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Photon interference and correlation effects produced by independent quantum sources

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Interference effects produced by independent quantum sources are investigated, when the state of the field is not describable as a simple mixture of coherent states. The results of classical and quantum-mechanical calculations are compared. Whereas correlation effects are predicted both classically and quantum mechanically when the sources have random phases, there are important differences when the number of atoms is small. In particular, when each source consists of just one atom, the joint probability of detecting two photons at two different points in the receiving plane is found to vanish when the distance between them is an odd number of half fringes. Finally it is shown that when the number of atoms of each source is subject to Poisson fluctuations, one recovers the solution given by classical optics for thermal light, no matter how weak the sources may be on the average.

I. INTRODUCTION

Ever since it was demonstrated experimentally that interference effects are observable with light produced by two independent, unrelated sources,[1–3] the quantum theory of the process has been the subject of attention from time to time.[4–7] Later experiments showing that interference effects exist even at the level of a few photons,[8–10] and even when each emitted photon passes through the interferometer and is absorbed at the detector with high probability before the next one is emitted, brought renewed interest. The interpretation of the experiments and some proposed modifications thereof have again been discussed recently.[11,12]

So far most of the experiments made use of laser sources,[2,3,8–10] which means that the coherent state of the electromagnetic field,[13] or a randomly phased mixture thereof, could be used as an adequate representation of the quantum state. However, with the development of experimental techniques for studying resonance fluorescence from single atoms,[14,15] other quantum states of the field have become accessible for interference experiments, and this raises new possibilities.

In the following we calculate both the probability of photon detection and the joint probability of two-photon detection as a function of position in the region of the interference pattern, when the sources consist of a finite number of atoms in various quantum states. We approach the problem first semiclassically and heuristically and then through the quantized field. We show that there are important differences in the results predicted by the two approaches, although they agree in the limit of a large number of atoms, and also when the number of atoms is not large but subject to Poisson fluctuations.

We find that even when each source consists of a single excited atom, so that there is no definite phase relationship between the sources, interference effects still show up in the two-photon detection probability. In particular, this probability is zero when the two detectors are separated by a distance corresponding to $n+\frac{1}{2}$ interference fringes $(n=0,1,2,\ldots)$. Two photons can therefore never be found at certain pairs of points. This prediction has no classical analog, and its confirmation would represent an interesting test of the quantum theory of the electromagnetic field.

II. INTERFERENCE IN THE CLASSICAL DOMAIN

We start off by discussing the interference problem in terms of a completely classical electromagnetic field. The conclusions will serve as reference for the quantum-mechanical calculations to follow. Although the answers in the two approaches have a good deal in common, they also exhibit some important differences.

We consider two polarized, approximately plane electromagnetic waves at positions $\vec{r}_1$ and $\vec{r}_2$ described by complex scalar amplitudes $V(\vec{r}_1,t)\equiv V_1(t)$ and $V(\vec{r}_2,t)=V_2(t)$ which are superposed at some position $\vec{R}$ (see Fig. 1). For simplicity, the angle between the $\vec{r}_1-\vec{R}$ and $\vec{r}_2-\vec{R}$ vectors will be assumed to be very small. Then the resultant complex field amplitude at $\vec{R}$ at time t is given by[16]

$$V(\vec{R},t)=V_1(t-\tau_1)+V_2(t-\tau_2)\ , \tag{1}$$

where

$$\tau_1=|\vec{r}_1-\vec{R}|/c\ ,$$
$$\tau_2=|\vec{r}_2-\vec{R}/c \tag{2}$$

are transit times for the light. The instantaneous light intensity I at $\vec{R},t$ is then

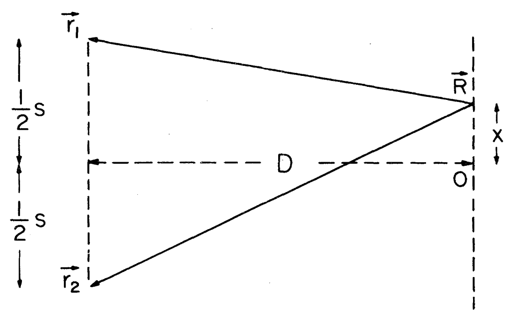

FIG. 1. The geometry for two point sources.

$$\begin{aligned} I(\vec{R},t) &= V^*(\vec{R},t)V(\vec{R},t) \\ &= |V_1(t-\tau_1)|^2+|V_2(t-\tau_2)|^2+V_1^*(t-\tau_1)V_2(t-\tau_2)+\text{c.c.} \\ &= I_1(t-\tau_1)+I_2(t-\tau_2)+V_1^*(t-\tau_1)V_2(t-\tau_2)+\text{c.c.} \end{aligned} \tag{3}$$

For quasimonochromatic light $I_1(t)$ and $I_2(t)$ do not vary much over short time intervals or small path lengths, while $V_1(t)$ and $V_2(t)$ are almost periodic in t with period $2\pi/\omega_0$, where ω_0 is the midfrequency. We can therefore write

$$\begin{aligned} V_1(t-\tau_1) &= V_1(t-\tau_0+\tau_0-\tau_1)\approx V_1(t-\tau_0)\exp[-i\omega_0(\tau_0-\tau_1)] \\ &= \sqrt{I_1(t-\tau_0)}e^{i\phi_1}\exp[-i\omega_0(\tau_0-\tau_1)]\ , \\ V_2(t-\tau_2) &= V_2(t-\tau_0+\tau_0-\tau_2)\approx V_2(t-\tau_0)\exp[-i\omega_0(\tau_0-\tau_1)] \\ &= \sqrt{I_2(t-\tau_0)}e^{i\phi_2}\exp[-i\omega_0(\tau_0-\tau_2)]\ , \end{aligned} \tag{4}$$

where τ_0 is the transit time from $\vec{r}_1$ or $\vec{r}_2$ to the point $O(x=0)$ in Fig. 1, and ϕ_1, ϕ_2 are phase angles that characterize the field at $x=0$ at time t. Equation (3) then becomes

$$I(\vec{R},t)=I_1(t-\tau_0)+I_2(t-\tau_0)+2[I_1(t-\tau_0)I_2(t-\tau_0)]^{1/2}\cos[\omega_0(\tau_1-\tau_2)+\phi_1-\phi_2]\ . \tag{5}$$

Reference to Fig. 1 shows that for small inclinations of the beams

$$c(\tau_1-\tau_2)\approx xs/D$$

so that we can write

$$I(x,t)=[I_1(t-\tau_0)+I_2(t-\tau_0)]\left\{1+2\left[\left(\frac{I_1(t-\tau_0)}{I_2(t-\tau_0)}\right)^{1/2}+\left(\frac{I_2(t-\tau_0)}{I_1(t-\tau_0)}\right)^{1/2}\right]^{-1}\cos\left[\frac{2\pi xs}{\lambda_0 D}+\phi_1-\phi_2\right]\right\}\ . \tag{6}$$

The light intensity therefore exhibits a periodic variation with position x, with periodicity $\lambda_0 D/s$, that we refer to as interference, over time intervals for which the phase difference $\phi_1-\phi_2$ remains reasonably constant. If the two light beams are mutually highly coherent, then $\phi_1-\phi_2$ remains constant indefinitely. But even if the two beams are derived from separate sources and have no fixed phase relationship, $\phi_1-\phi_2$ will not change appreciably over time intervals short compared with the coherence time, or the reciprocal bandwidth. Over such short times Eq. (6) predicts interference even for independent light beams, and this has been observed.[3] The "visibility" σ_{12} of the interference pattern is given by the coefficient of the cosine in Eq. (6), and for almost equal light intensities it is close to unity even for independent light beams. However, if we calculate the average light intensity from Eqs. (5) or (6) by averaging over the ensemble of all realizations, we find that when the phase difference $\phi_1-\phi_2$ is random

$$\langle I(x,t)\rangle=\langle I_1(t-\tau_0)\rangle+\langle I_2(t-\tau_0)\rangle$$

because the cosine term averages to zero, and this gives no inkling of interference effects. The ensemble average of the light intensity is therefore not the most relevant quantity in this case.

Nevertheless, we can demonstrate that interference effects are present even in an average sense. For this purpose we consider two neighboring points $\vec{R},\vec{R}'$ in the receiving plane, with x coordinates x,x', and we use Eq. (5) to calculate $I(x,t)$ and $I(x',t)$. Finally, we evaluate the average product or the two-point correlation function $\langle I(x,t)I(x',t)\rangle$ under the assumption that the two light beams are independent and the phase difference $\phi_1-\phi_2$ is random. We then obtain

$$\begin{aligned}\langle I(x,t)I(x',t)\rangle &= \langle I_1^2(t-\tau_0)\rangle + \langle I_2^2(t-\tau_0)\rangle + 2\langle I_1(t-\tau_0)\rangle\langle I_2(t-\tau_0)\rangle \\ &\quad + 2\langle I_1(t-\tau_0)\rangle\langle I_2(t-\tau_0)\rangle \cos\omega_0(\tau_1-\tau_1'-\tau_2+\tau_2') \\ &= \langle I_1^2(t-\tau_0)\rangle + \langle I_2^2(t-\tau_0)\rangle + 2\langle I_1(t-\tau_0)\rangle\langle I_2(t-\tau_0)\rangle \\ &\quad + 2\langle I_1(t-\tau_0)\rangle\langle I_2(t-\tau_0)\rangle \cos[2\pi(x-x')s/\lambda_0 D] \ . \end{aligned} \tag{7}$$

