PHYSICAL REVIEW A VOLUME 28, NUMBER 2 AUGUST 1983

Temporal aspects of absorptive optical bistability

T. Erneux and Paul Mandel
Université Libre de Bruxelles, Service de Chimie-Physique II, Campus de la Plaine, C.P. 231, 1050 Bruxelles, Belgium
(Received 29 July 1982)

We study the semiclassical mean-field theory of absorptive optical bistability in the limit of a fully developed hysteresis cycle, i.e., in the limit of large values of the bistability parameter C. We show by multiple time-scale perturbation analysis that it is possible to describe the transition between the two stable branches by simple equations. Near the low-transmission branch, the behavior of the system is governed by an equation for the atomic population, whereas, near the high-transmission branch, it is an equation for the field amplitude which determines the long-time evolution of the system. We then present an analytic study, completed by numerical results whenever necessary, of the time-dependent response of the system when the input field is swept across the bistable region. We study the influence of the initial conditions, the sweeping velocity, and the ratio of the atomic relaxation times on the dynamical hysteresis as well as its relation with the stationary hysteresis.

I. INTRODUCTION

The idea of optical bistability (OB) has been proposed by Szöke *et al.*[1] McCall[2] gave the first detailed theoretical discussion based mainly on numerical solutions of the steady-state coupled Maxwell-Bloch equations. Finally, Bonifacio and Lugiato[3] showed how the tools developed in laser theory could be applied in OB with a simple and elegant model. The experimental observation of OB[4,5] and the realization of a bistable optical device[6] (BOD) have given an additional impetus and new perspectives to theoretical research.

To be specific, we shall study in this paper the simplest possible model exhibiting OB. We therefore consider a ring cavity containing an absorbing nonlinear medium. An external signal is injected into the cavity and one studies how the output-field amplitude depends on the input-field amplitude. A standard approach to this problem is first to derive the stationary solutions and second to study their linear stability. In the simplest case of absorptive OB, such a program can be carried out analytically[7] within the mean-field semiclassical description. Although much energy has been devoted to the realization of the same program for more complicated situations which take into account dispersion,[8] radial mode variation,[9] propagation effect,[10] or boundary conditions[11] to name but a few directions of exploration, the steady-state regime *does not* correspond to most experimental situations. Indeed the experimentalist usually sweeps either the amplitude or the phase of the input field. What is the relation between the stationary bistable characteristic and the effective time-dependent response of the BOD is by no means a simple problem. The aim of this paper is precisely to study the effects of time-dependent control parameters and the relation between temporal and stationary responses.

Previous mathematical studies of transitions between steady states have been devoted mainly to systems where the parameters are constant in time.[12] However, authors motivated by stability questions in chemical reactors have recently considered bistability problems with time-varying parameters.[13] When the time variation of these parameters is slow compared with typical chemical reaction times, the method of matched asymptotic expansions can be used to reduce the full set of equations to simple nonlinear canonical equations.[14] In this problem the change in time of the control parameter is *not* necessarily slow compared with the characteristic relaxation times of the BOD. Hence we must examine how the BOD's response is modified as we modify the sweeping velocity of the control parameter. We shall use the Maxwell-Bloch equations to describe OB. When dealing with these equations in quantum optics, it is customary[15] to resort to some version of the adiabatic elimination scheme in order to simplify the theory. Such a scheme relies on strong inequalities that are assumed to exist between the cavity and the atomic decay rates. It was applied in OB in the good cavity limit by Benza and Lugiato[16] for the absorptive case and by Ikeda[11] for the dispersive case; it was also applied recently by Drummond[17] in one of the bad cavity limits. Unfortunately, the atomic decay rates are fixed constants and modifications of the cavity quality factor are not that easy. As a consequence it is desirable to have a theory that does not make use of such adiabatic schemes. Furthermore, a parameter which is easily varied between two experiments is the bistability parameter C; we recall that $C>4$ is the condition for OB to occur. We have found that in the limit of fully developed OB (i.e., in the limit of large C) it is possible to analyze the problem without any adiabatic elimination of variables. Our approach is a multiple time-scale perturbation method[14] to solve the time-dependent problem. We show that in the large-C domain the transition regions where the output field undergoes a jump are still governed by simple nonlinear equations whose solutions diverge in a finite time.

In Sec. II we recall the basic Maxwell-Bloch equations on which this paper is based. We also recall the stationary solutions, their scaling laws, and the results of the linear stability analysis. In Secs. III and IV we present the asymptotic theory of the low- and high-transmission branches, respectively. These two sections primarily contain a derivation of the basic equations and a first discussion of their properties based on numerical solutions. Section V is devoted to an analytic study of the equation derived in Sec. III to describe the vicinity of the limit point where the jump to the high-transmission branch occurs. A discussion and concluding remarks are found in Sec. VI.

II. MAXWELL-BLOCH EQUATIONS

Let us consider a very simple situation which is that of single-mode mean-field semiclassical absorptive OB in a ring cavity. In this case the nonlinear absorbing medium is modeled by a set of homogeneously broadened two-level atoms. Since absorption is taken as the dominant atomic mechanism, we may assume that the coherent external driving-field frequency, the atomic frequency, and the cavity-mode frequency are all identical. As discussed in Refs. 4 and 7 this situation can be described by the following set of Maxwell-Bloch equations:

$$\frac{\partial E}{\partial t'}=-\kappa\left[E-\frac{1}{\sqrt{T}}E_I\right]-gS, \tag{2.1}$$

$$\frac{\partial S}{\partial t'}=-\gamma_\perp S+\frac{\mu}{\hbar}E\Delta\ , \tag{2.2}$$

$$\frac{\partial \Delta}{\partial t'}=-\gamma_{\parallel}\left[\Delta-\frac{N}{2}\right]-\frac{\mu}{\hbar}ES\ , \tag{2.3}$$

where E is the real field amplitude produced in the cavity when a generally time-dependent field amplitude E_I is injected into the ring cavity via a mirror of transmittivity T. The injected signal is attenuated by the cavity, which has a decay rate κ, and by the interaction with the atomic medium in which it induces a polarization S. The atom-field coupling constant g and the modulus of the dipole moment μ are related by

$$g=4\pi\omega\mu/V\ ,$$

with ω being the unique frequency in the system. The atomic polarization S has a decay rate $\gamma_\perp$ and induces a modification of the population difference between the lower (N_-) and the upper (N_+) level defined by

$$\Delta=\tfrac{1}{2}[N_-(t')-N_+(t')]\ .$$

This population difference relaxes with a decay rate $\gamma_{\parallel}$ and the total number of atoms N is an invariant:

$$N=N_+(t')+N_-(t')\ .$$

Initially, $N_->N_+$.

