} \tag{2.18}$$

with $n\Delta\Omega=\Omega$, follows similarly,

$$\Delta\phi(\Omega)=\frac{1}{a_0}\frac{\left[-i\Omega+\frac{1}{\tau_a}\right]N_s(\Omega)+(\omega_0-\omega)N_c(\Omega)}{\left[-i\Omega+\frac{1}{\tau_a}\right]\left[-i\Omega+\frac{1}{\tau_p}\right]+(\omega_0-\omega)^2}\ . \tag{2.19}$$

The spectrum $W_{\Delta a}(\Omega)$ is obtained from (2.15) by taking the average of $\Delta a^\dagger(\Omega)\Delta a(\Omega)$ and by dividing by the frequency interval $\Delta\Omega=2\pi/T$ in the limit as $T\to\infty$, $\Delta\Omega\to 0$:

$$W_{\Delta a}(\Omega)=\lim_{\Delta\Omega\to 0}\frac{1}{\Delta\Omega}\langle \Delta a^{\dagger}(\Omega)\Delta a(\Omega)\rangle=\lim_{\Delta\Omega\to 0}\frac{1}{\Delta\Omega}\frac{1}{|\mathscr{D}|^2}\left[\left[\Omega^2+\frac{1}{\tau_p^2}\right]\langle N_c^{\dagger}(\Omega)N_c(\Omega)\rangle+(\omega_0-\omega)^2\langle N_s^{\dagger}(\Omega)N_s(\Omega)\rangle -\left[i\Omega+\frac{1}{\tau_p}\right](\omega_0-\omega)\langle N_c^{\dagger}(\Omega)N_s(\Omega)\rangle -\left[-i\Omega+\frac{1}{\tau_p}\right](\omega_0-\omega)\langle N_s^{\dagger}(\Omega)N_c(\Omega)\rangle\right], \tag{2.20}$$

where

$$|\mathscr{D}|^2=\left[\Omega^2-(\omega_0-\omega)^2-\frac{1}{\tau_a\tau_p}\right]^2+\left[\frac{1}{\tau_a}+\frac{1}{\tau_p}\right]^2\Omega^2 . \tag{2.21}$$

The noise source $G(t)$ is uncorrelated with itself; $G^{\dagger}(t)G(t')$ is correlated in an impulselike manner (1.2) and, hence, its spectrum is flat. Denote the spectrum of Δb by

$$W_{\Delta b}(\Omega)=\lim_{\Delta\Omega\to 0}\frac{1}{\Delta\Omega}\langle \Delta b^{\dagger}(\Omega)\Delta b(\Omega)\rangle \tag{2.22}$$

and similarly for $\Delta\psi$. We further assume that the amplitude and phase fluctuations of the injection signal are uncorrelated. Then using (2.16) and (2.17) the spectra of N_c and N_s are

$$\lim_{\Delta\Omega\to 0}\frac{1}{\Delta\Omega}\langle N_c^{\dagger}(\Omega)N_c(\Omega)\rangle =(\cos^2\phi_0)\kappa^2W_{\Delta b}(\Omega)+(\sin^2\phi_0)\kappa^2|b_0|^2W_{\Delta\psi}(\Omega) +\frac{1}{4\pi}\left[\frac{1}{2}\frac{\omega_0}{Q}+\frac{g^2}{\gamma}(N_2+N_1)\right], \tag{2.23}$$

$$\lim_{\Delta\Omega\to 0}\frac{1}{\Delta\Omega}\langle N_s^{\dagger}(\Omega)N_s(\Omega)\rangle =(\sin^2\phi_0)\kappa^2W_{\Delta b}(\Omega)+(\cos^2\phi_0)\kappa^2|b_0|^2W_{\Delta\psi}(\Omega) +\frac{1}{4\pi}\left[\frac{1}{2}\frac{\omega_0}{Q}+\frac{g^2}{\gamma}(N_2+N_1)\right]. \tag{2.24}$$

The cross spectrum of N_c and N_s is

$$\lim_{\Delta\Omega\to 0}\frac{1}{\Delta\Omega}\langle N_c^{\dagger}(\Omega)N_s(\Omega)\rangle =\kappa^2\sin\phi_0\cos\phi_0[W_{\Delta b}(\Omega)-|b_0|^2W_{\Delta\psi}(\Omega)] . \tag{2.25}$$

The spectra of the excess noise of the injection signal may possess structure, whereas the internal noise has a flat spectrum, the last terms in (2.23) and (2.24). The internal noise spectrum has an intensity that is simply related to the inversion as one may ascertain by eliminating the atom-field interaction parameter g^2/γ with the relation[11]

$$\frac{g^2}{\gamma}=\left[2(N_2-N_1)\frac{\omega_0}{Q}\right]^{-1} . \tag{2.26}$$

Combining (2.20) and (2.22)–(2.26) one may write for the amplitude spectrum

$$W_{\Delta a}(\Omega)=\frac{1}{|\mathscr{D}|^2}\left\{\left[\left[\Omega^2+\frac{1}{\tau_p^2}\right]\cos^2\phi_0+(\omega_0-\omega)^2\sin^2\phi_0+\frac{2}{\tau_p}(\omega_0-\omega)\sin\phi_0\cos\phi_0\right]\kappa^2W_{\Delta b}(\Omega) +\left[\left[\Omega^2+\frac{1}{\tau_p^2}\right]\sin^2\phi_0+(\omega_0-\omega)^2\cos^2\phi_0-\frac{2}{\tau_p}(\omega_0-\omega)\sin\phi_0\cos\phi_0\right]\kappa^2b_0^2W_{\Delta\psi}(\omega) +\frac{1}{2\pi}\left[\Omega^2+\frac{1}{\tau_p^2}+(\omega_0-\omega)^2\right]\frac{\omega_0}{Q}\left[\frac{1}{4}+\frac{1}{4}\frac{N_2+N_1}{N_2-N_1}\right]\right\}. \tag{2.27}$$

The phase spectrum is

$$W_{\Delta\phi}(\Omega)=\frac{1}{|\mathscr{D}|^2a_0^2}\left\{\left[\left[\Omega^2+\frac{1}{\tau_a^2}\right]\cos^2\phi_0+(\omega_0-\omega)^2\sin^2\phi_0+\frac{2}{\tau_a}(\omega_0-\omega)\sin\phi_0\cos\phi_0\right]\kappa^2b_0^2W_{\Delta\psi}(\Omega) +\left[\left[\Omega^2+\frac{1}{\tau_a^2}\right]\sin^2\phi_0+(\omega_0-\omega)\cos^2\phi_0-\frac{2}{\tau_a}(\omega_0-\omega)\sin\phi_0\cos\phi_0\right]\kappa^2W_{\Delta b}(\Omega) +\frac{1}{2\pi}\left[\Omega^2+\frac{1}{\tau_a^2}+(\omega_0-\omega)^2\right]\frac{\omega_0}{Q}\left[\frac{1}{4}+\frac{1}{4}\frac{N_2+N_1}{N_2-N_1}\right]\right\}. \tag{2.28}$$