Once again we notice a periodic variation with position, with the same periodicity $L=\lambda_0 D/s$ as before, showing that interference effects are present even though the phase difference $\phi_1-\phi_2$ is random. The correlation is partly a reflection of the random motion of the interference pattern, and it offers an alternative procedure for establishing the existence of interference effects.[8-10] The relative modulation amplitude is given by

$$\rho_{12}=2\langle I_1\rangle\langle I_2\rangle/[\langle I_1^2\rangle+\langle I_2^2\rangle+2I_1\rangle\langle I_2\rangle] \tag{8}$$

when the light beams are stationary, so that single-time averages are time-independent. Unlike the relative modulation amplitude obtained from Eq. (6), this has a maximum possible value of $\frac{1}{2}$ when $\langle I_1\rangle=\langle I_2\rangle$ in the absence of any fluctuations of $I_1(t)$ and $I_2(t)$, in which case Eq. (7) yields

$$\langle I(x,t)I(x',t)\rangle=\langle I(x,t)\rangle\langle I(x',t)\rangle \times[1+\tfrac{1}{2}\cos 2\pi(x-x')/L] \ . \tag{9}$$

When $I_1(t)$ and $I_2(t)$ fluctuate, the relative modulation amplitude is even smaller. For example, if the light beams obey thermal statistics, then $\langle I_1^2\rangle=2\langle I_1\rangle^2$, $\langle I_2^2\rangle=2\langle I_2\rangle^2$, and the maximum value of ρ_{12} is $\frac{1}{3}$, which is reached when $\langle I_1\rangle=\langle I_2\rangle$. Equation (7) then gives

$$\langle I(x,t)I(x',t)\rangle=\tfrac{3}{2}\langle I(x,t)\rangle\langle I(x',t)\rangle \times[1+\tfrac{1}{3}\cos 2\pi(x-x')/L] \ . \tag{10}$$

The correlation is smallest when

$$|x-x'|=(n+\tfrac{1}{2})L(n=0,1,2,\dots) \ ,$$

but it can never vanish, unlike the light intensity $I(x,t)$ given by Eq. (6), which can vanish for certain positions x.

III. INTERFERENCE EFFECTS IN LOCALIZED PHOTON STATES

We now turn to the quantum treatment of the phenomenon. We start by showing that interference effects are produced also in nonclassical states of the electromagnetic field, and even when only a few photons are present. However, the customary Fock states having definite photon occupation numbers in certain modes are inappropriate for our purpose, because they correspond to photons distributed over all space, whereas we wish to emphasize effects produced by strongly localized sources. States in which the modes are defined in a more general way have been discussed.[17] However, we shall find it convenient to make use of the quasilocalized photon states of the general form[18,19]

$$|\Phi_N\rangle\equiv\hat{V}_{i_1}^\dagger(\vec{r}_1,t_1)\hat{V}_{i_2}^\dagger(\vec{r}_2,t_2)\cdots\hat{V}_{i_N}^\dagger(\vec{r}_N,t_N)|\text{vac}\rangle \tag{11}$$

in which $|\text{vac}\rangle$ is the vacuum state of the field, and the $\hat{\vec{V}}^\dagger(\vec{r},t)$ are configuration space creation operators, that can be given the plane-wave mode expansion

$$\hat{\vec{V}}^\dagger(\vec{r},t)=\frac{1}{L^{3/2}}\sum_{\{\vec{k}\},s}\hat{a}_{\vec{k}s}^\dagger(t)\vec{\epsilon}_{\vec{k}s}^{\,*}e^{-i\vec{k}\cdot\vec{r}} \ . \tag{12}$$

Here $\{\vec{k}\}$ stands for a particular set of wave vectors $\vec{k}$ and s is a polarization index. The state $|\Phi_N\rangle$ may be regarded as an N-photon state, in which one photon is created approximately at position $\vec{r}_1$ at time t_1, one at $\vec{r}_2$ at time t_2, etc., provided no attempt is made to define the position to better than several optical wavelengths or the time to better than several optical periods.[19] For example, an excited two-level atom in the process of decay can be regarded as giving rise to the one-photon state $\hat{\vec{V}}^\dagger(\vec{r},t)\,|\,\text{vac}\rangle$, when $\{\vec{k}\}$ stands for the set of field modes to which the atom couples. $\{\vec{k}\}$ may include only a narrow band of frequencies centered on the atomic frequency ω_0.

We now consider a set of N atoms located around some point $\vec{r}$, and another group of M atoms around some point $\vec{r}\,'$ (see Fig. 2). These constitute two "point sources" of light separated by a distance s. If the atoms of one group are separated by less than a wavelength, we are justified in associating them with the same position coordinate $\vec{r}$ or $\vec{r}\,'$. We suppose that the atoms have been prepared at time $t=0$ in a product state, each of which is characterized by the same Bloch vector $|\,\theta,\phi\rangle$ with polar angle θ and azimuthal angle ϕ, while the field is in the vacuum state $|\,\text{vac}\rangle$. We then write for the initial state of the combined system

$$|\,\text{vac}\rangle\otimes\prod_{n=1}^{N}|\,\theta,\phi\rangle_n\otimes\prod_{m=1}^{M}|\,\theta,\phi'\rangle_m\ .$$

We use the index n to label atoms belonging to one source, and m those belonging to the other. The method of preparing this state will not concern us but it might be done with a short coherent pulse of light, for example. For simplicity we have assumed that the atomic excitation described by the polar angle θ is similar for all atoms. However, the phase angles ϕ of the atoms of one source may differ from the phase angles ϕ' of the others.

If the atoms interact with the field through an electric dipole interaction, then after a short time δt the state in the interaction picture will be of the form

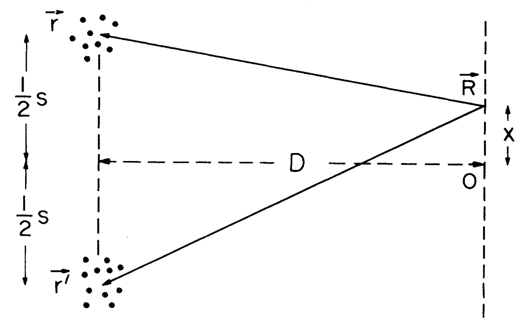

FIG. 2. The geometry for distributed atomic sources.

$$|\,\Psi\rangle=\exp(-i\hat{H}_I\delta t/\hbar)\,|\,\text{vac}\rangle\otimes\prod_{n=1}^{N}|\,\theta,\phi\rangle_n\otimes\prod_{m=1}^{M}|\,\theta,\phi'\rangle \tag{13}$$

with

$$\hat{H}_I=-\vec{\mu}\cdot\left[\sum_{n=1}^{N}\hat{\vec{E}}^{(-)}(\vec{r},0)\hat{b}_n(0)+\sum_{m=1}^{M}\hat{\vec{E}}^{(-)}(\vec{r}\,',0)\hat{b}_m(0)\right]+\text{H.c.} \tag{14}$$

Here $\vec{\mu}$ is the transition dipole moment of each atom, $\hat{b}_n(t)$ and $\hat{b}_n^\dagger(t)$ are atomic lowering and raising operators for atom n, and $\hat{\vec{E}}^{(+)}(\vec{r},t)$ and $\hat{\vec{E}}^{(-)}(\vec{r},t)$ are positive and negative frequency parts of the electric field. Because of the quasimonochromatic character of the field excitation, we may take $\hat{\vec{E}}^{(-)}(\vec{r},t)$ to be proportional to $\hat{\vec{V}}^\dagger(\vec{r},t)$ defined by Eq. (12), and we may write up to terms of order $(\delta t)^2$,

$$\begin{aligned}|\,\Psi\rangle=\Bigg[&1+K_j\left[\sum_{n=1}^{N}\hat{V}_j^\dagger(\vec{r},0)\hat{b}_n(0)+\sum_{m=1}^{M}\hat{V}_j^\dagger(\vec{r}\,',0)\hat{b}_m(0)+\text{H.c.}\right]\\&+\tfrac{1}{2}K_iK_j\left[\sum_{n=1}^{N}\sum_{n'=1}^{N}\hat{V}_i^\dagger(\vec{r},0)\hat{V}_j^\dagger(\vec{r},0)\hat{b}_n(0)\hat{b}_{n'}(0)+\sum_{m=1}^{M}\sum_{m'=1}^{M}\hat{V}_i^\dagger(\vec{r}\,',0)\hat{V}_j^\dagger(\vec{r}\,',0)\hat{b}_m(0)\hat{b}_{m'}(0)\right.\\&\left.\qquad+2\sum_{n=1}^{N}\sum_{m=1}^{M}\hat{V}_i^\dagger(\vec{r},0)\hat{V}_j^\dagger(\vec{r}\,',0)\hat{b}_n(0)\hat{b}_m(0)+\text{H.c.}\right]+\cdots\Bigg]\\&\times|\,\text{vac}\rangle\otimes\prod_{n=1}^{N}|\,\theta,\phi\rangle_n\otimes\prod_{m=1}^{M}|\,\theta,\phi'\rangle\ ,\end{aligned}$$

where

$$K_j\equiv(\vec{\mu})_j(\hbar\omega_0/2\epsilon_0)^{1/2}\delta t/\hbar\ .$$

Now the atomic state $|\,\theta,\phi\rangle_n$ can be expanded in terms of lower and upper states $|\,1\rangle_n, |\,2\rangle_n$ in the form[20]

$$|\,\theta,\phi\rangle_n=\sin\tfrac{1}{2}\theta\, e^{-1/2i\phi}\,|\,1\rangle_n+\cos\tfrac{1}{2}\theta\, e^{1/2i\phi}\,|\,2\rangle_n\ , \tag{15}$$