It is useful to introduce a set of reduced variables as follows:

$$x=\frac{\mu}{\hbar\sqrt{\gamma_{\parallel}\gamma_\perp}}E,\quad y=\frac{\mu}{\hbar\sqrt{T\gamma_{\parallel}\gamma_\perp}}E_I,$$

$$s=\frac{2}{N}\left[\frac{\gamma_\perp}{\gamma_{\parallel}}\right]^{1/2}S,\quad \delta=\frac{2}{N}\Delta,\quad t=\kappa t', \tag{2.4}$$

$$C=g\mu N/4\hbar\kappa\gamma_\perp,$$

in terms of which the Maxwell-Bloch equations become

$$\frac{\partial x}{\partial t}=\dot{x}=-x+y-2Cs, \tag{2.5}$$

$$\frac{\kappa}{\gamma_\perp}\dot{s}=-s+x\delta\ , \tag{2.6}$$

$$\frac{\kappa}{\gamma_{\parallel}}\dot{\delta}=-\delta+1-xs\ . \tag{2.7}$$

Consequently the number of relevant parameters is reduced to four for the general time-dependent situation and to two (C and y) for the stationary state. For a constant input field y the system will eventually reach a stationary state in which

$$\delta=(1+x^2)^{-1}\ , \tag{2.8}$$

$$s=x(1+x^2)^{-1}\ , \tag{2.9}$$

$$y=x+2Cx(1+x^2)^{-1}\ . \tag{2.10}$$

From the last equation we get a cubic equation for x as a function of y and C. When $C<4$ the cubic has a single real positive root whereas when $C>4$ there are three real positive roots. The ring cavity then displays a multistable characteristic as exemplified in Fig. 1. A linear-stability analysis of the stationary solutions[3] easily indicates that when the slope of the curve $x = x(y)$ is positive, the stationary solution is stable, whereas when the slope is negative the stationary solution is unstable. Hence when $C > 4$ and $y_m < y < y_M$, there corresponds two possible states to each single value of y. It is important to realize that the two branches of the hysteresis are associated with very different mechanisms. In the low-transmission branch defined for $0 < y < y_M$ most of the energy which is sent into the cavity is reflected by the atomic medium which is mostly opaque and manifests properties usually associated with cooperative behavior.[7] For instance, the fluorescence intensity I_f which is proportional to N_+ scales like N^{-1}. On the contrary, in the high-transmission branch defined for $y > y_m$ the energy is stored in the field mode of the cavity and the fluorescence intensity scales like N: the atomic system behaves like a collection of N weakly interacting atoms. This indicates that the physics of the high- and low-transmission branches is very different. For instance, consider a system which is initially in a stable stationary state. A modification of y will induce a time evolution and a relaxation towards a new stationary state if y reaches a constant value. If the initial state pertains to the lower branch it is the atomic system which relaxes and is likely to provide the dominant relaxation mechanism. On the contrary, if the initial state pertains to the upper branch the situation is drastically modified since it will be the field in the cavity mode which relaxes and is likely to dominate the time evolution.

The expression of the extremal values of the input field y_m and y_M and the ouput field x_m and x_M are fairly complicated. However, they become simpler in the limit of fully developed OB, i.e., in the limit of large C:

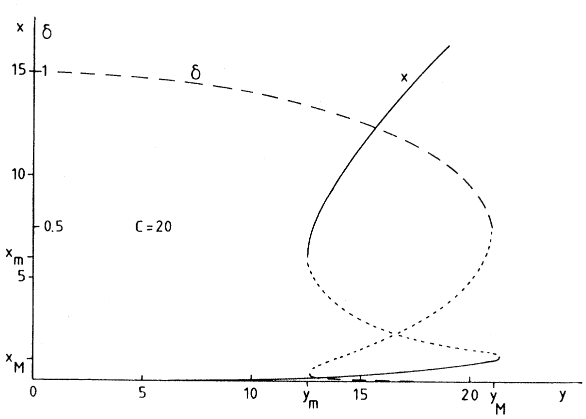

FIG. 1. Steady-state output-field amplitude (x) and atomic population (δ) vs input-field amplitude.

$$y_M = C,\quad x_M = 1,\quad z_M = C, \tag{2.11}$$

$$y_m = (8C)^{1/2},\quad x_m = (2C)^{1/2},\quad z_m = (2/C)^{1/2}, \tag{2.12}$$

where z_M (z_m) is the value of x on the upper (lower) branch corresponding to $y_M(y_m)$.

III. LOW-TRANSMISSION BRANCH

In this section we examine the time-evolution of the system in the neighborhood of the lower branch of the steady-state solutions in the limit $C \to \infty$. On the basis of the stationary scalings [(2.11) and (2.12)] we can divide the lower branch in two domains. In domain I, which includes the point (x_M, y_M), we have $y = O(C)$ and

$$x = O(1),\quad s = O(1),\quad \delta = O(1)\ . \tag{3.1}$$

In domain II, which includes the point (z_m, y_m) the external field amplitude is $y = O(C^{1/2})$ and the stationary solutions scale as

$$x = O(C^{-1/2}),\quad s = O(C^{-1/2}),\quad \delta = O(1)\ . \tag{3.2}$$

In this section we shall only consider domain I leaving the study of domain II for Appendix A. Hence we shall concentrate on the response of the system as the parameter $y(t)$ varies in time and crosses the critical value y_M, i.e., when x jumps from the lower to the upper branch of the steady-state solutions. Accordingly, we define

$$\epsilon = C^{-1} \ll 1, \tag{3.3}$$

$$Y(t) = y(t)C^{-1} = O(1), \tag{3.4}$$

and we write the Maxwell-Bloch equations (2.5)—(2.7) in terms of ϵ and Y:

$$\dot{x} = -x + \epsilon^{-1}(Y - 2s)\ , \tag{3.5}$$

$$\frac{\kappa}{\gamma_\perp}\dot{s} = -s + x\delta\ , \tag{3.6}$$

$$\frac{\kappa}{\gamma_{\parallel}}\dot{\delta} = -\delta + 1 - sx\ . \tag{3.7}$$

Since we are restricting our analysis to domain I, all three time-dependent variables x, s, and δ must be of order one. In particular, we must restrict the initial conditions as follows:

$$x(0) = x_i + O(\epsilon^{1/2}), \tag{3.8}$$

$$\delta(0) = \delta_i + O(\epsilon^{1/2}), \tag{3.9}$$

$$s(0) = \tfrac{1}{2}Y(0) + \epsilon^{1/2}s_i + O(\epsilon)\ . \tag{3.10}$$

The particular restriction (3.10) guarantees that the short time evolution of $x(t)$ remains $O(1)$: if

$s(0)-\frac{1}{2}Y(0)=O(1)$, then $x(t)$ quickly moves to large positive values when $s_i<\frac{1}{2}Y(0)$ or becomes negative when $s_i>\frac{1}{2}Y(0)$.

We may use (3.5) to express $s(t)$ in terms of x and Y:

$$s=\frac{1}{2}Y-\frac{\epsilon}{2}(x+\dot{x}). \tag{3.11}$$

Introducing this equation in Eqs. (3.6) and (3.7) leads to

$$\frac{\kappa}{\gamma_\perp}\epsilon\ddot{x}+\epsilon\dot{x}\left[1+\frac{\kappa}{\gamma_\perp}\right]+x(2\delta+\epsilon)-Y-\frac{\kappa}{\gamma_\perp}\dot{Y}=0\,, \tag{3.12}$$

$$\frac{\kappa}{\gamma_{\|}}\dot{\delta}+\delta-1+\frac{1}{2}xY-\frac{\epsilon}{2}(x^2+x\dot{x})=0, \tag{3.13}$$

with the initial conditions:

$$\begin{aligned}&x(0)=x_i+O(\epsilon^{1/2}),\\&\dot{x}(0)=-2\epsilon^{-1/2}s_i+O(1)\,,\\&\delta(0)=\delta_i+O(\epsilon^{1/2}).\end{aligned} \tag{3.14}$$

The stationary solutions are

$$\begin{aligned}&x_{1,2}=[1\pm(1-Y^2)^{1/2}]/Y+O(\epsilon),\\&\delta_{1,2}=(1+x_{1,2}^2)^{-1}=\frac{Y}{2x_{1,2}}+O(\epsilon),\\&s_{1,2}=x_{1,2}(1+x_{1,2}^2)^{-1}=\frac{1}{2}Y+O(\epsilon),\end{aligned} \tag{3.15}$$

with $Y\le 1+O(\epsilon)$. The linear stability of these solutions yields the following eigenvalues:

$$\begin{aligned}&\lambda_a=-\frac{\gamma_{\|}}{\kappa}(1-x_{1,2}^2)+O(\epsilon^{1/2}),\\&\lambda_{b,c}=\pm i\epsilon^{-1/2}(2\gamma_\perp\delta_{1,2}/\kappa)^{1/2}\\&\qquad-\frac{1}{2}\left[1+\frac{\gamma_\perp}{\kappa}+x_{1,2}^2\frac{\gamma_{\|}}{\kappa}\right]+O(\epsilon^{1/2}).\end{aligned} \tag{3.16}$$

The real root λ_a is positive for $x>1$ and negative for $x<1$. Hence, the steady-state solution $x_2(<1)$ is stable, whereas $x_1\,(>1)$ is unstable. Furthermore, there exists a limit point located at $Y_c=1+O(\epsilon)$ and $x_c=1+O(\epsilon)$. When $Y(t)$ reaches Y_c from below, the eigenvalue λ_a vanishes and small amplitude perturbations of the steady state x_2 exhibit a critical slowing down. Note that the same eigenvalues could have been derived directly from Eqs. (3.5)–(3.7).