Let us consider first the effect of the oscillator noise, assuming zero excess noise of the injection signal. In this case both amplitude and phase spectra simplify greatly. Consider the phase spectrum

$$a_0^2 W_{\Delta\phi}(\Omega)=\frac{\left[\Omega^2+\dfrac{1}{\tau_a^2}\right]+(\omega_0-\omega)^2}{|\mathscr{D}|^2}\frac{1}{2\pi}\times\frac{\omega_0}{Q}\left[\frac{1}{4}+\frac{1}{4}\frac{N_2+N_1}{N_2-N_1}\right]. \tag{2.29}$$

In the amplitude spectrum $1/\tau_a$ is interchanged with $1/\tau_p$.

$$W_{\Delta a}(\Omega)=\frac{\left[\Omega^2+\dfrac{1}{\tau_p^2}\right]+(\omega_0-\omega)^2}{|\mathscr{D}|^2}\frac{1}{2\pi}\times\frac{\omega_0}{Q}\left[\frac{1}{4}+\frac{1}{4}\frac{N_2+N_1}{N_2-N_1}\right]. \tag{2.30}$$

Because the two relaxation rates are not the same, the spectra differ.

The phase measurement on the oscillator at a large amplitude $\underline{a}_0$ may be viewed as a measurement of the phase of the injection signal. In the process noise is introduced. The minimum amount of uncertainty, or mean-square phase deviation within the observation time T, must be compatible with the uncertainty principle that does not permit the measurement of phase better than given by the inequality

$$\langle \Delta\phi^2\rangle_{\text{av}} \geq \frac{1}{4\langle \Delta n_s^2\rangle_{\text{av}}}, \tag{2.31}$$

where $\langle \Delta n_s^2\rangle_{\text{av}}$ are the fluctuations of the signal photon number. For a coherent state $\langle \Delta n_s^2\rangle_{\text{av}}=\langle n_s\rangle_{\text{av}}$ and therefore

$$\langle \Delta\phi^2\rangle_{\text{av}} \geq \frac{1}{4\langle n_s\rangle_{\text{av}}}.$$

The spectrum of the injection signal is at, and near, the frequency ω. A measurement of the phase must be centered at $\Omega=0$ and have a bandwidth $4\pi B=2\pi/T$ (two-sided spectrum). To detect an undistorted phase signal, the phase response of the locked oscillator must not vary

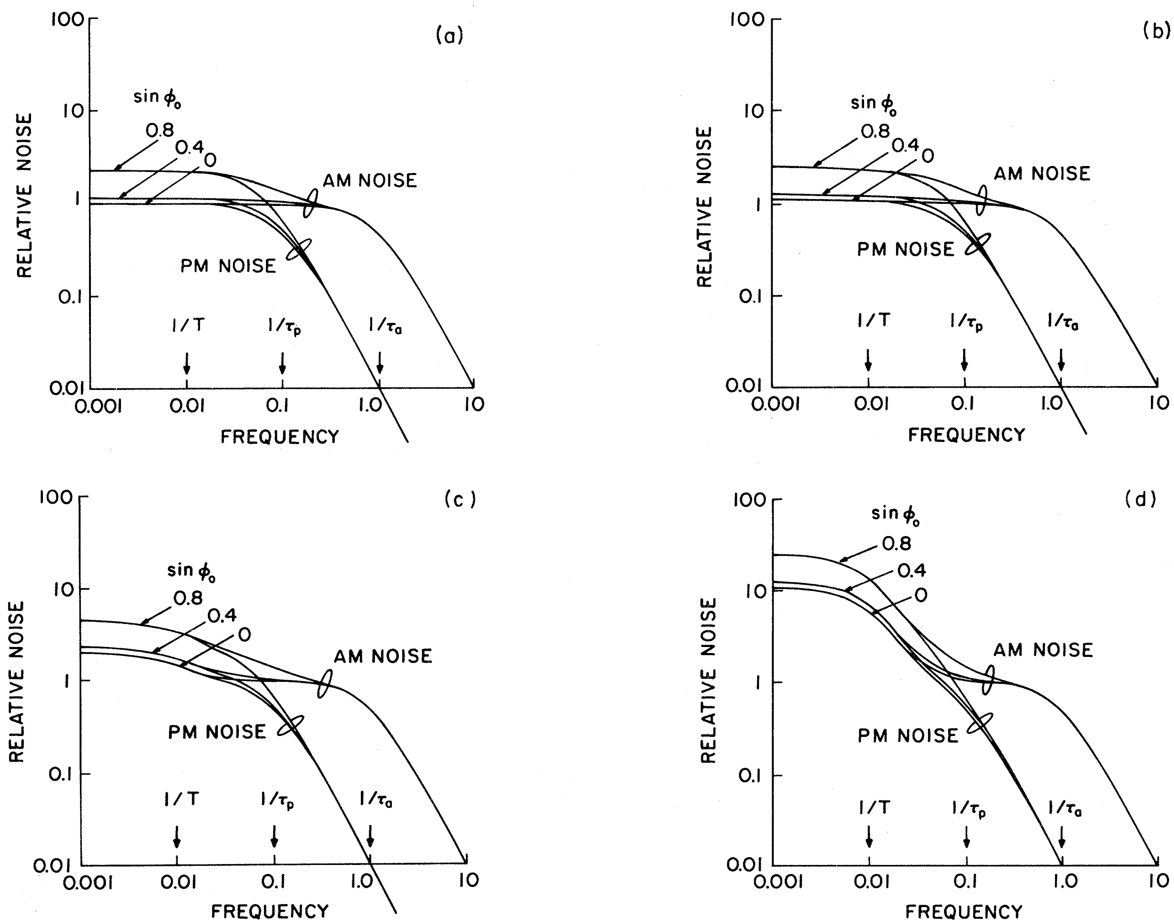

FIG. 3. Spectrum of amplitude and phase noise. Amplitude and phase noise are normalized by $\omega_0\tau_a^2/4\pi Q$ and $\omega_0\tau_p^2\cos^2\phi_0/4\pi Q a_0^2$, respectively. Frequency is normalized by the amplitude noise bandwidth, $1/\tau_a$. Phase noise bandwidth is assumed to be $1/\tau_p=1/10\tau_a$ and signal bandwidth $1/T=1/100\tau_a$. Excess noise of input signal is (a) $\langle \Delta b^2\rangle=b_0^2\langle \Delta\psi^2\rangle=0$, coherent state, (b) $\langle \Delta b^2\rangle=b_0^2\langle \Delta\psi^2\rangle=0.1/4$, (c) $\langle \Delta b^2\rangle=b_0^2\langle \Delta\psi^2\rangle=\frac{1}{4}$, equal to zero-point fluctuation, and (d) $\langle \Delta b^2\rangle=b_0^2\langle \Delta\psi^2\rangle=\frac{10}{4}$.