so that

$$
\begin{aligned}
(\Psi\rangle = {} & |\text{vac}\rangle \otimes \prod_{n=1}^{N} |\theta,\phi\rangle_n \otimes \prod_{m=1}^{M} |\theta,\phi'\rangle_m \\
& + K_i \hat{V}_i^\dagger(\vec{r},0)\,|\text{vac}\rangle \otimes \cos\tfrac{1}{2}\theta\, e^{-1/2 i\phi} \sum_{n=1}^{N} |1_n\rangle \prod_{n'\neq n}^{N} |\theta,\phi\rangle_{n'} \prod_{m=1}^{M} |\theta,\phi'\rangle_m \\
& + K_i \hat{V}_i^\dagger(\vec{r}\,',0)\,|\text{vac}\rangle \otimes \cos\tfrac{1}{2}\theta\, e^{-1/2 i\phi'} \sum_{m=1}^{M} |1_m\rangle \prod_{m'\neq m}^{M} |\theta,\phi'\rangle_{m'} \prod_{n=1}^{N} |\theta,\phi\rangle_n \\
& + \tfrac{1}{2} K_i K_j \cos^2\tfrac{1}{2}\theta \Bigg[\hat{V}_i^\dagger(\vec{r},0)\hat{V}_j^\dagger(\vec{r},0)\,|\text{vac}\rangle \otimes e^{-1/2 i\phi} \sum_{n\neq n'}^{N}\sum^{N} |1\rangle_n |1\rangle_{n'} \prod_{n''\neq n,n'}^{N} |\theta,\phi\rangle_{n''} \prod_{m=1}^{M} |\theta,\phi'\rangle_m \\
& \qquad + \hat{V}_i^\dagger(\vec{r}\,',0)\hat{V}_j^\dagger(\vec{r}\,',0)\,|\text{vac}\rangle \\
& \qquad \otimes e^{-1/2 i\phi'} \sum_{m\neq m'}^{M}\sum^{M} |1\rangle_m |1\rangle_{m'} \prod_{m''\neq m,m'}^{M} |\theta,\phi'\rangle_{m''} \prod_{n=1}^{N} |\theta,\phi\rangle_n \\
& \qquad + 2\hat{V}_i^\dagger(\vec{r},0)\hat{V}_j^\dagger(\vec{r}\,',0)\,|\text{vac}\rangle \\
& \qquad \otimes \exp[-\tfrac{1}{2} i(\phi+\phi')] \sum_{n=1}^{N}\sum_{m=1}^{M} |1\rangle_n |1\rangle_m \prod_{n'\neq n}^{N} |\theta,\phi\rangle_{n'} \prod_{m'\neq m}^{M} |\theta,\phi'\rangle_{m'} \Bigg] + \cdots \, . \qquad (16)
\end{aligned}
$$

We now wish to detect the field at some position $\vec{R}$ with the help of a photoelectric detector (see Fig. 2). For this purpose we equip the detector with an aperture of width δx and height δy (the points $\vec{r},\vec{r}\,',\vec{R}$ are taken to define the xz plane), and we assume that the distance D is so great compared with the source separation s that the light is incident almost normally on the photodetector. In a short time δT the photodetector then detects photons by effectively sweeping out a volume $\mathscr{V}$ of space in the form of a cylinder whose base is the photocathode area $\delta x\,\delta y$ and whose height is $c\,\delta T$. Now it has been shown[19] that the operator

$$\hat{n}_{\mathscr{V},t} \equiv \int_{\mathscr{V}} \hat{\vec{V}}^\dagger(\vec{r},t)\cdot\hat{\vec{V}}(\vec{r},t)\,d\vec{r} \qquad (17)$$

represents the number of photons within the volume $\mathscr{V}$ at time t in a certain sense, provided the linear dimensions of $\mathscr{V}$ are large compared with optical wavelengths. Moreover, the one-photon state $\hat{\vec{V}}^\dagger(\vec{r},t)\,|\text{vac}\rangle$ is an eigenstate of $\hat{n}_{\mathscr{V},t}$ with eigenvalue 1 if $\vec{r}\in\mathscr{V}$, and with eigenvalue 0 if $\vec{r}\notin\mathscr{V}$.[19] A slightly different operator for the photon number in a volume $\mathscr{V}$ has recently been introduced by Cook,[21] but we shall not use it here. The probability that a photon is detected at $\vec{R}$ at time t within δT is then proportional to the expectation $\langle \hat{n}_{\mathscr{V},t}\rangle$. We shall take t to be the propagation time of the light from the source to the detector. The measurement interval δT will not show up explicitly in the answer if we ensure that every photon produced by the source reaches the detector. Also, the joint probability for a photon to be detected at position $\vec{R}_1$ at time t_1 and one at position $\vec{R}_2$ at time t_2 is proportional to the normally ordered correlation $\langle :\hat{n}_{\mathscr{V}_1 t_1}\hat{n}_{\mathscr{V}_2 t_2}:\rangle$, where $\mathscr{V}_2$ is the correspondingly small volume located at position $\vec{R}_2$.

In calculating these expectations we shall have occasion to use the following free-field commutation rule, which holds so long as the linear dimensions of $\mathscr{V}$ are large compared with the wavelength,[19]

$$\hat{\vec{Q}}(\vec{r},0;\mathscr{V},t) \equiv [\hat{\vec{V}}(\vec{r},0),\hat{n}_{\mathscr{V},t}] = \frac{1}{L^{3/2}} \sum_{\{\vec{k}\},s} \hat{a}_{\vec{k}s}\vec{\epsilon}_{\vec{k}s} e^{i\vec{k}\cdot\vec{r}} U(\vec{r}+c\vec{k}t/k,\mathscr{V}) \, , \qquad (18)$$

where $U(\vec{r},\mathscr{V})$ is the discontinuous function that is unity when $\vec{r}\in\mathscr{V}$, and zero otherwise. It follows that if $\vec{r},0$ and $\mathscr{V},t$ are disjoint space-time regions then the commutator vanishes, whereas it reproduces $\hat{\vec{V}}(\vec{r},0)$ if $t=0$ and $\vec{r}\in\mathscr{V}$. However, in other cases the answer is generally less simple.

We now use these results to calculate the photon detection probability at $\vec{R}$ at time t to the second order in δt or K. To this order the only contributions to $|\Psi\rangle$ come from one-photon states in Eq. (16), and we find

$$\langle\overline{\Psi}\,|\,\hat{n}_{\mathscr{V},t}\,|\,\overline{\Psi}\rangle = K_iK_j\cos^2\tfrac{1}{2}\theta\{[N+N(N-1)\sin^2\tfrac{1}{2}\theta]\langle\text{vac}\,|\,\hat{V}_j(\vec{r},0)\hat{n}_{\mathscr{V},t}\hat{V}_i^\dagger(\vec{r},0)\,|\,\text{vac}\rangle$$

$$+[M+M(M-1)\sin^2\tfrac{1}{2}\theta]\langle\text{vac}\,|\,\hat{V}_j(\vec{r}\,',0)\hat{n}_{\mathscr{V},t}\hat{V}_i^\dagger(\vec{r}\,',0)\,|\,\text{vac}\rangle$$

$$+NM\sin^2\tfrac{1}{2}\theta e^{i(\phi-\phi')}\langle\text{vac}\,|\,\hat{V}_j(\vec{r}\,',0)\hat{n}_{\mathscr{V},t}\hat{V}_i^\dagger(\vec{r},0)\,|\,\text{vac}\rangle+\text{c.c.}\}\;. \tag{19}$$

The matrix elements of the field are easily evaluated with the help of Eq. (18). We obtain

$$\langle\text{vac}\,|\,\hat{V}_j(\vec{r}\,',0)\hat{n}_{\mathscr{V},t}\hat{V}_i^\dagger(\vec{r},0)\,|\,\text{vac}\rangle = D_{ji}(\vec{r}\,',0;\vec{r},0;\mathscr{V},t)\;, \tag{20}$$

where $D_{ji}(\vec{r}\,',0;\vec{r},0;\mathscr{V},t)$ is given by

$$D_{ji}(\vec{r}\,',0;\vec{r},0;\mathscr{V},t) \equiv [\hat{Q}_j(\vec{r}\,',0;\mathscr{V},t),\hat{V}_i^\dagger(\vec{r},0)]$$

$$=\frac{1}{(2\pi)^3}\int_{\{\vec{k}\}} d^3k(1-k_jk_i/k^2)\exp[i\vec{k}\cdot(\vec{r}-\vec{r}\,')]U(\vec{r}\,'+c\vec{k}t/k,\mathscr{V})\;, \tag{21}$$

as is shown in Appendix A. If $ct=|\vec{R}-\vec{r}\,'|$, and if the lines $\vec{R}-\vec{r}\,',\vec{R}-\vec{r}$ are almost perpendicular to the line $\vec{r}-\vec{r}\,'$ and to the induced dipole moment vector $\vec{\mu}$, then it can be seen by reference to Fig. 2 that $\vec{k}\cdot(\vec{r}-\vec{r}\,')$ in the exponent can be replaced by

$$-ks^2/2D+ksx/D\;,$$

because the factor $U(\vec{r}\,'+c\vec{k}t/k,\mathscr{V})$ restricts the direction of $\vec{k}$ to lie almost along $\vec{R}-\vec{r}\,'$. Hence

$$K_iK_jD_{ji}(\vec{r}\,',0;\vec{r},0;\mathscr{V},t)\approx\frac{K^2}{(2\pi)^3}\delta\Omega\int_{\{k\}}k^2dk\,\exp i(-\tfrac{1}{2}ks^2/D+ksx/D)\;. \tag{22}$$