From the linear-stability analysis we also observe that there are two natural time scales which appear in this problem; they are related to the real and imaginary parts of the eigenvalues which scale as

$$\begin{aligned}&\mathrm{Re}\lambda\sim O(1),\\&\mathrm{Im}\lambda\sim O(\epsilon^{-1/2}).\end{aligned}$$

This suggests the use of a multiple time-scale perturbation method[14] with solutions which are functions of t and T, where

$$T=\epsilon^{-1/2}\omega(t)\,. \tag{3.17}$$

The formal procedure consists in rewriting Eqs. (3.12) and (3.13) in terms of t and T considered as independent variables and eliminating secular solutions on the long time scale (i.e., solutions diverging like T). To determine these solutions, we assume that $x(t)$, $\delta(t)$, and $\omega(t)$ have a perturbation expansion of the form

$$\begin{aligned}&x(t)=x(t,T,\epsilon^{1/2})=x_0(t,T)\\&\qquad+\epsilon^{1/2}x_1(t,T)+\cdots\,,\\&\delta(t)=\delta(t,T,\epsilon^{1/2})=\delta_0(t,T)\\&\qquad+\epsilon^{1/2}\delta_1(t,T)+\cdots\,,\\&\omega(t)=\omega(t,\epsilon^{1/2})=\omega_0(t)\\&\qquad+\epsilon^{1/2}\omega_1(t)+\cdots\,.\end{aligned} \tag{3.18}$$

Introducing these power series into Eqs. (3.12) and (3.13) leads to the following results:

$$\delta_0(t,T)=\delta_0(t),$$

$$\frac{\kappa}{\gamma_{\|}}\dot{\delta}_0=1-\delta_0-\frac{1}{4\delta_0}\left[Y^2+\frac{\kappa}{\gamma_\perp}Y\dot{Y}\right], \tag{3.19}$$

$$x_0(t,T)=\frac{1}{2\delta_0}\left[Y+\frac{\kappa}{\gamma_\perp}\dot{Y}\right]+2\alpha(t)\cos T, \tag{3.20}$$

where

$$T=\epsilon^{-1/2}\int^t\{[2\gamma_\perp\delta_0(t'')/\kappa]^{1/2}+O(\epsilon^{1/2})\}dt'' \tag{3.21}$$

and

$$\dot{\alpha}=-\frac{\alpha}{4\delta_0}\left[\frac{Y^2}{2\delta_0}\frac{\gamma_{\|}}{\kappa}\left[1+\frac{\kappa}{\gamma_\perp}\frac{\dot{Y}}{Y}\right]+\dot{\delta}_0+2\delta_0\left[1+\frac{\gamma_\perp}{\kappa}\right]\right]. \tag{3.22}$$

To derive (3.19) and (3.20) we have assumed that $Y=Y(t)$, i.e., Y varies only on the slow time scale.

A few remarks concerning our results [(3.19) and (3.20)] are in order at this point. First, we notice that x_0 has a slow variation induced by the input field (proportional to Y and $\dot{Y}$) and a fast oscillatory variation (proportional to α) whereas δ_0 has no oscillations. This result may seem odd on the basis of the linear-stability analysis that lead to (3.16) because we would expect either all solutions or none of them to oscillate. The explanation of this difficulty is that the eigenvalues (3.16) give only part of the solution; a complete solution requires the determination of the corresponding eigenfunctions to give the relative weight of each eigenvalue for each solution x, s, or δ. By solving the system (3.12) and (3.13) which is equivalent to our starting Eqs. (3.5)—(3.7) we have determined at once the eigenvalues and the eigenfunctions, albeit in implicit form because all expressions depend on δ_0. Our result shows that to leading order in ϵ, the oscillations do not contribute to δ_0. They do contribute, however, to the next order, i.e., to δ_1. Second, the fact that δ_0 is still given by a closed differential equation whereas x_0 is explicitly given in terms of time-dependent functions does not imply that we have performed a procedure tantamount to an adiabatic elimination of x_0. Consider, for the sake of the argument, a constant Y. Since $t=\kappa t'$ the time evolution of δ_0 will solely depend on $\gamma_{\parallel}$. Consequently the nonoscillating part of x_0 will indeed follow adiabatically the time evolution of δ_0. This is no longer true for the oscillating part since $\alpha(t)$ may relax on a much different scale. The picture which emerges from (3.19) and (3.20) is therefore the following: To leading order in ϵ there appears a hierarchy between the atomic and the field dynamics in the sense that the atomic dynamic depends only on $\gamma_\perp$, $\gamma_{\parallel}$, and Y in the general case ($\dot{Y}\neq 0$) whereas the field dynamics depends on all parameters and on δ_0. Because of this fact, the atomic dynamics is unaffected by the cavity quality; it is the amplitude of the oscillations of the field intensity which will be affected by the cavity quality. Third, we may define a new time $\tau'=\gamma_{\parallel}t'$ and recast the equation for δ_0 in the form

$$\dot{\delta}=1-\delta-\frac{1}{4\delta}\left[Y^2+\frac{\gamma_{\parallel}}{\gamma_\perp}Y\dot{Y}\right], \tag{3.23}$$

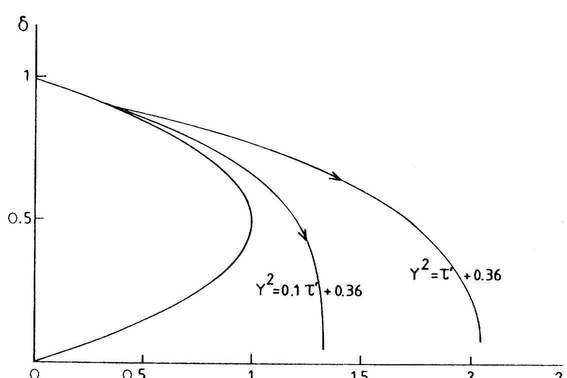

FIG. 2. Stationary vs time-dependent atomic population for two sweeping velocities.

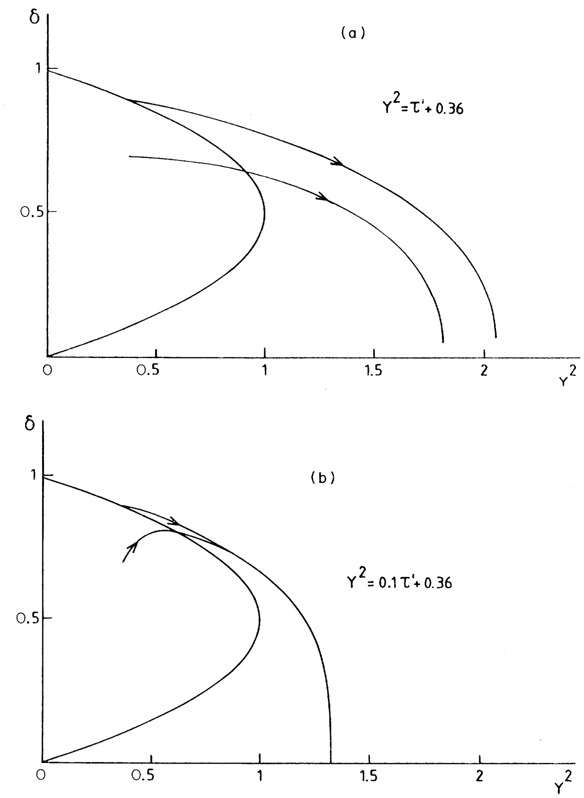

FIG. 3. Influence of the initial condition on the dynamic response of the system. (a) For $Y^2=\tau'+0.36$ the two curves remain different. (b) For $Y^2=0.1\tau'+0.36$ the two curves differ only for very short times.