over the bandwidth $4\pi B$. Therefore, we may evaluate the mean-square fluctuations $\langle \Delta\phi_T^2 \rangle_{av}$ within the observation time T from the product

$$4\pi B W_{\Delta\phi}(\Omega=0) .$$

Using (2.11) and (1.9)

$$\begin{aligned}\langle \Delta\phi_T^2 \rangle_{av} &= 4\pi B W_{\Delta\phi}(\Omega=0)\\ &= \frac{1}{2\langle n_s\rangle_{av}} \frac{N_2}{N_2-N_1}\left[1+\frac{\tau_e}{\tau_0}\right]\\ &\quad\times\frac{[1+\tau_a^2(\omega_0-\omega)^2][1+\tau_p^2(\omega_0-\omega)^2]}{[1+\tau_a\tau_p(\omega_0-\omega)^2]^2} ,\end{aligned} \tag{2.32}$$

where $\langle n_s \rangle_{av} = b_0^2$ is the average number of signal photons. The frequency-dependent factor has a minimum value of 1 at $(\omega-\omega_0)=0$ and $(\omega-\omega_0)=\infty$ (the latter value is uninteresting because it corresponds to no gain). Thus, the minimum value of $\langle \Delta\phi_T^2 \rangle_{av}$ occurs at $\omega=\omega_0$, when the injection signal has the natural frequency of the oscillator, and is equal to twice the value imposed by the uncertainty principle enhanced by the inversion factor $N_2/(N_2-N_1)$, and the cavity loss factor $[1+(\tau_e/\tau_0)]$. The two factors approach unity for complete inversion of the atomic systems, and for a highly over-coupled cavity $\tau_e/\tau_0 \ll 1$.

Next, consider the noise in the presence of excess amplitude and phase noise of the injection signal. The spectra of amplitude and phase are plotted in Fig. 3 for different detunings and input excess noise spectra. With increasing detuning both the amplitude and phase noise are increased due to amplitude-to-phase conversion and vice versa. The phase noise relaxation rate $1/\tau_p$ is assumed to be $\frac{1}{10}$ times the amplitude relaxation rate $1/\tau_a$ which corresponds to a signal gain of 20 dB. The excess noise is assumed to have the same bandwidth as the signal.

III. AMPLITUDE AND PHASE NOISE OF LINEAR AMPLIFIER

We have found that a locked oscillator measures the phase of the injection signal with an uncertainty twice that of the ideal measurement. This is, at first sight, surprising because one may have expected that the locking of an oscillator constitutes a measurement of the *phase* of the injection signal without a simultaneous measurement of the amplitude. As such, it ought not to incur the 3 dB penalty imposed by a simultaneous measurement.[23–25] In order to understand our result, it is useful to study the linear amplifier.

The fluctuations of the linear amplifier differ from those of the locked oscillator only by the fact that the phase and amplitude relaxation times are identical in the amplifier case

$$\frac{1}{\tau}=\frac{1}{\tau_a}=\frac{1}{\tau_p}=\frac{1}{2}\left[\frac{\omega_0}{Q}-\mathscr{A}\right] \tag{3.1}$$

and the saturation term $\mathscr{B}a_0^2$ is ignored. The result of Sec. II can be taken over with the interpretation of ϕ_0 [compare (2.5)]

$$\tan\phi_0=(\omega_0-\omega)\tau . \tag{3.2}$$

Introduction of these relations into (2.27) and (2.28) gives

$$\begin{aligned}W_{\Delta a}(\Omega)=\frac{1}{|\mathscr{D}|^2}\Bigg[&\left[\Omega^2\cos^2\phi_0+\frac{1}{\tau^2\cos^2\phi_0}\right]\kappa^2W_{\Delta b}(\Omega)+\kappa^2b_0^2\Omega^2(\sin^2\phi_0)W_{\Delta\psi}(\Omega)\\ &+\frac{\omega_0}{Q}\left[\Omega^2+\frac{1}{\tau^2}+(\omega_0-\omega)^2\right]\frac{1}{2\pi}\left[\frac{1}{4}+\frac{1}{4}\frac{N_2+N_1}{N_2-N_1}\right]\Bigg]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}a_0^2W_{\Delta\phi}(\Omega)=\frac{1}{|\mathscr{D}|^2}\Bigg[&\left[\Omega^2\cos^2\phi_0+\frac{1}{\tau^2\cos^2\phi_0}\right]\kappa^2b_0^2W_{\Delta\psi}(\Omega)+\kappa^2\Omega^2\sin^2\phi_0W_{\Delta b}(\Omega)\\ &+\frac{\omega_0}{Q}\left[\Omega^2+\frac{1}{\tau^2}+(\omega_0-\omega)^2\right]\frac{1}{2\pi}\left[\frac{1}{4}+\frac{1}{4}\frac{N_2+N_1}{N_2-N_1}\right]\Bigg] ,\end{aligned} \tag{3.4}$$

where

$$|\mathscr{D}|^2=\left[[\Omega+(\omega_0-\omega)]^2+\frac{1}{\tau^2}\right]\left[[\Omega-(\omega_0-\omega)]^2+\frac{1}{\tau^2}\right] . \tag{3.5}$$

It is of interest to ascertain the uncertainty in the determination of amplitude and phase of a coherent state, with $W_{\Delta\psi}=W_{\Delta b}=0$. In this limit, the amplitude spectrum referred to the "input" by division by the power gain

$$G^2=\frac{\kappa^2}{(\omega_0-\omega)^2+\frac{1}{\tau^2}}$$

gives, at $\Omega=0$,

$$\frac{W_{\Delta a}(\Omega=0)}{G^2}=\frac{1}{\kappa^2}\frac{\omega_0}{Q}\frac{1}{4\pi}\left[\frac{N_2}{N_2-N_1}\right] . \tag{3.6}$$

The same result is obtained for $b_0^2W_{\Delta\phi}(\Omega=0)$. Again the use of the definition of the coupling coefficient gives for the amplitude uncertainty $\langle \Delta b_T^2 \rangle_{av}$ measured in a time interval T

$$\langle \Delta b_T^2 \rangle_{\rm av} = \frac{2\pi}{T} \frac{W_{\Delta a}(\Omega=0)}{G^2} = \frac{1}{2} \frac{N_2}{N_2-N_1} \left[1+\frac{\tau_e}{\tau_0} \right] \quad (3.7)$$

and for the phase uncertainty

$$\langle \Delta\phi_T^2 \rangle_{\rm av} = \frac{2\pi}{T} W_{\Delta\phi}(\Omega=0) = \frac{1}{2} \frac{1}{b_0^2} \frac{N_2}{N_2-N_1} \left[1+\frac{\tau_e}{\tau_0} \right] . \quad (3.8)$$