The element of solid angle subtended by the detector aperture at $\vec{r}$ or $\vec{r}\,'$ is

$$\delta\Omega\equiv\delta y\,\delta x/D^2\;. \tag{23}$$

If we use these results in Eq. (19) we readily obtain

$$\langle\overline{\Psi}\,|\,\hat{n}_{\mathscr{V},t}\,|\,\overline{\Psi}\rangle = K^2\frac{\delta x\,\delta y}{D^2}\cos^2\tfrac{1}{2}\theta\frac{1}{(2\pi)^3}\int_{\{k\}}k^2dk\left[N+M+[N(N-1)+M(M-1)]\sin^2\tfrac{1}{2}\theta\right.$$

$$\left.+2NM\sin^2\tfrac{1}{2}\theta\cos\left[\frac{ksx}{D}-\frac{ks^2}{2D}+\phi'-\phi\right]\right]\;. \tag{24}$$

This shows that, as long as the phase difference $\phi'-\phi$ is fixed, and provided the range of $\{k\}$, which determines the coherence time of the light, is not too great, there is a periodic variation of the photon detection probability with position x, that we call "interference fringes." This result corresponds to Eq. (6) which was derived classically. The fringe spacing is given by

$$L\approx\frac{2\pi D}{k_0 s}=\frac{\lambda_0}{s/D} \tag{25}$$

exactly as before. If $\phi'-\phi$ is random and uniformly distributed over 0 to 2π, then the interference term averages to zero. However, even if $\phi'-\phi=0$, the interference term vanishes if $\theta=0$, i.e., if the atoms are in the fully excited state. This conclusion has no classical counterpart. The reason for it is that the fully excited atomic state has no well-defined phase, so that cross-terms vanish when we calculate the quantum expectation. The term proportional to $[N(N-1)+M(M-1)]$ in Eq. (24) is associated with cooperative atomic emissions from each source or super-radiance. It is a consequence of the fact that all the atoms of one source are initially in phase.

In physical terms, we may attribute the appearance of interference fringes when $\theta\neq0$, despite the fact that the sources are independent, to the fact that the two sources are unresolved at the detector. For, in determining the position of the photon at the detector to a precision $\delta x<L$, we can fix the x component of the momentum to no better than $h/L=(h/\lambda_0)(s/D)$, which means that it is impossi-

ble to tell from which source the photon was emitted. There are, therefore, two probability amplitudes which interfere. However, if the atoms start from the fully excited state, it is possible, in principle, to determine by an examination of the sources from which of the two any photon was emitted. One of the two probability amplitudes then vanishes, and the interference fringes disappear.

Whenever the interference pattern is present, we see from Eq. (24) that its visibility, or the relative modulation amplitude σ_{12}, is given by

$$\sigma_{12}=\frac{2NM\sin^2\frac{1}{2}\theta}{N+M+[N(N-1)+M(M-1)]\sin^2\frac{1}{2}\theta} \ . \tag{26}$$

The θ dependence of σ_{12} has no classical counterpart. When N and M are equal and large, $\sigma_{12}\approx 1$ irrespective of θ, except for the excited state $\theta\approx 0$. In the other extreme, when each source consists of but a single atom, $N=1=M$, and

$$\sigma_{12}=\sin^2\tfrac{1}{2}\theta \ ,$$

which varies between 0 when the atoms are fully excited ($\theta=0$) and 1 when they are almost unexcited ($\theta\approx\pi$). Although the visibility is greatest in the latter case, because the atoms then behave somewhat like classical oscillators, the photon detection probability is then close to zero because $\cos^2\frac{1}{2}\theta\approx 0$.

IV. TWO-PHOTON CORRELATIONS

Although the interference term in Eq. (24) vanishes when we average over phases and the phase difference $\phi'-\phi$ is random, or when $\theta=0$ and the atoms are fully excited, again it must not be thought that all interference effects then disappear. The effects are still there, but they are reflected in expectations of higher-order operator products, which require correlation measurements of two or more photons to be revealed. We now calculate the two-photon detection probability.

For this purpose we consider two detectors located at positions $\vec{R}_1$ and $\vec{R}_2$ in the receiving plane (the xy plane), with x coordinates x_1 and x_2. We denote by $\mathscr{V}_1$ and $\mathscr{V}_2$ the volumes ($\delta x\, \delta y\, c\delta T$) that are effectively sampled by the two detectors at time t, and by $\hat{n}_{\mathscr{V}_1 t}, \hat{n}_{\mathscr{V}_2 t}$ the associated localized photon number operators. The joint probability of two-photon detection is then proportional to the normally ordered correlation[13] $\langle :\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\rangle$ in the quantum state $|\overline{\Psi}\rangle$ given by Eq. (16).

It is apparent that nonvanishing contributions to $\langle :\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\rangle$ can come only from the two-photon and higher-order states in Eq. (16). We readily find to order $(\delta t)^4$ or K^4, after multiplying out and evaluating the atomic products,

$$\begin{aligned}
\langle\overline{\Psi}\,|\,:\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\,|\,\overline{\Psi}\rangle = {} & \tfrac{1}{4}K_iK_jK_pK_q\cos^4\tfrac{1}{2}\theta \\
& \times\{(2N^{(2)}+4N^{(3)}\sin^2\tfrac{1}{2}\theta+N^{(4)}\sin^4\tfrac{1}{2}\theta) \\
& \times\langle\text{vac}\,|\,\hat{V}_i(\vec{r},0)\hat{V}_j(\vec{r},0):\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\hat{V}_p^\dagger(\vec{r},0)\hat{V}_q^\dagger(\vec{r},0)\,|\,\text{vac}\rangle \\
& +(2M^{(2)}+4M^{(3)}\sin^2\tfrac{1}{2}\theta+M^{(4)}\sin^4\tfrac{1}{2}\theta) \\
& \times\langle\text{vac}\,|\,\hat{V}_i(\vec{r}\,',0)\hat{V}_j(\vec{r}\,',0):\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\hat{V}_p^\dagger(\vec{r}\,',0)\hat{V}_q^\dagger(\vec{r}\,',0)\,|\,\text{vac}\rangle \\
& +4[NM+(N^{(2)}M+NM^{(2)})\sin^2\tfrac{1}{2}\theta+N^{(2)}M^{(2)}\sin^4\tfrac{1}{2}\theta] \\
& \times\langle\text{vac}\,|\,\hat{V}_i(\vec{r},0)\hat{V}_j(\vec{r}\,',0):\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\hat{V}_p^\dagger(\vec{r}\,',0)\hat{V}_q^\dagger(\vec{r},0)\,|\,\text{vac}\rangle \\
& +N^{(2)}M^{(2)}\sin^4\tfrac{1}{2}\theta\, e^{2i(\phi-\phi')} \\
& \times\langle\text{vac}\,|\,\hat{V}_i(\vec{r},0)\hat{V}_j(\vec{r},0):\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\hat{V}_p^\dagger(\vec{r}\,',0)\hat{V}_q^\dagger(\vec{r}\,',0)\,|\,\text{vac}\rangle+\text{c.c.} \\
& +2M(2N^{(2)}\sin^2\tfrac{1}{2}\theta+N^{(3)}\sin^4\tfrac{1}{2}\theta)e^{i(\phi-\phi')} \\
& \times\langle\text{vac}\,|\,\hat{V}_i(\vec{r},0)\hat{V}_j(\vec{r},0):\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\hat{V}_p^\dagger(\vec{r},0)\hat{V}_q^\dagger(\vec{r}\,',0)\,|\,\text{vac}\rangle+\text{c.c.} \\
& +2N(2M^{(2)}\sin^2\tfrac{1}{2}\theta+M^{(3)}\sin^4\tfrac{1}{2}\theta)e^{i(\phi'-\phi)} \\
& \times\langle\text{vac}\,|\,\hat{V}_i(\vec{r}\,',0)\hat{V}_j(\vec{r}\,',0):\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\hat{V}_p^\dagger(\vec{r},0)\hat{V}_q^\dagger(\vec{r}\,',0)\,|\,\text{vac}\rangle+\text{c.c.}\} \ .
\end{aligned} \tag{27}$$

In writing this equation we have used the abbreviation

$$N^{(r)} \equiv N(N-1)(N-2)\ldots(N-r+1) .$$

It is worth noting that only the first three terms in this equation survive if the phase difference $\phi-\phi'$ is random; the other terms all average to zero. Moreover, irrespective of whether the phases are random, only the third term makes a nonzero contribution when each source consists of a single atom ($N=1=M$).