where the dot stands for the derivative with respect to τ' (since no confusion is possible we now drop the subscript 0). Hence the atomic dynamics is governed by a generic equation if the input field is constant. Otherwise, only the ratio of the atomic decay rates enters as a relevant parameter apart from Y and $\dot{Y}$. In Sec. V we shall present an analytic discussion of (3.23). Nevertheless, we shall close this section by commenting on some general properties of Eq. (3.23) when Y is time dependent and $\gamma_{\parallel}=\gamma_\perp$. We have integrated this equation for various time-dependent input-field amplitudes. The results are shown in Figs. 2—4. On all these figures we have drawn the stationary solution as a reference; the time-dependent solutions are labeled with an arrow pointing in the direction of increasing time. Initially the system is in a steady state.

In Fig. 2 we consider the influence of the sweeping velocity. The initial condition is $Y(0)=0.6$ and $\delta(0)=0.9$ which is on the stationary curve. As expected the dynamical curves will lie near the stationary curve outside the vicinity of the critical point at $Y=1$ (i.e., $y=y_M$); the slowest Y varies, the nearest $\delta(\tau')$ is from its stationary value. But in the vicinity

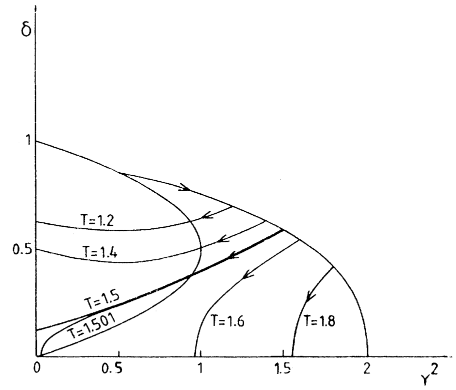

FIG. 4. Back and forth sweep. Input-field intensity is first increased according to $Y^2=0.5+\tau'$ until $Y^2=T$. It is then decreased according to $Y^2=T-\tau'$.

of $Y=1$, there occurs a critical slowing down. As a result, no matter how slowly Y varies, there exists a domain around $Y=1$ where the relaxation time of the system is so long that $\delta(\tau')$ will no longer be able to follow the stationary curve; hence the departure between the dynamical and the stationary curves seen in Fig. 2. This means that a time dependent Y induces a dynamical hysteresis which may be very different from the stationary hysteresis. This question will be treated analytically in more detail in Sec. V.

In Fig. 3 we consider the influence of the initial conditions. Since $Y=Y(\tau')$ there is no stationary solution and therefore the whole evolution of δ will remain dependent on the initial condition. However, this dependence may be washed out by the fact that for $0<Y\ll 1$ all solutions are attracted by the stationary solution. Hence if Y varies slowly enough, the dependence on the initial conditions will be negligible; as the sweeping velocity is increased, this dependence will be more and more pronounced. For the particular values considered in Fig. 3, $\delta(\tau')$ remains different during its whole evolution when $Y^2=\tau'+0.36$, whereas the two curves practically coincide, except for very short times, when $Y^2=0.1\tau'+0.36$.

Figure 4 refers to a class of experiments where the field intensity is swept back and forth. Since the actual state of the BOD may seriously depart from its stationary characteristic, one must ask how this property affects the switching between the two branches of the hysteresis. A typical example is shown in Fig. 4. The initial condition is $Y^2(0)=0.5$ and $\delta(0)=0.85$. The input-field intensity increases linearly in time $Y_1^2(\tau')=Y_{(0)}^2+\tau'$. At some point $Y^2=T$ the field suddenly begins to decrease in time according to a similar law, i.e., $Y_2^2(\tau')=T-\tau'$. This shows a typical back and forth sweep. The trajectories of the backward sweep fall into two domains. If the change $\tau'\to-\tau'$ occurs "early enough" there will be no switching and the system will reach a state with $0<\delta(2T)<1$. As a matter of fact, this may be a possible way to prepare the states necessary to discuss the influence of the initial conditions (see Fig. 3). On the contrary, if the decrease of the input intensity occurs "too late" the switching process will nevertheless take place. These two domains have one curve in common, a separatrix. For the situation considered in Fig. 4, the separatrix corresponds to $T=1.5$.

Since Figs. 2—4 refer to the time evolution of δ_0 as given by Eq. (3.23) it is normal that no value of C be given. Nevertheless the choice of C does affect in an indirect way these figures because it restricts the domain of variation of both $\delta(t)$ and $Y(t)$ [see (3.4) and (3.14)]. These restrictions imply that there is a strip along both axes in which the requirement (3.4) and (3.14) no longer hold. Hence the actual domain in which (3.23) is a good approximation of the full set (2.5)—(2.7) is limited by these two strips whose width is a function of ϵ.

IV. HIGH-TRANSMISSION BRANCH

In this section we consider the time evolution of the system near the upper branch of the steady-state solution. In particular, we are interested in the dynamic response of the system when y progressively decreases. As in the preceding section, we expect that the system will, more or less, follow the stable branch until the limit point is reached. Then a jump to the lower state will occur. Our purpose in this section is to analyze the initial stage of this transition process. The remaining part of the upper branch is discussed in Appendix B.

When $C\to\infty$ and $y=O(C^{1/2})$, we know from the steady-state relation (2.10) that the upper branch is given by

$$x=C^{1/2}\tfrac{1}{2}[Y+(Y^2-8)^{1/2}]+O(1), \qquad (4.1)$$

where Y is defined through

$$y=C^{1/2}Y,\quad Y=O(1)<8^{1/2}, \qquad (4.2)$$

and

$$\delta=O(C^{-1}),\quad s=O(C^{-1/2}),$$

are related to (4.1) by (2.8) and (2.9). Since we assume that $x(t)$, $s(t)$, and $\delta(t)$ are initially in the vicinity of x, s, and δ it will be convenient to define X, S, and D by

$$x=C^{1/2}X,\quad s=C^{-1/2}S,\quad \delta=C^{-1}D, \qquad (4.3)$$

where X, S, and D are all $O(1)$ variables. Rewriting the evolution equations (2.5)—(2.7) in terms of X, S, and D, we obtain

$$\frac{\kappa}{\gamma_\perp}\dot{S}=DX-S, \tag{4.4}$$

$$\frac{\kappa}{\gamma_{||}}\dot{D}=\epsilon^{-1}(1-SX)-D, \tag{4.5}$$

$$\dot{X}=-X-2S+Y, \tag{4.6}$$

where $\epsilon\equiv C^{-1}\ll 1$.