The product of the two uncertainties is

$$\langle \Delta b_T^2 \rangle_{\rm av} \langle \Delta\phi_T^2 \rangle_{\rm av} = \frac{1}{4} \frac{1}{\langle n_s \rangle_{\rm av}} \left[\frac{N_2}{N_2-N_1} \right]^2 \left[1+\frac{\tau_e}{\tau_0} \right]^2 . \quad (3.9)$$

When the inversion is complete this product reaches the ideal limit, imposed on a simultaneous measurement of amplitude and phase. The rms value of the ideal limit is two times larger than that imposed by the Heisenberg principle on an ideal measurement of *either* amplitude *or* phase. This is the penalty incurred by a simultaneous measurement.[23–25]

Injection-locked oscillators also permit the simultaneous measurement of amplitude and phase variations; however, the gains for the in-phase and quadrature components are not the same. It is of interest, therefore, to ascertain whether the locked oscillator obeys the uncertainty principle as it applies to a simultaneous measurement. The amplitude gain follows from (2.7) with $\Omega \to 0$:

$$G_a^2 = \left| \frac{\langle \Delta a \rangle}{\langle \Delta b \rangle} \right|^2 = |\kappa\tau_a \cos\phi_0|^2 = \frac{\kappa^2\tau_a^2}{1+(\omega-\omega_0)^2\tau_p^2} . \quad (3.10)$$

The measurement of b within an observation time T is subject to fluctuations referred to the input by division by G_a^2:

$$\begin{aligned} \langle \Delta b_T^2 \rangle_{\rm av} &= W_{\Delta a}(\Omega=0) \frac{2\pi}{T} \frac{1}{G_a^2} \\ &= \frac{1}{2} \frac{N_2}{N_2-N_1} \left[1+\frac{\tau_e}{\tau_0} \right] \frac{[1+(\omega-\omega_0)^2\tau_p^2]^2}{[1+(\omega-\omega_0)^2\tau_a\tau_p]^2} . \end{aligned} \quad (3.11)$$

For $(\omega-\omega_0) \to 0$ the above reduces to the result of the linear amplifier. The product $\langle \Delta b_T^2 \rangle_{\rm av} \langle \Delta\phi_T^2 \rangle_{\rm av}$ can be made to approach the ideal limit for a simultaneous measurement in the limit of no detuning, strong overcoupling, and complete inversion. The product increases with increased detuning, because $\tau_p > \tau_a$ for a locked oscillator. [Compare the definitions of $1/\tau_a$ and $1/\tau_p$, (2.10) and (2.11), respectively.]

IV. FOKKER-PLANCK EQUATION FOR c-NUMBER INJECTION SIGNAL

In the preceding sections we used the Langevin approach to obtain expressions for the spectra of the fluctuations. The Fokker-Planck equation leads to the probability distributions of amplitude and phase. The Fokker-Planck equation for the P distribution of Ref. 11, pp. 294–295, Eq. (25), corrected for the erroneously omitted term describing diffusion in the radial direction, reads

$$\begin{aligned} \frac{\partial P}{\partial t} &= -\frac{1}{2} \frac{1}{r} \frac{\partial}{\partial r} \left[r^2 \left[\mathscr{A} - \frac{\omega_0}{Q} - \mathscr{B} r^2 \right] P \right] \\ &\quad + \frac{\mathscr{A}}{4} \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial P}{\partial r} \right] + \frac{\mathscr{A}}{4} \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} P . \end{aligned} \quad (4.1)$$

Here, polar coordinates are used and the α parameter is written

$$\alpha = r e^{i\theta} .$$

Equation (4.1) was obtained by assuming an "injection" of active particles in the upper state. The lower state is populated as the result of the interaction of the particles with the field, resulting in the reduction by $\mathscr{B} r^2$ of the gain parameter $\mathscr{A}$.

The equation of motion is a diffusion equation with a forcing term in the radial direction, the first derivative with respect to r, that tends to confine the P distribution in the radial direction. An initial delta function distribution $P(\alpha)=\delta(\alpha-\alpha_0)$, diffuses from α_0 and, as a function of time, spreads in both θ directions. The natural time dependence $\exp(-i\omega_0 t)$, where ω_0 is the frequency of the oscillator, has been factored out.

Next, recall the origin of the Fokker-Planck equation which was derived through integration by parts of the equation for the density matrix

$$\rho \equiv \int d^2\alpha \, P(\alpha) |\alpha\rangle\langle\alpha| ,$$

where

$$\begin{aligned} \dot{\rho} &= \int d^2\alpha \frac{\partial P}{\partial t} |\alpha\rangle\langle\alpha| \\ &= \int d^2\alpha \, \alpha \left[\mathscr{A} - \frac{\omega_0}{Q} - \mathscr{B} |\alpha|^2 \right] P \frac{\partial}{\partial \alpha} |\alpha\rangle\langle\alpha| + \text{c.c.} \\ &\quad + \int d^2\alpha \mathscr{A} P \frac{\partial^2}{\partial\alpha \, \partial\alpha^*} |\alpha\rangle\langle\alpha| . \end{aligned} \quad (4.2)$$

We shall use both (4.1) and (4.2) in extending the analysis to an injection signal.

Suppose that the Hamiltonian used in the derivation of the equation of motion of the density matrix

$$\rho = \int d^2\alpha \, P(\alpha) |\alpha\rangle\langle\alpha| \quad (4.3)$$

is supplied by the coupling Hamiltonian (1.3). In this section we shall treat the annihilation operator b of the injection signal as a c number. In Sec. V we shall generalize the analysis, treating b as an operator. The time dependence of the density matrix

$$\dot{\rho} = -\frac{i}{\hbar} [\mathscr{H}, \rho] \quad (4.4)$$

is supplemented by

$$[(\kappa b a^\dagger e^{-i(\omega-\omega_0)t}-\kappa b^* a\, e^{i(\omega-\omega_0)t})P(\alpha)\,|\alpha\rangle\langle\alpha|-P(\alpha)\,|\alpha\rangle\langle\alpha|\,(\kappa b a^\dagger e^{-i(\omega-\omega_0)t}-\kappa b^* a e^{i(\omega-\omega_0)t})]$$
$$=P(\alpha)\left[\kappa b^* e^{i(\omega-\omega_0)t}\frac{\partial}{\partial\alpha^*}+\kappa b\, e^{-i(\omega-\omega_0)t}\frac{\partial}{\partial\alpha}\right]|\alpha\rangle\langle\alpha| \;, \tag{4.5}$$

where we have used the identities[11]

$$a\,|\alpha\rangle\langle\alpha|=\alpha\,|\alpha\rangle\langle\alpha| \;,$$
$$a^\dagger|\alpha\rangle\langle\alpha|=\left[\frac{\partial}{\partial\alpha}+\alpha^*\right]|\alpha\rangle\langle\alpha| \;,$$
$$|\alpha\rangle\langle\alpha|\,a=\left[\frac{\partial}{\partial\alpha^*}+\alpha\right]|\alpha\rangle\langle\alpha| \;,$$
$$|\alpha\rangle\langle\alpha|\,a^\dagger=\alpha^*\,|\alpha\rangle\langle\alpha| \;.$$