The matrix elements of the field can again be evaluated by repeated application of the commutation rules. It is shown in Appendix B that when the lines joining the sources to the detectors are almost perpendicular to the xy plane

$$\begin{aligned}
K_iK_jK_pK_q\langle \text{vac} \,|\, \hat{V}_i(\vec{r}_1,0)\hat{V}_j(\vec{r}_2,0)\!:\!\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}\!:\!\hat{V}_p^\dagger(\vec{r}_3,0)\hat{V}_q^\dagger(\vec{r}_4,0) \,|\, \text{vac}\rangle
&= K^4\frac{1}{(2\pi)^6}\int_{\{\vec{k}\}} d^3k \int_{\{\vec{k}'\}} d^3k' (\exp\{i[\vec{k}\cdot(\vec{r}_1-\vec{r}_4)+\vec{k}'\cdot(\vec{r}_2-\vec{r}_3)]\} \\
&\quad \times U(\vec{r}_4+c\vec{k}t/k,\mathscr{V}_1)U(\vec{r}_2+c\vec{k}'t/k',\mathscr{V}_2) \\
&\quad +\exp\{i[\vec{k}\cdot(\vec{r}_1-\vec{r}_3)+\vec{k}'\cdot(\vec{r}_2-\vec{r}_4)]\} \\
&\quad \times U(\vec{r}_3+c\vec{k}t/k,\mathscr{V}_1)U(\vec{r}_2+c\vec{k}'t/k',\mathscr{V}_2) \\
&\quad +\exp\{i[\vec{k}\cdot(\vec{r}_2-\vec{r}_4)+\vec{k}'\cdot(\vec{r}_1-\vec{r}_3)]\} \\
&\quad \times U(\vec{r}_4+c\vec{k}t/k,\mathscr{V}_1)U(\vec{r}_1+c\vec{k}'t/k',\mathscr{V}_2) \\
&\quad +\exp\{i[\vec{k}\cdot(\vec{r}_2-\vec{r}_3)+\vec{k}'\cdot(\vec{r}_1-\vec{r}_4)]\} \\
&\quad \times U(\vec{r}_3+c\vec{k}t/k,\mathscr{V}_1)U(\vec{r}_1+c\vec{k}'t/k',\mathscr{V}_2)) .
\end{aligned} \tag{28}$$

If we apply this rule to each of the matrix elements in Eq. (27) and refer to Fig. 2 for the evaluation of $\vec{k}\cdot(\vec{r}-\vec{r}')$ as before, we obtain for quasimonochromatic sources, after discarding the terms in $\phi-\phi'$, and for suitable times t,

$$\begin{aligned}
\langle\bar{\Psi}\,|\,:\hat{n}_{\mathscr{V}_1 t}\hat{n}_{\mathscr{V}_2 t}:\bar{\Psi}\rangle &= K^4\left[\frac{\delta x\,\delta y}{D^2}\right]^2\cos^4\tfrac{1}{2}\theta\frac{1}{(2\pi)^6}\int_{\{k\}}k^2dk\int_{\{k'\}}k'^2dk' \\
&\quad \times\{2(N^{(2)}+M^{(2)})+4(N^{(3)}+M^{(3)})\sin^2\tfrac{1}{2}\theta+(N^{(4)}+M^{(4)})\sin^4\tfrac{1}{2}\theta \\
&\quad +2[NM+(N^{(2)}M+M^{(2)}N)\sin^2\tfrac{1}{2}\theta+N^{(2)}M^{(2)}\sin^4\tfrac{1}{2}\theta] \\
&\quad \times[1+\cos ks(x_1-x_2)/D]\} .
\end{aligned} \tag{29}$$

We note that this correlation function or joint probability exhibits a periodic variation with position, with the same periodicity $\lambda_0 D/s$ as the interference pattern described by Eq. (24). Moreover, the periodic variation is there even if the atoms start off in the fully excited state ($\theta=0$), and even if the phase difference $\phi-\phi'$ is random, when the interference term in Eq. (24) averages to zero. The periodic correlation may be regarded as reflecting the random motion of the interference pattern, and Eq. (29) can be compared with the classical equation (7). Photoelectric measurements of this correlation were used in the experiments of Refs. 8 and 9 to establish the existence of interference effects.

From Eq. (29) the relative modulation amplitude ρ_{12} is given by

$$\rho_{12}=\frac{NM+(N^{(2)}M+M^{(2)}N)\sin^2\tfrac{1}{2}\theta+N^{(2)}M^{(2)}\sin^4\tfrac{1}{2}\theta}{NM+N^{(2)}+M^{(2)}+(N^{(2)}M+M^{(2)}N+2N^{(3)}+2M^{(3)})\sin^2\tfrac{1}{2}\theta+(N^{(2)}M^{(2)}+\tfrac{1}{2}N^{(4)}+\tfrac{1}{2}M^{(4)})\sin^4\tfrac{1}{2}\theta}\ . \tag{30}$$

Unlike the corresponding classical expression given by Eq. (8), this can be as large as unity when $N=1=M$, although it tends to the classical value $\frac{1}{2}$ when $N=M$ and both numbers are large.

The case when there is just one atom at each source is exceptional and particularly interesting. We find from Eq. (29) on setting $N=1=M$,

$$\langle\Psi|:\hat{n}_{\mathscr{Y}_1 t}\hat{n}_{\mathscr{Y}_2 t}:\Psi\rangle=2K^4\left[\frac{\delta x\,\delta y}{D^2}\right]^2\cos^4\tfrac{1}{2}\theta\frac{1}{(2\pi)^6}\int_{[k]}k^2dk\int_{[k']}k'^2dk'[1+\cos ks(x_1-x_2)/D]\ , \tag{31}$$

and we note that this vanishes whenever $|x_1-x_2|=(n+\frac{1}{2})L$ $(n=0,1,2,\ldots)$, where $L=\lambda_0 D/s$ is the interference fringe spacing. In other words, it is impossible to detect the two emitted photons at two points separated by an odd number of half fringes. Needless to say, this conclusion has no classical analogy. It is a reflection of the fact that one photon must have come from one source and one from the other, but we cannot tell which came from which. There are therefore only two probability amplitudes that interfere destructively. However, as soon as one or the other source has two or more atoms, the two detected photons could have come from the same source and the additional probability amplitudes lower the relative interference effect.

V. Randomly Distributed Atoms

Although our simple model is able to bring out certain essential differences between the quantum mechanical and the classical situations, it is not very realistic. In practice the atoms of the sources are likely to be distributed over regions that are large compared with the wavelength, and because of variations in atomic position, the phases of the different atoms are likely to be different. Moreover, with moving atoms even the number of atoms that constitutes a source may fluctuate from one trial to the next. Although the previous calculation can, in principle, be adapted to this more general situation, it becomes cumbersome. It is a little simpler to express the electromagnetic field $\hat{\vec{E}}(\vec{R},t)$ at position $\vec{R}$ in the far field of the atoms directly in terms of atomic variables and to calculate the expectations from these.

We again consider two sources of N and M identical two-level atoms, located at positions $\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N$ and $\vec{r}\,'_1,\vec{r}\,'_2,\ldots,\vec{r}\,'_M$, respectively. The starting point of our calculation this time is the expression for the positive frequency part $\hat{\vec{E}}^{(+)}(\vec{R},t)$ of the electromagnetic field at position $\vec{R}$ in the far field of an atom at $\vec{r}$,[22]

$$\hat{\vec{E}}^{(+)}(\vec{R},t)=\frac{\omega_0^2}{4\pi\epsilon_0 c^2}\left[\frac{\vec{\mu}}{|\vec{R}-\vec{r}|}-\frac{\vec{\mu}\cdot(\vec{R}-\vec{r})(\vec{R}-\vec{r})}{|\vec{R}-\vec{r}|^3}\right]\hat{b}\left[t-\frac{|\vec{R}-\vec{r}|}{c}\right]+\hat{\vec{E}}^{(+)}_{\text{free}}(R,t)\ . \tag{32}$$

$\hat{\vec{E}}_{\text{free}}(\vec{R},t)$ is the positive frequency part of the free field, in the absence of the atom. Equation (32) is nothing but the quantum-mechanical version of the field of a classical oscillating dipole. We may readily adapt the equation to the problem in which there are $N+M$ atoms at different positions, and we write

$$\hat{\vec{E}}^{(+)}(\vec{R},t)=\sum_{n=1}^{N}\vec{C}^{(n)}\hat{b}_n(t-|\vec{R}-\vec{r}_n|/c)+\sum_{m=1}^{M}\vec{C}^{\prime(m)}\hat{b}_m(t-|\vec{R}-\vec{r}\,'_m|/c)+\hat{\vec{E}}^{(+)}_{\text{free}}(\vec{R},t) \tag{33}$$

with

$$\vec{C}^{(n)}\equiv\frac{\omega_0^2}{4\pi\epsilon_0 c^2}\left[\frac{\vec{\mu}}{|\vec{R}-\vec{r}_n|}-\frac{\vec{\mu}\cdot(\vec{R}-\vec{r}_n)(\vec{R}-\vec{r}_n)}{|\vec{R}-\vec{r}_n|^3}\right] \tag{34}$$

and $\vec{C}\,'(m)$ given by a similar expression with $\vec{r}_n$ replaced by $\vec{r}\,'_m$. The probability that a photon is detected at position $\vec{R}$ at time t is then proportional to $\langle\hat{\vec{E}}^{(-)}(\vec{R},t)\cdot\hat{\vec{E}}^{(+)}(\vec{R},t)\rangle$. We shall assume, either because the field that was used to excite the atoms is turned off at time t, or because the point $\vec{R}$ is located outside the region of the exciting field, that

$$\hat{\vec{E}}^{(+)}_{\text{free}}(\vec{R},t)|\ \rangle_{\text{field}}=0\ . \tag{35}$$

In the Heisenberg picture the operators $\hat{b}_n(t)$ oscillate at the atomic frequency ω_0 as well as evolving more slowly in time at a rate determined by the coupling. If the transit time $|\vec{R}-\vec{r}_n|/c$ (or at least its variation from one atom to another) is very short compared with the atomic lifetime, as we may assume, then to a good approximation

$$\hat{b}_n(t-|\vec{R}-\vec{r}_n|/c)\approx\hat{b}_n(t-\tau_0)e^{i\omega_0(\tau_n-\tau_0)}\,,\quad \tau_n\equiv|\vec{R}-\vec{r}_n|/c\,, \tag{36}$$

where τ_0 is the transit time from the midpoint of either source to point O in Fig. 2.