Equations (4.4)–(4.6) admit the following initial conditions:

$$S(0)=S_i+O(\epsilon^{1/2}),$$
$$X(0)=X_i+O(\epsilon^{1/2}), \tag{4.7}$$
$$D(0)=D_i+O(\epsilon^{1/2}),$$

where S_i, D_i, and X_i are specified $O(1)$ quantities verifying

$$1-X_iS_i=0. \tag{4.8}$$

This last relation guarantees that the initial evolution of the solution remains $O(1)$. From (4.5) we may express S as a function of D and X:

$$S=\frac{1}{X}\left[1-\epsilon D-\epsilon\frac{\kappa}{\gamma_{||}}\dot{D}\right]. \tag{4.9}$$

Introducing (4.9) into (4.5)–(4.7) we obtain two equations relating D and X:

$$-\epsilon\frac{\kappa^2}{\gamma_{||}\gamma_\perp}\ddot{D}+\epsilon\dot{D}\left[-\frac{\kappa}{\gamma_\perp}-\frac{\kappa}{\gamma_{||}}+\frac{\kappa^2}{\gamma_{||}\gamma_\perp}\frac{\dot{X}}{X}\right]+D\left[\epsilon\frac{\kappa}{\gamma_\perp}\frac{\dot{X}}{X}-X^2-\epsilon\right]-\frac{\kappa}{\gamma_\perp}\frac{\dot{X}}{X}+1=0, \tag{4.10}$$

$$\dot{X}+\frac{2}{X}\left[1-\epsilon D-\frac{\kappa}{\gamma_{||}}\epsilon\dot{D}\right]-Y+X=0, \tag{4.11}$$

$$X(0)=X_i+O(\epsilon^{1/2}),$$
$$D(0)=D_i+O(\epsilon^{1/2}), \tag{4.12}$$
$$\dot{D}(0)=\epsilon^{-1/2}\xi+O(1),$$

where $\xi=O(1)$ is obtained from (4.5) and (4.7).

Analyzing the linear stability of the stationary solutions of either (4.10) and (4.11) or (4.4)–(4.6) yields the following eigenvalues:

$$\lambda_a=-1+2D+O(\epsilon),$$

$$\lambda_{b,c}=-\frac{1}{2}\left[\frac{\gamma_{||}+\gamma_\perp}{\kappa}+\frac{2}{X_0^2}\right]\pm i\epsilon^{-1/2}X(\gamma_{||}\gamma_\perp/\kappa^2)^{1/2}+O(\epsilon^{1/2}).$$

The imaginary part of these eigenvalues is the Rabi frequency.

Motivated by the linear-stability analysis of the upper branch of the steady-state solution when $C\to\infty$, we seek a solution to Eqs. (4.10) and (4.11) of the form

$$X(T,t,\epsilon^{1/2})=X_0(T,t)+\epsilon^{1/2}X_1(T,t)+\cdots, \tag{4.13}$$

$$D(T,t,\epsilon^{1/2})=D_0(T,t)+\epsilon^{1/2}D_1(T,t)+\cdots, \tag{4.14}$$

where T is a rapid time variable defined by

$$T=\epsilon^{-1/2}\sigma(\epsilon,t)=\epsilon^{-1/2}[\sigma_0(t)+\epsilon^{1/2}\sigma_1(t)+\cdots] \tag{4.15}$$

and $\sigma(\epsilon,t)$ is an unknown function of t which will be determined by the multiscale perturbation procedure. In a first approximation, i.e., to leading order in $\epsilon^{1/2}$, we find the following results:

$$\dot{X}_0=Y(t)-X_0-2/X_0, \tag{4.16}$$

$$X_0(0)=X_i,$$

$$D_0=\frac{1}{X_0^2}\left[1-\frac{\kappa}{\gamma_\perp}\frac{\dot{X}_0}{X_0}\right]+2\alpha(t)\cos T, \tag{4.17}$$

where

$$T=\epsilon^{-1/2}(\gamma_{||}\gamma_\perp/\kappa^2)^{1/2}\int^t X_0(s)ds=\epsilon^{-1/2}\sigma_0(t), \tag{4.18}$$

and $\alpha(t)$ represents a decaying function of time:

$$\dot{\alpha}=-\frac{1}{2}\alpha\left[\frac{2}{X_0^2}+\frac{\gamma_{||}+\gamma_\perp}{\kappa}\right]. \tag{4.19}$$

Both variables X and D rapidly oscillate on the fast time scale T. However, the amplitude of these oscillations is $O(\epsilon)$ for X but $O(1)$ for D. The long-time behavior of the system is determined by Eq. (4.16) for X_0 while D_0 is passively related to X_0 by Eq. (4.18). Equation (4.16) cannot be solved exactly (except when $Y=0$) and has been analyzed numerically. Figure 5 represents a typical evolution of $X_0(t)$. As previously noted the jump is considerably delayed when $Y=Y(t)$.

V. ANALYTIC RESULTS

In this section, we analyze Eq. (3.23) in detail for three different time dependences of $\dot{Y}(\tau')$. First, we consider the stationary case $\dot{Y}=0$. We show that in a vicinity of the limit point $Y=1$, $\delta=\frac{1}{2}$, the system exhibits a critical slowing down, i.e., the relaxation time of the system decreases and eventually vanishes. Our analysis complements earlier investiga-

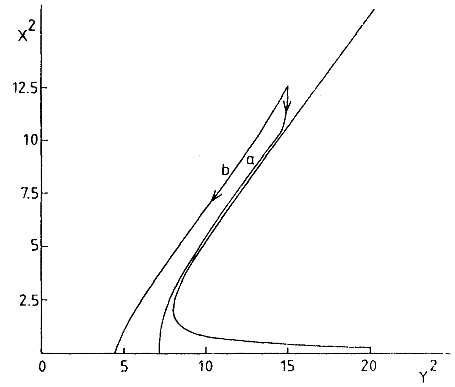

FIG. 5. Down switching under influence of an output intensity decreasing according to $Y^2(\tau')=Y^2(0)-\alpha\tau'$. Initial conditions: $Y^2(0)=15$ and $X^2(0)=12.5$. Sweeping velocities: $\alpha=0.1$ for curve a and $\alpha=1.0$ for curve b.

tions by Bonifacio and Meystre.[18] There the emphasis was on the linear evolution of small perturbations of the steady state when Y approached the limit point from the left. The present analysis considers nonlinear evolution of the system on both sides of the critical point. Second, we examine the case of a gradual, slow increase of $Y(\tau')$ (with $\dot{Y}<1$). As expected the system will approximately follow the lower branch until the turning point is reached. Then a jump to the upper branch will occur. We show that this jump is not always possible if Y is slowly decreased after it has crossed the turning point. Third, we concentrate on the behavior of the system when Y is rapidly changed ($\dot{Y}>1$). Finally, we mention that similar results can be found for the transition from the upper to the lower branches by analyzing Eq. (4.16). This will not be done here.

A. Critical slowing down

We consider Eq. (3.23) with the initial point located near the turning point. More precisely the system is first located near the low-transmission branch. The external field Y is then suddenly changed to a new fixed value near (but greater or smaller than) the limit point $Y=1$. We characterize this vicinity by a small parameter η:

$$\eta=[(Y-1)/e]^{1/2}\ll 1, \tag{5.1}$$

where $e=\pm 1$ when $Y-1\gtrless 0$ and by assuming the following expansion for the initial condition:

$$\delta_i(\eta)=\tfrac{1}{2}+\eta\xi_1+\eta^2\xi_2+\cdots\ . \tag{5.2}$$

We then seek a solution of Eq. (3.23) of the form

$$\delta(\tau')=\delta(\tau,\eta)=\tfrac{1}{2}+\eta\delta_1(\tau)+\eta^2\delta_2(\tau)+\cdots\ , \tag{5.3}$$

where τ is a slow time defined by

$$\tau=\eta\tau'\ . \tag{5.4}$$

We introduce (5.1)—(5.4) into Eq. (3.23); to leading order in η, we find

$$\frac{d\delta_1}{d\tau}=-e-2\delta_1^2\ ,\qquad \delta_1(0)=\xi_1\ . \tag{5.5}$$

The solution of Eq. (5.5) depends on the sign of e.