Equation (4.5) introduced into (4.2) to supplement the interaction Hamiltonian, subsequent integration by parts, lead to the Fokker-Planck equation:

$$\frac{\partial P}{\partial t}=-\left\{\frac{\partial}{\partial\alpha}\left[\alpha\left[\mathscr{A}-\frac{\omega_0}{Q}-\mathscr{B}\,|\alpha|^2\right]P\right]+\text{c.c.}\right\}$$
$$+\mathscr{A}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}P-\left[\kappa b^* e^{i(\omega-\omega_0)t}\frac{\partial}{\partial\alpha^*}P+\text{c.c.}\right] . \tag{4.6}$$

One may now introduce polar coordinates to clarify the meaning of the above equation. Define the argument of b as $-\psi$, $b=|b|\,e^{-i\psi}$

$$\frac{\partial P}{\partial t}=-\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\left[r^2\left[\mathscr{A}-\frac{\omega_0}{Q}-\mathscr{B}r^2\right]P\right]+\frac{\mathscr{A}}{4}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial P}{\partial r}\right]$$
$$+\frac{\mathscr{A}}{4}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}P-\kappa\,|b|\left[\frac{1}{r}\frac{\partial}{\partial r}r\cos[\theta+(\omega-\omega_0)t+\psi]P-\frac{1}{r}\frac{\partial}{\partial\theta}\sin[\theta+(\omega-\omega_0)t+\psi]P\right] . \tag{4.7}$$

By introducing a new angular variable

$$\phi\equiv(\omega_0-\omega)t-\psi-\theta \tag{4.8}$$

we may transform (4.7) into

$$\frac{\partial}{\partial t}P(r,\phi,t)=-\frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\left[r^2\left[\mathscr{A}-\frac{\omega_0}{Q}-\mathscr{B}r^2\right]P\right]+\frac{\mathscr{A}}{4}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}P\right]+\frac{\mathscr{A}}{4}\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}P$$
$$+(\omega-\omega_0)\frac{\partial P}{\partial\phi}-\kappa\,|b|\left[\frac{1}{r}\frac{\partial}{\partial r}(r\cos\phi\, P)-\frac{1}{r}\frac{\partial}{\partial\phi}(\sin\phi\, P)\right] . \tag{4.9}$$

This is the desired locking equation.

Before we proceed with its analysis we want to show that it is of more general validity than its derivation implies. We have assumed that the injection signal was a c-number source. Yet, we are interested in a full quantum-mechanical analysis of a locked oscillator. One may question, therefore, whether the results obtained from (4.9) ignore some quantum noise effects. This is not the case. In Sec. V we shall show that Eq. (4.9) is valid, if the injection signal is a coherent state of another system separated from the oscillator by an isolator at zero temperature so that the coherent state can be defined independent of the evolution of the excitation in the oscillator. Of course, the isolator is responsible, in part, for the zero-point fluctuations of the injection source. The parameter b in (4.9) has to be interpreted as the eigenvalue β of the coherent state of the injection signal, $b=\beta$.

V. FOKKER-PLANCK EQUATION FOR OPERATOR INJECTION SIGNAL

In Sec. IV, we have treated the injection signal amplitude b as a c number. Consider now the case when the injection signal is treated as an operator. The density matrix of the combined system is now

$$\rho=P(\alpha,\beta)\,|\alpha\rangle\langle\alpha|\otimes|\beta\rangle\langle\beta| \;,$$

where $|\beta\rangle\langle\beta|$ is the matrix of the states of the injected signal. The coupling term (4.5) has to be generalized to account for the operator nature of b. We note the property of the operator product $b^\dagger a$:

$$b^\dagger a\;|\alpha\rangle\langle\alpha|\otimes|\beta\rangle\langle\beta|=\left[\frac{\partial}{\partial\beta}+\beta^*\right]\alpha\,|\alpha\rangle\langle\alpha|\otimes|\beta\rangle\langle\beta|$$

and analogous relations for the pre- and post-multiplication by $ba^\dagger$ and $b^\dagger a$. The result is

$$(\kappa b a^{\dagger} e^{i(\omega_0-\omega)t} - \kappa a b^{\dagger} e^{i(\omega-\omega_0)t}) \,|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| - |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \,(\kappa b a^{\dagger} e^{i(\omega_0-\omega)t} - \kappa a b^{\dagger} e^{i(\omega-\omega_0)t})$$

$$= \left[-e^{i(\omega-\omega_0)t} \kappa\alpha \frac{\partial}{\partial\beta} + e^{i(\omega_0-\omega)t}\kappa\beta\frac{\partial}{\partial\alpha} + e^{i(\omega-\omega_0)t}\kappa\beta^*\frac{\partial}{\partial\alpha^*} - e^{i(\omega_0-\omega)t}\kappa\alpha^*\frac{\partial}{\partial\beta^*} \right] |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \; . \tag{5.1}$$

Integration by parts transfer the derivatives onto P, so that the equation of motion for P becomes [compare (4.6)]

$$\frac{\partial P}{\partial t} = \left\{ -\frac{\partial}{\partial\alpha}\left[\alpha\left[\mathscr{A} - \frac{\omega_0}{Q} - \mathscr{B}\,|\alpha|^2\right]P\right] + \text{c.c.}\right\} + \mathscr{A}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}P$$
$$+ \left[-\left[\kappa e^{i(\omega-\omega_0)t}\beta^*\frac{\partial}{\partial\alpha^*} - \kappa e^{i(\omega_0-\omega)t}\alpha^*\frac{\partial}{\partial\beta^*}\right]P + \text{c.c.}\right] . \tag{5.2}$$

We can integrate P over all β. Then, the terms containing derivatives with respect to β and β^* integrate to zero. The equation of motion for $\int d^2\beta P$ is

$$\frac{\partial}{\partial t}\int d^2\beta P = \left\{ -\frac{\partial}{\partial\alpha}\left[\alpha\left[\mathscr{A} - \frac{\omega_0}{Q} - \mathscr{B}\,|\alpha|^2\right]\int d^2\beta P\right] + \text{c.c.}\right\}$$
$$+ \mathscr{A}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\int d^2\beta P + \left[-\left[\kappa e^{i(\omega-\omega_0)t}\frac{\partial}{\partial\alpha^*}\right]\int d^2\beta\beta^* P + \text{c.c.}\right] . \tag{5.3}$$

This is the Fokker-Planck equation for an oscillator, described in terms of a P distribution of α states, locked by injection of β states. In general, the oscillator system reacts back onto the injection-signal system. Another equation would have to be written down for it. If an isolator (at zero degrees) is inserted between the oscillator and the injection-signal system, the latter may be treated as unaffected by the former, except of course that energy is being lost by it to the oscillator. One may imagine it to be continually prepared in sequences of β states assigned to time intervals T.