With the help of Eqs. (35) and (36) we then have for the expected light intensity, or the photon detection probability,

$$\begin{aligned}\langle\hat{I}(\vec{R},t)\rangle&=\langle\hat{\vec{E}}^{(-)}(\vec{R},t)\cdot\hat{\vec{E}}^{(+)}(\vec{R},t)\rangle\\ &=\left\langle\left[\sum_{n=1}^{N}C_i^{(n)*}\hat{b}_n^\dagger(t-\tau_0)e^{-i\omega_0(\tau_n-\tau_0)}+\sum_{m=1}^{M}C_i^{\prime(m)*}\hat{b}_m^\dagger(t-\tau_0)e^{-i\omega_0(\tau_m-\tau_0)}\right]\right.\\ &\quad\times\left.\left[\sum_{n=1}^{N}C_i^{(n)}\hat{b}_n(t-\tau_0)e^{i\omega_0(\tau_n-\tau_0)}+\sum_{m=1}^{M}C_i^{\prime(m)}\hat{b}_m(t-\tau_0)e^{i\omega_0(\tau_m-\tau_0)}\right]\right\rangle\\ &=\sum_{n\neq n'}^{N}\sum^{N}\vec{C}^{(n)*}\cdot\vec{C}^{(n')}\langle\hat{b}_n^\dagger(t-\tau_0)\hat{b}_{n'}(t-\tau_0)\rangle+\sum_{m\neq m}^{M}\sum^{M}\vec{C}^{\prime(m)*}\cdot\vec{C}^{\prime(m')}\langle\hat{b}_m^\dagger(t-\tau_0)\hat{b}_{m'}(t-\tau_0)\rangle\\ &\quad+\sum_{n=1}^{N}\sum_{m=1}^{M}\vec{C}^{(n)*}\cdot\vec{C}^{\prime(m)}\langle\hat{b}_n^\dagger(t-\tau_0)\hat{b}_m(t-\tau_0)\rangle e^{-i\omega_0(\tau_n-\tau_m)}+\text{c.c.}\end{aligned} \tag{37}$$

As before, we assume that the atomic excitation has been produced by exposing the atoms to a coherent optical field, and that the atomic state at time $t-\tau_0$ is a product state of the form

$$|\;\rangle_{\text{atoms}}=\prod_{n=1}^{N}|\theta_1,\phi_{1n}\rangle\otimes\prod_{m=1}^{M}|\theta_2,\phi_{2m}\rangle\,. \tag{38}$$

The polar angles of the Bloch vectors are taken to be equal for all atoms of one source, but the azimuthal angles in general are all different, because the atomic positions are different. Therefore each complex term in Eq. (37), such as

$$\langle\hat{b}_n^\dagger(t-\tau_0)\hat{b}_{n'}(t-\tau_0)\rangle\,,$$

in general has a different phase angle. In a short observation time δT these terms generate interference effects, as before. However, if we are dealing with moving atoms, it is to be expected that the phase angles change for each independent observation, because the atomic positions are random. When we average over the ensemble, all cross terms in Eq. (37) average to zero, with the result

$$\begin{aligned}\langle\hat{I}(\vec{R},t)\rangle&=|\vec{C}|^2\sum_{n=1}^{N}\langle\hat{b}_n^\dagger(t-\tau_0)\hat{b}_n(t-\tau_0)\rangle+|\vec{C}'|^2\sum_{m=1}^{M}\langle\hat{b}_m^\dagger(t-\tau_0)\hat{b}_m(t-\tau_0)\rangle\\ &\approx|\vec{C}|^2[N\cos^2\tfrac{1}{2}\theta_1+M\cos^2\tfrac{1}{2}\theta_2]\,.\end{aligned} \tag{39}$$

Under the small angle approximation $|\vec{C}^{(n)}|^2$ and $|\vec{C}^{\prime(m)}|^2$ have been taken to be almost the same for all atoms. This averaged equation of course shows no interference. Nevertheless, interference effects can be exhibited even by the ensemble average if we calculate the normally ordered two-point intensity correlation function,[13] which is proportional to the joint probability of photodetection at two points $\vec{R},\vec{R}'$ in the receiving plane.

On making use of the fact that atomic operators commute with free-field operators at later times,[22,23] we obtain from Eqs. (33) and (36)

$$\Gamma^{(2,2)}\equiv\langle \hat{E}_i^{(-)}(\vec{R},t)E_j^{(-)}(\vec{R}',t)\hat{E}_j^{(+)}(\vec{R}',t)\hat{E}_i^{(+)}(\vec{R},t)\rangle$$
$$=\Bigg\langle\left[\sum_{n=1}^{N}C_i^{(n)*}\hat{b}_n^{\dagger}(t-\tau_0)e^{-i\omega_0(\tau_{1n}-\tau_0)}+\sum_{m=1}^{M}C_i^{\prime(m)*}\hat{b}_m^{\dagger}(t-\tau_0)e^{-i\omega_0(\tau_{1m}-\tau_0)}\right]$$
$$\times\left[\sum_{n'=1}^{N}C_j^{(n')*}\hat{b}_{n'}^{\dagger}(t-\tau_0)e^{-i\omega_0(\tau_{2n'}-\tau_0)}+\sum_{m'=1}^{M}C_j^{\prime(m')*}\hat{b}_{m'}^{\dagger}(t-\tau_0)e^{-i\omega_0(\tau_{2m'}-\tau_0)}\right]$$
$$\times\left[\sum_{n''=1}^{N}C_j^{(n'')}\hat{b}_{n''}(t-\tau_0)e^{i\omega_0(\tau_{2n''}-\tau_0)}+\sum_{m''=1}^{M}C_j^{\prime(m'')}\hat{b}_{m''}(t-\tau_0)e^{i\omega_0(\tau_{2m''}-\tau_0)}\right]$$
$$\times\left[\sum_{n'''=1}^{N}C_i^{(n''')}\hat{b}_{n'''}(t-\tau_0)e^{i\omega_0(\tau_{1n'''}-\tau_0)}+\sum_{m'''=1}^{M}C_i^{\prime(m''')}\hat{b}_{m'''}(t-\tau_0)e^{i\omega_0(\tau_{1m'''}-\tau_0)}\right]\Bigg\rangle \quad (40)$$

with the abbreviations

$$c\tau_{1n}\equiv|\vec{R}-\vec{r}_n|\ ,$$
$$c\tau_{2n}\equiv|\vec{R}'-\vec{r}_n|\ . \quad (41)$$

Multiplying out generates 16 distinct terms, each of which consists of a quadruple sum. If we discard all terms with unpaired lowering and raising operators, on the grounds that the phase of each atom is random and averages to zero over the ensemble, and make the small angle approximation as before, we are left with the following nonvanishing contributions:

$$\Gamma^{(2,2)}=|\vec{C}|^4\Bigg\{2\sum_{n\neq n'}^{N}\sum^{N}\langle\hat{b}_n^{\dagger}(t-\tau_0)\hat{b}_{n'}^{\dagger}(t-\tau_0)\hat{b}_{n'}(t-\tau_0)\hat{b}_n(t-\tau_0)\rangle$$
$$+2\sum_{m\neq m'}^{M}\sum^{M}\langle\hat{b}_m^{\dagger}(t-\tau_0)\hat{b}_{m'}^{\dagger}(t-\tau_0)\hat{b}_{m'}(t-\tau_0)\hat{b}_m(t-\tau_0)\rangle$$
$$+\sum_{n=1}^{N}\sum_{m=1}^{M}\langle\hat{b}_n^{\dagger}(t-\tau_0)\hat{b}_m^{\dagger}(t-\tau_0)\hat{b}_m(t-\tau_0)\hat{b}_n(t-\tau_0)\rangle(1+e^{i\omega_0 s(x'-x)/D})+\text{c.c.}\Bigg\}\ . \quad (42)$$

In writing this equation we have made use of the relations (cf. Fig. 2)

$$c\tau_{1\,{}^{n}_{m}}=D+\tfrac{1}{2}(\tfrac{1}{2}s\pm x)^2/D+c\alpha_{{}^{n}_{m}}/\omega_0\ ,$$
$$c\tau_{2\,{}^{n}_{m}}=D+\tfrac{1}{2}(\tfrac{1}{2}s\pm x')^2/D+c\beta_{{}^{n}_{m}}/\omega_0\ , \quad (43)$$

where $\alpha_{{}^{n}_{m}}$, $\beta_{{}^{n}_{m}}$ are position-dependent phases that may be taken to be random. The matrix elements in Eq. (42) are readily evaluated with the help of Eq. (38), and we arrive at

$$\Gamma^{(2,2)}=2|\vec{C}|^4\{N^{(2)}\cos^4\tfrac{1}{2}\theta_1+M^{(2)}\cos^4\tfrac{1}{2}\theta_2+NM\cos^2\tfrac{1}{2}\theta_1\cos^2\tfrac{1}{2}\theta_2[1+\cos2\pi(x'-x)/L]\}\ , \quad (44)$$

where L given by Eq. (25) is again the spacing of the interference fringes. This result can be contrasted with that given by Eq. (29), in which the atoms were assumed to be very close together and there were super-radiant contributions. The two expressions agree when $N=1=M$ and $\theta_1=\theta_2$. Evidently interference effects show up in higher-order measurements even when the sources have random phases.