1. *Precritical region:* $e=-1$ ($Y<1$)

We have

$$\delta_1(\tau)=-\frac{1}{2^{1/2}}\frac{1+b\exp(c\tau)}{1-b\exp(c\tau)}\ , \tag{5.6}$$

where

$$b=\frac{2^{1/2}\xi_1+1}{2^{1/2}\xi_1-1},\quad c=2^{3/2}\ . \tag{5.7}$$

From (5.6) we note that

$$\delta_1(\tau)\to\frac{1}{2^{1/2}}\ \text{as}\ \tau\to\infty\ \text{if}\ \xi_1>-\frac{1}{2^{1/2}}\ , \tag{5.8a}$$

$$\delta_1(\tau)\simeq\frac{2^{1/2}}{c}\frac{1}{\tau-\tau_c}\ \text{as}\ \tau\to\tau_c=\frac{1}{c}\ln\left[\frac{1}{b}\right]\ \text{if}\ \xi_1<-\frac{1}{2^{1/2}}\ . \tag{5.8b}$$

Clearly, $\delta_1(\tau)$ is bounded for all τ if $\delta_1(0)>-1/2^{1/2}$ and, as $\tau\to\infty$, $\delta_1(\tau)$ approaches a stable steady state $\delta_1=1/2^{1/2}$. On the other hand, if $\delta_1(0)<-1/2^{1/2}$, $\delta_1(\tau)$ becomes unbounded at a critical time $\tau=\tau_c$. In other words, depending on the initial condition, the system will remain on the lower branch or will jump to the upper branch notwithstanding the fact that $Y<1$.

2. *Postcritical region:* $e=1$ ($Y>1$)

We have

$$\delta_1(\tau)=\frac{1}{2^{1/2}}\tan(b-2^{1/2}\tau)\ , \tag{5.9}$$

where

$$b=\tan^{-1}(2^{1/2}\xi_1)\ . \tag{5.10}$$

From (5.9) we note that

$$\delta_1(\tau)=-\frac{1}{2}\frac{1}{\tau-\tau_c}\quad \text{as}\ \tau\to\tau_c=\frac{1}{2^{1/2}}(b+\tfrac{1}{2}\pi)\ . \tag{5.11}$$

Thus $\delta_1(\tau)$ becomes unbounded when τ approaches the critical time τ_c, i.e., the transition to the upper branch will occur when $Y>1$ regardless of the initial condition.

B. Slow passage through criticality

Our purpose is to describe the dynamical response of the system to slow variations in Y. In particular, we shall examine the transition from the lower branch solution to the jump solution. We characterize the slow varitation of Y by introducing a small parameter μ and a slow time τ, where

$$\tau=\mu\tau', \quad 0<\mu\ll 1 \; . \tag{5.12}$$

With (5.12), Eq. (3.23) can be rewritten as

$$\mu\dot{\delta}=\mu\frac{d\delta}{d\tau}=1-\delta-\left[Y^2+\mu\frac{\gamma_{\parallel}}{\gamma_{\perp}}\dot{Y}Y\right]\frac{1}{4\delta} \; , \tag{5.13}$$

where $Y=Y(\tau)$. At an initial time $\tau=\tau_i<0$:

$$\delta(\tau_i)=\delta_i, \quad Y(\tau_i)=Y_i \; . \tag{5.14}$$

We choose $Y(\tau)$ to be a smooth monotonically increasing function which has the power-series representation

$$Y(\tau)=1+\tau+O(\tau^2) \; , \tag{5.15}$$

as $\tau\to 0$. This specifies $\tau=0$ to be the instant at which the limit point is reached. From the definition of μ there is no loss of generality in setting $\dot{Y}(0)=1$.

With the initial conditions (5.14), the system first rapidly evolves $[\tau=O(\mu)]$ to a slowly varying regime in the vicinity of the stationary solution; this slow regime can be described in the following two distinct stages.

1. *Precritical stage*

We first seek a solution of (5.13) of the form

$$\delta(\tau,\mu)=\delta_0(\tau)+\mu\delta_1(\tau)+O(\mu^2) \; . \tag{5.16}$$

After introducing (5.16) into (5.13) we obtain the following results. The leading-order solution has the same form as the static solution, i.e.,

$$\delta_0(\tau)=\tfrac{1}{2}\{1-[1-Y^2(\tau)]^{1/2}\} \; . \tag{5.17}$$

Knowing $\delta_0(\tau)$ we determine $\delta_1(\tau)$:

$$\delta_1(\tau)=\frac{4\delta_0^2}{Y^2-4\delta_0^2}\left[\dot{\delta}_0+\frac{1}{4\delta_0}\frac{\gamma_{\parallel}}{\gamma_{\perp}}\dot{Y}Y\right] . \tag{5.18}$$

The asymptotic expression (5.15) then shows that (5.17) is valid only for $\tau<0$. Indeed we find that

$$\delta_0(\tau)=\tfrac{1}{2}-\tfrac{1}{2}(-2\tau)^{1/2}+O(\tau) \; , \tag{5.19}$$

$$\delta_1(\tau)=\frac{1}{4(-2\tau)}+O((-\tau)^{-1/2}) \; , \tag{5.20}$$

as $\tau\to 0^-$. Since δ_1 becomes large when $\tau\to 0^-$, the expansion (5.16) is no longer valid. We therefore expect a different behavior of the solution near the limit point.

2. *Transition stage*

To describe the new development of the solution we introduce a new time scale s defined by

$$s=\mu^{-2/3}\tau \; . \tag{5.21}$$

Equations (5.13) and (5.15) are transformed into

$$\mu^{1/3}\dot{\delta}=\mu^{1/3}\frac{d\delta}{ds}=1-\delta-\frac{1}{4\delta}[1+2\mu^{2/3}s+O(\mu)] \; , \tag{5.22}$$

$$Y(s)=1+\mu^{2/3}s+O(\mu^{4/3}) \; , \tag{5.23}$$

and the new expansion for δ is

$$\delta=\tfrac{1}{2}+\mu^{1/3}d_1(s)+O(\mu^{2/3}) \; . \tag{5.24}$$

From (5.22) and (5.23) we observe that the transition solution (5.24) is restricted to a small area of the limit point $Y=1$, $\delta=\frac{1}{2}$. Substituting (5.24) into (5.22), we obtain the following Ricatti equation for d_1:

$$\dot{d}_1=-s-2d_1^2 \; . \tag{5.25}$$

This equation must be solved with the condition

$$d_1\to -2^{-1/2}(-s)^{1/2}-\frac{1}{8s} \quad \text{as } s\to-\infty \tag{5.26}$$

which comes from matching $\delta=\frac{1}{2}+\mu^{1/3}d_1(s)$ as $s\to-\infty$ with the precritical solution $\delta=\delta_0+\mu\delta_1$ as $\tau\to 0^-$. The solution of (5.25) and (5.26) is

$$d_1(s)=-2^{-2/3}\frac{\dot{\mathrm{Ai}}(\zeta)}{\mathrm{Ai}(\zeta)} \; , \tag{5.27}$$

where

$$\dot{\mathrm{Ai}}(\zeta)=\frac{d\,\mathrm{Ai}(\zeta)}{d\zeta}, \quad \zeta=-2^{1/3}s \; , \tag{5.28}$$

and $\mathrm{Ai}(\zeta)$ denotes the Airy function.[19] This solution is valid only for

$$\zeta>\zeta_0 \quad (s<s_0) \tag{5.29}$$

where ζ_0 is the first zero of $\mathrm{Ai}(\zeta)$:

$$\zeta_0\simeq -2.3381 \quad (s_0\simeq 1.8558) \; . \tag{5.30}$$

When $\xi \to \xi_0^+$ $(s \to s_0^-)$, $d_1(s)$ becomes singular:

$$d_1(s) \simeq -2^{-2/3}\frac{1}{\xi-\xi_0} = -\frac{1}{2}\frac{1}{s_0-s} . \tag{5.31}$$

In summary, we have shown that the solution follows the low-transmission branch with the slow time dependence $\mu\tau$ and passes through criticality with the slightly faster time dependence $\mu^{1/3}\tau'$. When $Y(\mu\tau')$ approaches the critical value

$$Y_c = 1+\mu^{2/3}s_0+O(\mu^{4/3}) , \tag{5.32a}$$

the solution leaves the slowly varying regime and goes on to the initial stage of the jump, varying on the fast scale τ'. However, (5.32) does not represent the critical limit for a transition between the lower and the upper branches of the steady states for the back and forth sweeping process. Indeed if $Y(\mu\tau')$ is slowly decreased after it has crossed the limit point of the steady states, the jump occurs provided $Y(0) > Y_c'$:

$$Y_c' = 1+\mu^{2/3}s_0'+O(\mu^{4/3}) , \tag{5.32b}$$

where $s_0' = -2^{-1/3}\xi_0'$ and ξ_0' is the first zero of $\mathrm{Ai}(\xi)$.