If the injection signal is in a β state, the integral over all β of $\beta^* P$ can be replaced by $\beta^* \int d^2\beta P$. Then, interpreting $\int d^2\beta P$ as the reduced P distribution one finds that (5.3) and (4.6) are in one-to-one correspondence if β is interpreted as b, and P as $\int d^2\beta P$ of the full P distribution. In the sequel we shall use the notation of Sec. IV with the understanding that we are treating the case of locking via a coherent β state.

VI. FLUCTUATIONS DERIVED FROM FOKKER-PLANCK EQUATION

In Sec. IV we have derived the Fokker-Planck equation for a c-number injection signal. In Sec. V we showed that the same equation follows for an injection signal that is in a coherent state; the eigenvalue β of the coherent state can be identified with the amplitude of the c-number source. In this section we shall study the mean-square fluctuations predicted by the Fokker-Planck equation and compare them with the results obtained from the operator Langevin equations. Because we assumed a coherent state injection signal we are covering only the case of zero excess noise of the injection signal.

Consider first (4.1), the equation of the free-running oscillator. In the steady state, $(\partial/\partial t)P=0$, the phase is random, $(\partial/\partial\theta)P=0$. The first derivative with respect to r provides a forcelike restoring action that makes P cluster around the equilibrium value $r=a_0$, for which

$$\mathscr{A} - \frac{\omega_0}{Q} - \mathscr{B}a_0^2 = 0 \; . \tag{6.1}$$

The equation is solved approximately by expanding

$$r\left[\mathscr{A} - \frac{\omega_0}{Q} - \mathscr{B}r^2\right]$$

around $r=a_0$:

$$-r\left[\mathscr{A} - \frac{\omega_0}{Q} - \mathscr{B}r^2\right] = 2\mathscr{B}a_0^2(r-a_0)$$
$$= 2\left[\mathscr{A} - \frac{\omega_0}{Q}\right](r-a_0) \; . \tag{6.2}$$

The P distribution can be integrated directly from (6.2)

$$P(r) = \frac{1}{\sqrt{2\pi}\sigma_r}\exp-\frac{(r-a_0)^2}{2\sigma_r^2} \tag{6.3}$$

with

$$\sigma_r^2 \equiv (4\mathscr{B}a_0^2/\mathscr{A})^{-1} \; . \tag{6.4}$$

The distribution is Gaussian around the average value $r=a_0$. Note that the first derivative term with respect to r provides a stabilizing effect around $r=a_0$ and that its coefficient is positive.

The locking of the oscillator by the injection signal has two effects represented by the new terms, derivatives with respect to r and ϕ. The derivative with respect to ϕ provides a forcelike restoring action on the phase analogous to the restoring "force" on the amplitude of the free-running oscillator. This term can be interpreted by expanding it around ϕ_0, the phase for which the argument of the derivative $(\partial/r\partial\phi)$ vanishes. Write $r=a_0+\delta r$, $\phi=\phi_0+\delta\phi$. Then

$$(\omega-\omega_0)r+\kappa\,|b\,|\sin\phi=(\omega-\omega_0)a_0+\kappa\,|b\,|\sin\phi_0+(\omega-\omega_0)\delta r+\kappa\,|b\,|\cos\phi_0\,\delta\phi\ . \tag{6.5}$$

The zeroth-order part of the above equation vanishes when

$$\omega_0-\omega=\frac{\kappa\,|b\,|\sin\phi_0}{a_0} \tag{6.6}$$

which is identical in form with (2.12).

The second effect of locking is the increase of the amplitude a_0 from the value imposed by (6.1) in the free-running case. Expansion of the terms under the derivative $(1/r)(\partial/\partial r)r$ in (4.1) around a_0 and ϕ_0 gives

$$\begin{aligned}\frac{1}{2}r\left[\mathcal{A}-\frac{\omega_0}{Q}-\mathcal{B}r^2\right]&+\kappa\,|b\,|\cos\phi\\&=\frac{1}{2}a_0\left[\mathcal{A}-\frac{\omega_0}{Q}-\mathcal{B}a_0^2\right]\\&\quad+\kappa\,|b\,|\cos\phi_0-\mathcal{B}a_0^2\delta r-\kappa\,|b\,|\sin\phi_0\,\delta\phi\ .\end{aligned} \tag{6.7}$$

The zeroth-order term gives the new equation for the amplitude a_0 as affected by the injection signal and corresponds to (2.7). The perturbation term may be approximated, in the limit of $|b\,|/a_0\ll 1$, large gain, by

$$-\mathcal{B}a_0^2\delta r-\kappa\,|b\,|\sin\phi_0\,\delta\phi=-\frac{1}{\tau_a}\delta r-(\omega-\omega_0)a_0\,\delta\phi\ , \tag{6.8}$$

where we have used (2.10) and (6.6). When (6.5)—(6.8) are introduced into (4.9) one obtains the "linearized" version of the Fokker-Planck equation with $x\equiv\delta r$ and $y\equiv a_0\delta\phi$ as the independent variables. A change of variables to the Cartesian coordinates x and y gives

$$\begin{aligned}\frac{\partial}{\partial t}P(x,y,t)=&\frac{\partial}{\partial x}\left[\frac{1}{\tau_a}x+(\omega_0-\omega)y\right]P\\&+\frac{\mathcal{A}}{4}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]P\\&+\frac{\partial}{\partial y}\left[\frac{1}{\tau_p}y-(\omega_0-\omega)x\right]P\ ,\end{aligned} \tag{6.9}$$

where we have set

$$\frac{\kappa\,|b\,|}{a_0}\cos\phi_0=\frac{1}{\tau_p} \tag{6.10}$$

in analogy with (2.11).