Once again we note that $\Gamma^{(2,2)}$, and therefore the joint probability of two-photon detection, vanishes when $N=1=M$ and $|x'-x|$ is an odd number of half fringes, for reasons we have already discussed. This conclusion cannot be obtained from any classical argument, e.g., from Eq. (7). On the other hand, when $\theta_1=\theta_2$ and N and M are both large and approximately equal, Eq. (44) can be expressed in the form

$$\Gamma^{(2,2)}\approx\tfrac{3}{2}\langle I(x,t)\rangle\langle Ix',t\rangle$$
$$\times[1+\tfrac{1}{3}\cos2\pi(x'-x)/L]\ , \quad (45)$$

which is exactly the classical Eq. (10), applicable to thermal light. In general, when $\theta_1=\theta_2$, the relative modulation amplitude ρ_{12} obtained from Eq. (44) is

$$\rho_{12}=\frac{NM}{N^{(2)}+M^{(2)}+NM}\ , \quad (46)$$

and this varies between 1 and $\frac{1}{3}$ for equal N,M, and becomes very small for greatly unequal N,M.

VI. FLUCTUATING NUMBER OF ATOMS

Finally we consider the situation in which the numbers of atoms N,M at each source are themselves random variables. This situation might be encountered with gaseous or atomic beam sources. We then have to average the previous results over the fluctuations of N,M.

In the simplest situation N and M are independent Poisson variates with means $\langle N\rangle$ and $\langle M\rangle$, respectively. After averaging over N and M we obtain from Eq. (39)

$$\langle \hat{I}(\vec{R},t)\rangle = |\vec{C}|^2[\langle N\rangle\cos^2\tfrac{1}{2}\theta_1 + \langle M\rangle\cos^2\tfrac{1}{2}\theta_2]\ , \quad (47)$$

and from Eq. (44), since

$$\langle N(N-1)\rangle=\langle N\rangle^2,\ \langle M(M-1)\rangle=\langle M\rangle^2\ ,$$

$$\Gamma^{(2,2)}=2|\vec{C}|^4\{\langle N\rangle^2\cos^4\tfrac{1}{2}\theta_1+\langle M\rangle^2\cos^4\tfrac{1}{2}\theta_2+\langle N\rangle\langle M\rangle\cos^2\tfrac{1}{2}\theta_1\cos^2\tfrac{1}{2}\theta_2[1+\cos 2\pi(x'-x)/L]\}\ . \quad (48)$$

When $\langle N\rangle=\langle M\rangle$, $\theta_1=\theta_2$ this can be written

$$\Gamma^{(2,2)}=\tfrac{3}{2}\langle I(x,t)\rangle\langle I(x',t)\rangle \times[1+\tfrac{1}{3}\cos 2\pi(x'-x)/L]\ , \quad (49)$$

which is the same as Eq. (45). However, this time the equation holds exactly, without the assumption that $\langle N\rangle,\langle M\rangle$ are large. Indeed, the answer holds even if $\langle N\rangle,\langle M\rangle \ll 1$, when one might have expected to encounter the strongly nonclassical situation represented by Eq. (31). Instead we find that the atomic number fluctuations cause the behavior of the system to move into the classical domain, because the source of any one photon is made more uncertain.

VII. SUMMARY

We have studied the interference effects produced by nonclassical sources, when the state of the field is not expressible as a simple mixture of coherent states, and have shown that interference effects are present even when the phases of the sources are random. We have used both classical and quantum mechanical approaches to calculate the two-point intensity correlation function, and have found that there are similarities as well as significant differences. But whereas the relative modulation amplitude for a classical field has a maximum value of $\frac{1}{2}$, it can be as large as 1 for a quantum field. We showed that when each source consists of one partly excited atom, the joint probability of photon detection at two points in the receiving plane is zero when the two points are separated by an odd multiple of the half-fringe spacing. Finally, we considered sources in which the number of atoms is governed by Poisson fluctuations, and we found the same interference effects as are exhibited by a classical, thermal light field.

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APPENDIX A: EVALUATION OF A MATRIX ELEMENT

We consider the following matrix element of the operator $\hat{n}_{\mathscr{V},t}$ between two localized one-photon states:

$$D_{ji}(\vec{r}',0;\vec{r},0;\mathscr{V},t)\equiv\langle \text{vac}|\hat{V}_j(\vec{r}',0)\hat{n}_{\mathscr{V},t}\hat{V}_i^\dagger(\hat{r},0)|\text{vac}\rangle\ .$$

The commutator of $\hat{V}_j(\vec{r}',0)$ and $\hat{n}_{\mathscr{V},t}$ is given by[19]

$$\hat{\vec{Q}}(\vec{r}'0;\mathscr{V},t)\equiv[\hat{\vec{V}}(\vec{r}',0),\hat{n}_{\mathscr{V},t}]=\frac{1}{L^{3/2}}\sum_{\{\vec{k}\},s}\hat{a}_{\vec{k}s}\vec{\epsilon}_{\vec{k}s}e^{i\vec{k}\cdot\vec{r}'}U(\vec{r}'+c\vec{k}t/k,\mathscr{V}) \quad (A1)$$

with $U(\vec{r},\mathscr{V})=1$ or 0 according as $\vec{r}\in\mathscr{V}$ or $\vec{r}\notin\mathscr{V}$. By commuting $\hat{V}_j(\vec{r}',0)$ and $\hat{n}_{\mathscr{V},t}$ we obtain

$$D_{ji}(\vec{r}',0;\vec{r},0;\mathscr{V},t)=\langle \text{vac} \mid \hat{Q}_j(\vec{r}',0;\mathscr{V},t)\hat{V}_i^{\dagger}(\vec{r},0) \mid \text{vac}\rangle$$
$$=\frac{1}{L^3}\sum_{\{\vec{k}\},s}\sum_{\{k'\},s'}[\hat{a}_{ks},\hat{a}^{\dagger}_{\vec{k}',s'}](\epsilon_{\vec{k}s})_j(\vec{\epsilon}^{\,*}{}_{\vec{k}'s'})_i e^{i(\vec{k}\cdot\vec{r}'-\vec{k}'\cdot\vec{r})}U(\vec{r}'+c\vec{k}t/k,\mathscr{V})$$
$$=\frac{1}{L^3}\sum_{\{\vec{k}\}}(\delta_{ji}-k_jk_i/k^2)e^{i\vec{k}\cdot(\vec{r}'-\vec{r})}U(\vec{r}'+c\vec{k}t/k,\mathscr{V})$$
$$\rightarrow\left[\frac{1}{2\pi}\right]^3\int_{\{\vec{k}\}}d^3k(\delta_{ji}-k_jk_i/k^2)e^{i\vec{k}\cdot(\vec{r}'-\vec{r})}U(\vec{r}'+c\vec{k}t/k,\mathscr{V})\ . \qquad \text{(A2)}$$

The last line is obtained from the previous one after we go to the continuum limit, when the quantization volume $L^3\rightarrow\infty$, and it coincides with Eq. (21) of the text.

APPENDIX B: EVALUATION OF HIGHER-ORDER MATRIX ELEMENTS

Consider the matrix element

$$M\equiv K_iK_jK_pK_q\langle \text{vac} \mid \hat{V}_i(\vec{r}_1,0)\hat{V}_j(\vec{r}_2,0){:}\hat{n}_{\mathscr{V}_1t}\hat{n}_{\mathscr{V}_2t}{:}\hat{V}^{\dagger}_p(\vec{r}_3,0)\hat{V}^{\dagger}_q(\vec{r}_4,0) \mid \text{vac}\rangle \qquad \text{(B1)}$$

with $\hat{n}_{\mathscr{V}_1t},\hat{n}_{\mathscr{V}_2t}$ defined by Eq. (17). We start by expanding $\hat{n}_{\mathscr{V}_1t}$ as an integral, and then apply the commutation rule

$$F_{lm}(\vec{r},t;\vec{r}',t')\equiv[\hat{V}_l(\vec{r},t),\hat{V}^{\dagger}_m(\vec{r}',t)]=\frac{1}{L^3}\sum_{\{\vec{k}\},s}(\vec{\epsilon}_{\vec{k}s})_l(\vec{\epsilon}^{\,*}{}_{\vec{k}s})_m e^{i[\vec{k}\cdot(\vec{r}-\vec{r}')-w(t-t')]}$$
$$\rightarrow\frac{1}{(2\pi)^3}\int_{\{\vec{k}\}}d^3k\left[\delta_{lm}-\frac{k_lk_m}{k^2}\right]e^{i[\vec{k}\cdot(\vec{r}-\vec{r}')-w(t-t')]} \qquad \text{(B2)}$$

twice, followed by the rule (A1). We then find

$$M=K_iK_jK_pK_q\left\langle\text{vac}\left|\hat{V}_i(\vec{r}_1,0)\hat{V}_j(\vec{r}_2,0)\int_{\mathscr{V}_1}d\vec{r}'\hat{V}^{\dagger}_n(\vec{r}',t)\hat{n}_{\mathscr{V}_2t}\hat{V}_n(\vec{r}',t)\hat{V}^{\dagger}_p(\vec{r}_3,0)\hat{V}^{\dagger}_q(\vec{r}_4,0)\right|\text{vac}\right\rangle$$
$$=K_iK_jK_pK_q\int_{\mathscr{V}_1}d\vec{r}'\langle \text{vac} \mid \hat{V}_i(\vec{r}_1,0)[\hat{V}^{\dagger}_n(\vec{r}',t)\hat{V}_j(\vec{r}_2,0)+F_{jn}(\vec{r}_2,0;\vec{r}',t)]\hat{n}_{\mathscr{V}_2t}$$
$$\times[\hat{V}^{\dagger}_p(\vec{r}_3,0)\hat{V}_n(\vec{r}',t)+F_{np}(\vec{r}',t;\vec{r}_3,0)]\hat{V}^{\dagger}_q(\vec{r}_4,0) \mid \text{vac}\rangle$$
$$=K_iK_jK_pK_q\int_{\mathscr{V}_1}d\vec{r}'\langle \text{vac} \mid [F_{in}(\vec{r}_1,0;\vec{r}',t)\hat{V}_j(\vec{r}_2,0)+F_{jn}(\vec{r}_2,0;\vec{r}',t)\hat{V}_i(\vec{r}_1,0)]\hat{n}_{\mathscr{V}_2t}$$
$$\times[F_{nq}(\vec{r}'t;\vec{r}_4,0)\hat{V}^{\dagger}_p(\vec{r}_3,0)+F_{np}(\vec{r}',t;\vec{r}_3,0)\hat{V}^{\dagger}_q(\vec{r}_4,0)] \mid \text{vac}\rangle$$
$$=K_iK_jK_pK_q\int_{\mathscr{V}_1}d\vec{r}'\langle \text{vac} \mid [F_{in}(\vec{r}_1,0;\vec{r}',t)\hat{Q}_j(\vec{r}_2,0;\mathscr{V}_2t)+F_{jn}(\vec{r}_2,0;\vec{r}',t)\hat{Q}_i(\vec{r}_1,0;\mathscr{V}_2t)]$$
$$\times[F_{nq}(\vec{r}',t;\vec{r}_4,0)\hat{V}^{\dagger}_p(\vec{r}_3,0)+F_{np}(\vec{r}'t;\vec{r}_3,0)\hat{V}^{\dagger}_q(\vec{r}_4,0)] \mid \text{vac}\rangle\ .$$