By examining Eq. (5.25) the problem can be studied in the phase plane (d_1, s). The latter is shown in Fig. 6 which represents d_1 as a function of time. From the practical viewpoint, our analysis thus shows that the transition between the steady states occurs only if Y has crossed a critical point larger than the natural limit point of the steady-state solutions. For slow $O(\mu)$ variations of Y this limit is given by (5.32) indicating a $O(\mu^{2/3})$ deviation from the limit point. For larger $O(1)$ variations of Y, this deviation can be important as suggested by the numerical results presented in Sec. III.

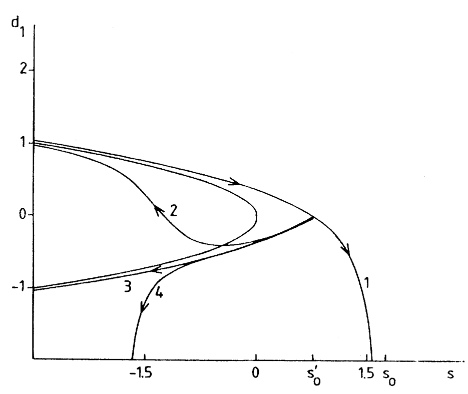

FIG. 6. Back and forth sweep in the vicinity of the limit point $Y=1$. Curve 1 verifies $\dot{d}_1 = -s-2d_1^2$; curves 2–4 verify $\dot{d}_1 = -(a-s)-2d_1^2$ with $a = s_0'-\epsilon$ for 2, $a = s_0'$ for 3, and $a = s_0'+\epsilon$ for 4 where s_0' is the first zero of the derivative of the Airy function.

C. Rapid variation of $Y(\tau')$

We now consider the situation where Y changes rapidly. We analyze two cases: $\gamma_{\parallel}/\gamma_{\perp} = O(1)$ and $\theta = \gamma_{\parallel}/\gamma_{\perp} < 1$. Let us first consider the case $\gamma_{\parallel}/\gamma_{\perp} = O(1)$. Let

$$Y(\tau') = \alpha + \beta\tau' ,$$

where $\alpha = O(1)$ and $\beta \gg 1$. We seek a solution of Eq. (3.23) of the form

$$\delta(\tau') = \delta(\tau, \beta^{-q}) = \delta_0(\tau) + \beta^{-q}\delta_1(\tau) + O(\beta^{-2q}) , \tag{5.33}$$

where

$$\tau = \beta^p\tau' = O(1) , \tag{5.34}$$

$$p > 0, \quad q > O . \tag{5.35}$$

Introducing (5.33)–(5.35) into (3.23), and identifying terms in Eq. (3.23) that may be neglected in the asymptotic limit $\beta \to \infty$, we find that

$$p = 1 \tag{5.36}$$

and the leading-order solution must satisfy the following equation:

$$\dot{\delta}_0 = -\frac{1}{4\delta_0}\frac{\gamma_{\parallel}}{\gamma_{\perp}}(\alpha+\tau) , \qquad \delta_{(0)} = \delta_i . \tag{5.37}$$

The solution of (5.37) is given by

$$\delta_0(\tau) = \left[\delta_i^2 - \frac{\gamma_{\parallel}}{\gamma_{\perp}}(\alpha\tau + \tfrac{1}{2}\tau^2)\right]^{1/2} . \tag{5.38}$$

From (5.38) we note that δ_0 approaches zero when

$$\tau \to \tau_c = -\alpha + \left[\alpha^2 + 4\frac{\gamma_{\perp}}{\gamma_{\parallel}}\delta_i^2\right]^{1/2} . \tag{5.39}$$

The critical time τ_c corresponds to the following value of Y:

$$Y_c = \alpha + \tau_c = \left[\alpha^2 + 4\frac{\gamma_{\perp}}{\gamma_{\parallel}}\delta_i^2\right]^{1/2} . \tag{5.40}$$

From (5.40) we conclude that the jump $(\delta_0 \to 0)$ is subcritical when $Y_c < 1$, i.e., when

$$\alpha^2 + 4\frac{\gamma_{\perp}}{\gamma_{\parallel}}\delta_i^2 < 1 , \tag{5.41a}$$

which implies the following conditions for α and δ_i:

$$\alpha < 1 , \qquad \delta_i < \left[\frac{\gamma_{\parallel}}{4\gamma_{\perp}}(1-\alpha^2)\right]^{1/2} ; \tag{5.41b}$$

conversely, the jump will be supercritical when $Y_c > 1$, i.e., when

$$\alpha^2+4\frac{\gamma_\perp}{\gamma_{\|}}\delta_i^2>1 , \tag{5.42a}$$

which implies the following conditions for α and δ_i:

$$\alpha<1 ,\quad \delta_i > \left[\frac{\gamma_{\|}}{4\gamma_\perp}(1-\alpha^2)\right]^{1/2} . \tag{5.42b}$$

Hence we have shown that although $Y(\tau')$ varies rapidly, $\delta(\tau')$ nevertheless becomes rapidly zero at a critical value of Y. The latter always remains at a $O(1)$ distance from the limit point $Y=1$. Under some conditions the jump even occurs before this point.

The previous analysis is valid for $\gamma_{\|}/\gamma_\perp=O(1)$. We now consider the case $\theta=\gamma_{\|}/\gamma_\perp<1$ and show that different behaviors may be expected. Again we assume

$$Y(\tau')=\alpha+\beta\tau' , \tag{5.43}$$

where $\alpha=O(1)$ and β admits the following expansion:

$$\beta=\theta^{-q}b+O(\theta^{-q+1}),\quad b=O(1),\quad q>0 . \tag{5.44}$$

We seek a solution of the form

$$\delta(\tau')=\delta(\tau,\theta^r)=\delta_0(\tau)+\theta^r\delta_1(\tau)+O(\theta^{2r}) , \tag{5.45}$$

where

$$\tau=\theta^{-p}\tau'=O(1) , \tag{5.46}$$

$$r>0,\quad p>0 . \tag{5.47}$$

Inserting (5.43)–(5.47) into (3.23) and identifying terms that may be neglected in the limit $\theta\to 0$, we find the following results:

(1) $q<\frac{3}{2}$, $p=2q/3$. The leading-order solution verifies the equation

$$\dot{\delta}_0=-\frac{1}{4\delta_0}b^2\tau^2 , \tag{5.48}$$

whose solution is

$$\delta_0(\tau)=\left[\delta_i^2-\frac{b^2\tau^3}{6}\right]^{1/2} . \tag{5.49}$$

(2) $q=\frac{3}{2}$, $p=1$. The evolution of δ_0 is now governed by the equation

$$\dot{\delta}_0=-\frac{1}{4\delta_0}(b^2\tau^2+b^2\tau) , \tag{5.50}$$

and its solution is

$$\delta_0(\tau)=\left[\delta_i^2-\frac{b^2}{2}\left[\frac{\tau^3}{3}+\frac{\tau^2}{2}\right]\right]^{1/2} . \tag{5.51}$$

(3) $q>\frac{3}{2}$, $p=q-\frac{1}{2}$. In this case δ_0 verifies

$$\dot{\delta}_0=-\frac{1}{4\delta_0}b^2\tau , \tag{5.52}$$

so that δ_0 is given by

$$\delta_0(\tau)=\left[\delta_i^2-\frac{b^2}{4}\tau^2\right]^{1/2} . \tag{5.53}$$

For each case, we observe that the jump $[\delta_0(\tau)\to 0]$ appears at a critical value of Y which is located at a large distance of the limit point $Y=1$. Indeed if $\tau=\tau_c$ is defined as the time for which $\delta_0(\tau)=0$, $Y_c\simeq\tau_c\theta^{-q/3}$ in the first case while $Y_c\simeq\tau_c\theta^{-1/2}$ in the two other cases. We have thus demonstrated that the effect of small $\gamma_{\|}/\gamma_\perp$ is to delay the jump.