The steady-state solution of (6.9) is obtained with the Gaussian ansatz

$$P\propto\exp[-\tfrac{1}{2}(A_{xx}x^2+2A_{xy}xy+A_{yy}y^2)]\ . \tag{6.11}$$

Equating equal powers of x and y one obtains four equations for the three unknowns A_{xx}, A_{yy}, and A_{xy}. These equations are not independent and have the solution:

$$A_{xx}=\frac{4}{\mathcal{A}}\frac{\left[\dfrac{1}{\tau_p}+\dfrac{1}{\tau_a}\right]\left[\dfrac{1}{\tau_a}\left[\dfrac{1}{\tau_a}+\dfrac{1}{\tau_p}\right]+2(\omega-\omega_0)^2\right]}{\left[\dfrac{1}{\tau_a}+\dfrac{1}{\tau_p}\right]^2+4(\omega-\omega_0)^2}\ , \tag{6.12}$$

$$A_{yy}=\frac{4}{\mathcal{A}}\frac{\left[\dfrac{1}{\tau_p}+\dfrac{1}{\tau_a}\right]\left[\dfrac{1}{\tau_p}\left[\dfrac{1}{\tau_a}+\dfrac{1}{\tau_p}\right]+2(\omega-\omega_0)^2\right]}{\left[\dfrac{1}{\tau_a}+\dfrac{1}{\tau_p}\right]^2+4(\omega-\omega_0)^2}\ , \tag{6.13}$$

and

$$A_{xy}=\frac{4}{\mathcal{A}}(\omega_0-\omega)\frac{\dfrac{1}{\tau_a^2}-\dfrac{1}{\tau_p^2}}{\left[\dfrac{1}{\tau_p}+\dfrac{1}{\tau_a}\right]^2+4(\omega-\omega_0)^2}\ . \tag{6.14}$$

The probability distribution is indicated in Fig. 4 in the (r,ϕ) plane. The x and y coordinates parallel to the amplitude and phase perturbations, respectively, are also indicated. When the injection signal is detuned from the

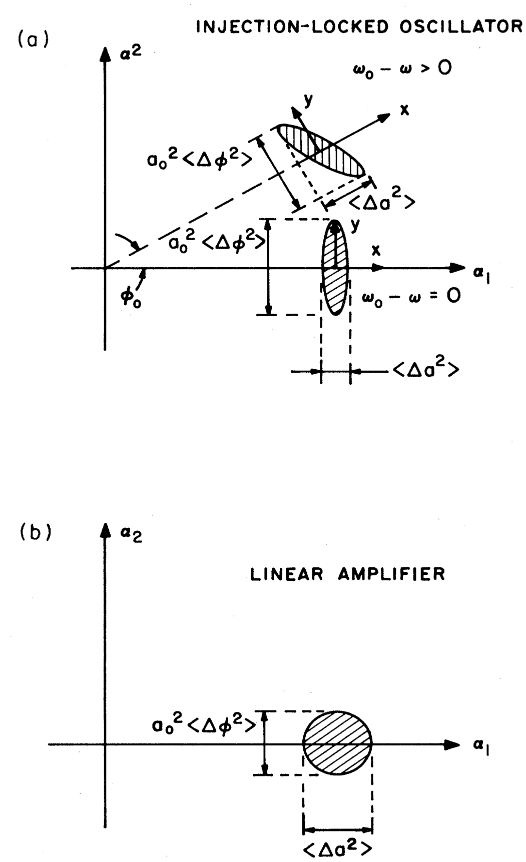

FIG. 4. P distribution of injection-locked oscillator (a) and linear amplifier (b) in α plane.

natural frequency of the oscillator, $\omega \neq \omega_0$, then the two-dimensional Gaussian has principal axes that are not parallel to x and y, respectively; the phase and amplitude fluctuations are correlated.

The present results are easily compared with the results of Sec. II when $\omega = \omega_0$. We shall look at this case in detail. Consider the mean-square phase deviations of (6.9) for $\omega_0 - \omega = 0$, $A_{xy} = 0$,

$$\langle \delta\phi^2 \rangle_{\rm av} = \frac{1}{A_{yy} a_0^2} = \frac{\mathscr{A}}{4a_0^2} \tau_p \ . \tag{6.15}$$

In the derivation of (4.1) it had been assumed that the laser medium is in the upper state in the absence of saturation. The saturation reduces the gain so that one may identify the unsaturated gain parameter $\mathscr{A}$, divided by the saturated gain, with the ratio of the upper level population N_2, to the difference between upper and lower level populations:

$$\frac{\mathscr{A}}{\mathscr{A} - \mathscr{B} a_0^2} = \frac{N_2}{N_2 - N_1}$$

and no level degeneracy has been considered. The saturated gain is approximately equal to ω_0/Q. Thus, we may write for (6.15)

$$\langle \delta\phi^2 \rangle_{\rm av} = \frac{\omega_0}{Q} \frac{1}{4a_0^2} \tau_p \frac{N_2}{N_2 - N_1} \ . \tag{6.16}$$

Next we introduce the average photon number of the injection signal, $|b|^2 = \langle n_s \rangle_{\rm av}$, using the gain at synchronism derived from (2.4)

$$\langle \delta\phi^2 \rangle_{\rm av} = \frac{\omega_0}{Q} \frac{1}{4\langle n_s \rangle_{\rm av}} \frac{1}{\kappa^2 \tau_p} \frac{N_2}{N_2 - N_1} \ . \tag{6.17}$$

The phase fluctuations have a Lorentzian spectral profile (see Sec. II) that occupies a bandwidth proportional to $1/\tau_p$. The phase measurement of the source requires only a bandwidth B equal to $1/2T$, where T is the time of "observation" of the source. Thus, if a filter is introduced that cuts out the unnecessary part of the spectrum, the mean-square fluctuations are reduced by the factor $4B\tau_p = 2\tau_p/T$ (see Appendix B) so that one has for the filtered mean-square fluctuations

$$\langle \delta\phi_T^2 \rangle_{\rm av} = \frac{\omega_0}{Q} \frac{1}{2\langle n_s \rangle_{\rm av}} \frac{1}{\kappa^2 T} \frac{N_2}{N_2 - N_1} = \frac{1}{2\langle n_s \rangle_{\rm av}} \left[1 + \frac{\tau_e}{\tau_0} \right] \frac{N_2}{N_2 - N_1} \ , \tag{6.18}$$

where we have used (1.9). This is the same result as (2.32) for $\omega_0 - \omega = 0$. An analogous investigation of the mean-square amplitude fluctuations $\langle \Delta r^2 \rangle_{\rm av}$ confirms the result of Sec. III at synchronism. The analysis of locking off synchronism is considerably more complicated and is not presented here. The following issues have to be confronted: (a) The mean-square deviation of the amplitude and phase are equal to the diagonal elements of the inverse of the matrix

$$\begin{bmatrix} A_{xx} A_{xy} \\ A_{xy} A_{yy} \end{bmatrix} .$$

(b) The spectrum of the fluctuations off synchronism is given by (2.29) and (2.30), respectively. The filter of bandwidth B selects the portion $4\pi B$ of the overall spectrum at $\Omega = 0$. When the analysis is carried out the results of Secs. II and III in the general case, $\omega \neq \omega_0$, are fully confirmed.