Next we make use of the commutation relation

$$[\hat{Q}_i(\vec{r},0;\mathscr{V}_2t),\hat{V}^{\dagger}_p(\vec{r}',0)]=\frac{1}{L^3}\sum_{\{\vec{k}\},s}(\vec{\epsilon}_{\vec{k}s})_i(\vec{\epsilon}^{\,*}_{\vec{k}s})_p e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}U(\vec{r}+c\vec{k}t/k,\mathscr{V}_2)$$
$$\rightarrow\frac{1}{(2\pi)^3}\int_{\{\vec{k}\}}d^3k\left[\delta_{ip}-\frac{k_ik_p}{k^2}\right]e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}U(\vec{r}+c\vec{k}t/k,\mathscr{V}_2)$$
$$=D_{ip}(\vec{r},0;\vec{r}',0;\mathscr{V}_2t)\ , \qquad \text{(B3)}$$

and obtain

$$M=K_iK_jK_pK_q\int_{\mathscr{V}_1}d\vec{r}\,'[F_{in}(\vec{r}_1,0;\vec{r}\,',t)F_{nq}(\vec{r}\,',t;\vec{r}_4,0)D_{jp}(\vec{r}_2,0;\vec{r}_3,0;\mathscr{V}_2,t)$$
$$+F_{in}(\vec{r}_1,0;\vec{r}\,',t)F_{np}(\vec{r}\,',t;\vec{r}_3,0)D_{jq}(\vec{r}_2,0;\vec{r}_4,0;\mathscr{V}_2,t)$$
$$+F_{jn}(\vec{r}_2,0;\vec{r}\,',t)F_{nq}(\vec{r}\,',t;\vec{r}_4,0)D_{ip}(\vec{r}_1,0;\vec{r}_3,0;\mathscr{V}_2,t)$$
$$+F_{jn}(\vec{r}_2,0;\vec{r}\,',t)F_{np}(\vec{r}\,',t;\vec{r}_3,0)D_{iq}(\vec{r}_1,0;\vec{r}_4,0;\mathscr{V}_2,t)]\ . \tag{B4}$$

We now evaluate the $\vec{r}\,'$-integral under the assumption that $\mathscr{V}_1$ is in the form of a box centered at $\vec{r}_0$ with sides l_1,l_2,l_3. For a typical element we have

$$\int_{\mathscr{V}_1}d\vec{r}\,'F_{in}(\vec{r}_1,0;\vec{r}\,',t)F_{nq}(\vec{r}\,',t;\vec{r}_2,0)$$
$$=\frac{1}{(2\pi)^6}\int_{\{\vec{k}\}}d^3k\int_{\{\vec{k}\,'\}}d^3k'\left[\delta_{in}-\frac{k_ik_n}{k^2}\right]\left[\delta_{nq}-\frac{k_n'k_q'}{k'^2}\right]$$
$$\times\int_{\mathscr{V}_1}d\vec{r}\,'\exp\{i[\vec{k}\cdot(\vec{r}_1-\vec{r}\,')+\vec{k}\,'\cdot(\vec{r}\,'-\vec{r}_2)+(\omega-\omega')t]\}$$
$$=\frac{1}{(2\pi)^6}\int_{\{\vec{k}\}}d^3k\int_{\{\vec{k}\,'\}}d^3k'\left[\delta_{in}-\frac{k_ik_n}{k^2}\right]\left[\delta_{nq}-\frac{k_n'k_q'}{k'^2}\right]\exp\{i[(\vec{k}\,'-\vec{k})\cdot\vec{r}_0-(\omega'-\omega)t]\}$$
$$\times\exp[i(\vec{k}\cdot\vec{r}_1-\vec{k}\,'\cdot\vec{r}_2)]\prod_{j=1}^{3}\left[\frac{\sin\frac{1}{2}(k_j'-k_j)l_j}{\frac{1}{2}(k_j'-k_j)}\right]\ .$$

When the dimensions l_1,l_2,l_3 are all large compared with the optical wavelength, because of the last factor, $\vec{k}$ has to be very close to $\vec{k}\,'$. We now put $\vec{k}\,'-\vec{k}=\vec{k}\,''$ and integrate over $\vec{k}\,''$. Then to a good approximation

$$\int_{\mathscr{V}_1}d\vec{r}\,'F_{in}(\vec{r}_1,0;\vec{r}\,',t)F_{nq}(\vec{r}\,',t;\vec{r}_2,0)=\frac{1}{(2\pi)^3}\int_{\{\vec{k}\}}d^3k\left[\delta_{iq}-\frac{k_ik_q}{k^2}\right]e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}$$
$$\times\frac{1}{(2\pi)^3}\int d^3k''e^{i[\vec{k}\,''\cdot(\vec{r}_0-\vec{r}_2)-(\omega'-\omega)t]}\prod_{j=1}^{3}\left[\frac{\sin\frac{1}{2}k_j''l_j}{\frac{1}{2}k_j''}\right]$$

and the $\vec{k}\,''$ integral becomes the Dirichlet integral,[19]

$$=\frac{1}{(2\pi)^3}\int_{\{\vec{k}\}}d^3k\left[\delta_{iq}-\frac{k_ik_q}{k^2}\right]e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}U(\vec{r}_2+c\vec{k}t/k,\mathscr{V}_1)$$
$$=D_{iq}^*(\vec{r}_2,0;\vec{r}_1,0;\mathscr{V}_1,t)\ . \tag{B5}$$

We now use this result repeatedly in Eq. (B4) and obtain finally

$$M=K_iK_jK_pK_q\frac{1}{(2\pi)^6}$$
$$\times\int_{\{\vec{k}\}}d^3k\int_{\{\vec{k}\,'\}}d^3k'\left[\left[\delta_{iq}-\frac{k_ik_q}{k^2}\right]\left[\delta_{jp}-\frac{k_j'k_p'}{k'^2}\right]\right.$$
$$\times e^{i[\vec{k}\cdot(\vec{r}_1-\vec{r}_4)]+\vec{k}\,'\cdot(\vec{r}_2-\vec{r}_3)]}U\left[\vec{r}_4+\frac{c\vec{k}t}{k},\mathscr{V}_1\right]U\left[\vec{r}_2+\frac{c\vec{k}\,'t}{k'},\mathscr{V}_2\right]$$
$$+\left[\delta_{ip}-\frac{k_ik_p}{k^2}\right]\left[\delta_{jq}-\frac{k_j'k_q'}{k'^2}\right]e^{i[\vec{k}\cdot(\vec{r}_1-\vec{r}_3)+\vec{k}\,'\cdot(\vec{r}_2-\vec{r}_4)]}$$

$$\times U\left[\vec{r}_3+\frac{c\vec{k}t}{k},\mathscr{V}_1\right]U\left[\vec{r}_2+\frac{c\vec{k}'t}{k'},\mathscr{V}_2\right]$$

$$+\left[\delta_{jq}-\frac{k_jk_q}{k^2}\right]\left[\delta_{ip}-\frac{k_i'k_p'}{k'^2}\right]e^{i[\vec{k}\cdot(\vec{r}_2-\vec{r}_4)+\vec{k}'\cdot(\vec{r}_1-\vec{r}_3)]}$$

$$\times U\left[\vec{r}_4+\frac{c\vec{k}t}{k},\mathscr{V}_1\right]U\left[\vec{r}_1+\frac{c\vec{k}'t}{k'},\mathscr{V}_2\right]$$

$$+\left[\delta_{jp}-\frac{k_jk_p}{k^2}\right]\left[\delta_{iq}-\frac{k_i'k_q'}{k'^2}\right]e^{i[\vec{k}\cdot(\vec{r}_2-\vec{r}_3)+\vec{k}'\cdot(\vec{r}_1-\vec{r}_4)]}$$

$$\times U\left[\vec{r}_3+\frac{c\vec{k}t}{k},\mathscr{V}_1\right]U\left[\vec{r}_1+\frac{c\vec{k}'t}{k'},\mathscr{V}_2\right]\Bigg]\ . \quad \text{(B6)}$$

This becomes Eq. (28) of the main text when the vector $\vec{K}$ is almost perpendicular to the lines joining the sources at $\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4$ to either of the detectors, i.e., when the scalar products $\vec{K}\cdot\vec{k}$ are very small.

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