We have used the spectral information obtained from the Langevin equations to derive the filtered mean-square fluctuations from the total mean-square fluctuations of the steady-state solution of the Fokker-Planck equation. Alternately, one could have derived the spectrum from the time-dependent solution of the Fokker-Planck equation. The time-dependent Fokker-Planck equation yields the time evolution of an initial impulse of the probability distribution in the x-y plane. The autocorrelation functions

$$\langle x(t) x(t+\tau) \rangle_{\rm av}$$

and

$$\langle y(t) y(t+\tau)_{\rm av}$$

can be derived from this information. The spectra are the Fourier transforms of the autocorrelation functions. The same result is obtained as from the Langevin equations, albeit with considerably more effort.

Finally, we note that according to (4.6) we have related the results of the Langevin equation to the mean-square fluctuations of $\alpha = r e^{i\theta}$, and not those of the field operator a. In doing so we have ignored the mean-square fluctuations associated with an α state. Because the mean-square fluctuations found are large compared with the mean-square spread associated with an α state, the approximation, applicable in the limit of high gain, is a legitimate one.

VII. DISCUSSION

The operator Langevin equation leads rather directly to the spectra of the amplitude and phase fluctuations of a locked oscillator. The two spectra differ in the case of the oscillator, become identical for the modulated amplifier. The locked oscillator provides a means for the quantum measurement of the phase of the injection signal. It does not give minimum uncertainty. In the limit of complete inversion and negligible internal loss the excess fluctuation is 3 dB higher than the ideal limit. We interpreted this result with the aid of the linear amplifier which does offer an ideal *simultaneous* quantum measurement of amplitude and phase in the limit of complete inversion and negligible internal loss. Such a measurement requires doubling of the minimum uncertainty of each of the complementary variables. The phase noise of the locked oscillator can be understood from another point of view: It can be interpreted as frequency-noise to phase-noise conversion of the oscillator. The spectrum of the frequency modulation noise $W_{\Delta\omega}(\Omega)$ of the self-oscillating laser follows from (2.29), with $1/\tau_p = 0$ and $\omega - \omega_0 = 0$

$$W_{\Delta\omega}(\Omega) = \Omega^2 W_{\Delta\phi}(\Omega) = \frac{\omega_0/Q}{4\pi a_0^2} \frac{N_2}{N_2 - N_1} \ . \tag{7.1}$$

The frequency-to-phase conversion factor is obtained from (2.5) and (2.11)

$$\frac{d\phi}{d\omega_0} = -\frac{a_0/\kappa b_0}{\cos\phi_0} . \quad (7.2)$$

Combining (7.1) and (7.2) we find

$$W_{\Delta\phi}(\Omega) = W_{\Delta\omega}(\Omega) \left[\frac{d\phi}{d\omega_0}\right]^2 = \frac{\omega_0/Q}{4\pi\kappa^2 b_0^2 \cos^2\phi_0} \frac{N_2}{N_2 - N_1}$$

and thus

$$\langle \Delta\phi^2 \rangle = \frac{2\pi}{T} W_{\Delta\phi}(\Omega) = \frac{1}{2b_0^2 \cos^2\phi_0} \frac{N_2}{N_2 - N_1} \left[1 + \frac{\tau_e}{\tau_0}\right]$$

which is equal to the phase noise of the locked oscillator (6.18) for $\cos\phi_0 = 1$.

The Fokker-Planck approach gives the probability distribution of amplitude and phase. The uncertainty of a measurement cannot be determined from it directly without information on the spectra of phase and amplitude which is obtained most conveniently from the Langevin approach. Of course, the information on the spectrum is contained in the Fokker-Planck equation as well but requires a greater effort of extrication.

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APPENDIX A: THE EXPECTATION OF EQ. (1.8)

We explore the expectation of Eq. (1.8) as applied to the cavity with no gain

$$\frac{d}{dt}\langle a \rangle = -\left[\frac{1}{\tau_e} + \frac{1}{\tau_0}\right]\langle a \rangle + \kappa\langle b \rangle . \quad (A1)$$

The system is conservative when $1/\tau_0 = 0$. In this limit one may apply time reversal considerations to the system. Consider the unexcited cavity, with $\langle b \rangle = 0$. Thus, $\langle a \rangle$ decays because energy escapes from the resonator at the rate $2/\tau_e$.

The rate of energy escape is $2\hbar\omega_0/\tau_e \mid \langle a \rangle \mid^2$. This has to equal to the power carried away from the resonator (see Fig. 1). When this solution is time reversed, the power flow is reversed and travels toward the resonator. The buildup rate of energy is now $2/\tau_e$. From (A1)

$$\frac{d}{dt} \mid \langle a \rangle \mid^2 = \frac{2}{\tau_e} \mid \langle a \rangle \mid^2 = \frac{\kappa^2 \mid \langle b \rangle \mid^2}{2/\tau_e} . \quad (A2)$$

But

$$\mid \langle b \rangle \mid^2 = \frac{2}{\tau_e} \langle a \rangle^2 T , \quad (A3)$$

where T is the sampling time for b. Introducing (A3) into (A2) we find

$$\kappa^2 = \frac{2}{\tau_e T} .$$

This is the desired relation.

APPENDIX B: THE FILTERING OF PHASE FLUCTUATIONS

We have stated in the text that a filter of bandwidth B reduces the mean-square fluctuations of the phase by a factor of $4B\tau_p$, where τ_p is the response time of the phase. We prove this statement here.

In Sec. II we find that the spectrum of the phase is Lorentzian of the form

$$H(\Omega) = \frac{1}{1 + \Omega^2 \tau_p^2} , \quad (B1)$$

where τ_p is the decay time of the phase and $H(\Omega)$ is assigned unity amplitude at $\Omega = 0$. The area of $H(\Omega)$ is

$$\int_{-\infty}^{\infty} H(\Omega) d\Omega = \int_{-\infty}^{\infty} \frac{d\Omega}{1 + \Omega^2 \tau_p^2} = \pi/\tau_p . \quad (B2)$$

A filter with flat response over a bandwidth B ($\ll 1/\tau_p$) in Hz passes a portion $4\pi B$ of the (two-sided) spectrum. Thus, the ratio of the total mean-square fluctuations [integral over all Ω of $H(\Omega)$] to the mean-square fluctuations passed by the filter is

$$\frac{4\pi B}{\pi/\tau_p} = 4B\tau_p . \quad (B3)$$

The sampling time T is related to the bandwidth B by the Nyquist criterion

$$B = 1/2T .$$

Thus, the fraction $4B\tau_p$ can be written

$$4B\tau_p = 2\tau_p/T . \quad (B4)$$

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