VI. DISCUSSION

Our main result is that in the limit of large C the Maxwell-Bloch equations can be decoupled and the study of absorptive OB may be reduced to the analysis of a single nonlinear differential equation. Furthermore, a salient feature of this procedure is that it leads to an equation for Δ, the population difference between the two levels, in the low-transmission domain and to an equation for x, the output-field amplitude, in the high-transmission domain. These two equations are not entirely new. Indeed, recently Drummond[14] derived an equation for Δ assuming $\gamma_{\|}\ll\kappa$ and $\gamma_\perp$; when we take into account that $\Delta\gg 1$ in the low-transmission domain and assume $\dot{Y}=0$, his equation reduces to our Eq. (3.19). Another result in the derivation by Benza and Lugiato[13] of an equation for x assuming $\kappa\ll\gamma_{\|}$, and $\gamma_\perp$; when we take into account that $x\gg 1$ is the high-transmission domain, their equation reduces to our Eq. (4.16). However, there is an essential difference between the present work and the two studies just mentioned. Indeed in the treatment of Drummond and of Benza and Lugiato an adiabatic elimination of two variables is performed, leading to a unique differential equation and therefore a unique relaxation mechanism for both high- and low-transmission branches. By contrast our analysis leads to a nonlinear differential equation for a different dynamical variable in each transmission domain. Our analysis also indicates that in first approximation the fundamental variable is not oscillating whereas the other two variables display oscillations on a short time scale. More precisely the remaining variables can be expressed as the sum of two terms: the first one adiabatically follows the behavior of the fundamental variable and the second represents rapid oscillations decaying on a different time scale. Although these oscillations have a frequency proportional to $\epsilon^{-1/2}$ or to ϵ^{-1}, they are damped over a time which, to leading order, is independent of ϵ and therefore comparable with the relaxation time of the fundamental variable [see, for instance, Eq. (B13) in Appendix B]. Since the two fundamental variables are different for the low- and high-transmission branches, the transition between the stable states necessarily presents a change from a monotonic (oscillatory) to an oscillating (monotonic) behavior for Δ and x. Furthermore, the frequency of the oscillations gradually varies with time and is proportional to the fundamental variable δ near the low-transmission branch and proportional to x in the vicinity of the high-transmission branch.

Two typical kinds of situations can be considered in OB: either the input field is abruptly changed, or it is swept through the bistability region. In the former case, the main interest lies in a description of the jump. In Sec. V A we have analyzed analytically Eq. (3.23) and described the behavior of the system when the initial point is located near the limit point. In addition to the nonlinear critical slowing down, we have shown that the dynamic variables are slowly changing during an induction period before presenting an abrupt jump process. Numerical solutions of Eq. (3.23) indicate that this phenomenon appears significantly when $Y-1 \leq 0.1$ The existence of this induction time has already been reported earlier[17,20] but only on the basis of numerical results and in the frame of adiabatic elimination schemes. In a separate publication[21] the two fundamental equations (3.23) and (4.16) have been solved and the up- and down-switching time have been evaluated analytically. A much richer situation results when $y(t)$ is swept across the bistable domain. We have only considered sweeps in which the input intensity varies linearly in time. The question to ask is what factors may influence and how do they influence the dynamical (i.e., the time-dependent) response of the system. This question is studied analytically in Secs. V B and V C for the low-transmission branch and the beginning of the jump towards the high-transmission branch. On the basis of these analytic results we have made a numerical study of the time dependence of $\delta(\tau')$. The results are summarized in Figs. 7 and 8. Figure 7 indicates that a subcritical jump is possible provided the sweeping velocity is sufficiently large. This phenomenon was already predicted by Eq. (5.40). Subcritical jumps were not observed earlier because they require unusual initial conditions. However, Fig. 4 shows that in a series of back and forth sweeps such "initial conditions"

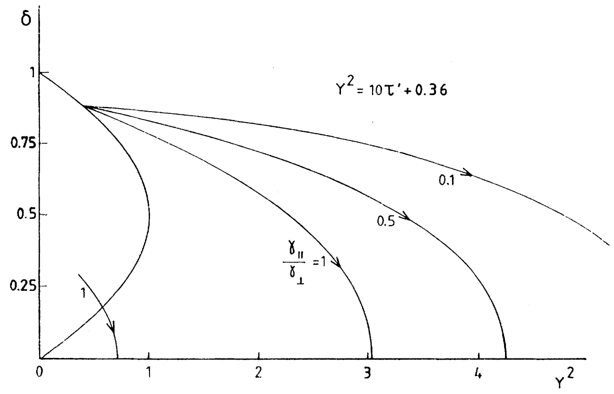

FIG. 7. Examples of subcritical ($Y<1$) and supercritical ($Y>1$) transitions; influence of the ratio $\gamma_{\parallel}/\gamma_{\perp}$ for a sweeping velocity ten times greater than in Fig. 8. Two initial conditions are $Y^2(0)=0.36$ with $\delta(0)=0.9$ and 0.3.

may be generated. The remaining curves in Fig. 7 represent the response of the system as the ratio $\gamma_{\parallel}/\gamma_{\perp}$ is progressively decreased. We observe that the jump is considerably delayed. This result is verified when the sweeping velocity is $O(1)$ as shown in Fig. 8.

In conclusion, we have proved that the time dependence of y introduces a very serious bias by creating a dynamical hysteresis which differs from the stationary hysteresis. The dynamical hysteresis may be smaller or larger than the stationary cycle. In the whole domain of variation of the input field $y(t)$, two relevant parameters are the initial condition and the sweeping velocity. In addition, the ratio $\gamma_{\parallel}/\gamma_{\perp}$ influences the non-oscillatory part of the response in the vicinity of the low-transmission branch but not near the high-transmission branch. All other parameters will only influence the relation (2.4) between the original and the reduced variables.

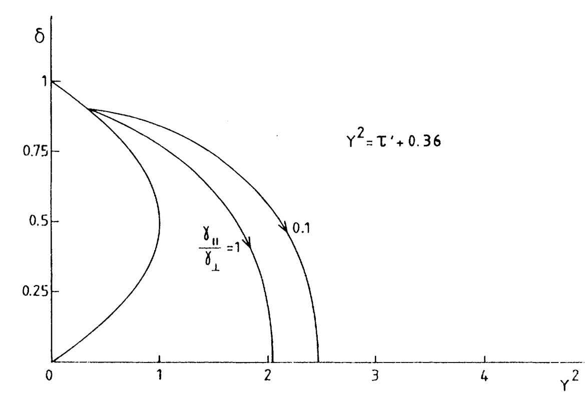

FIG. 8. Influence of the ratio $\gamma_{\parallel}/\gamma_{\perp}$ on the dynamical response.

ACKNOWLEDGMENTS

We wish to acknowledge fruitful discussions with P. D. Drummond, L. A. Lugiato, and P. Meystre. This work has been supported by the Fonds National de la Recherche Scientifique (Belgium) and the Association Euratom-Etat Belge.

APPENDIX A

In this appendix we carry through the study of the domain $x<1$ by considering the domain II in which the stationary scaling is given by (3.2). We proceed along the lines followed to investigate the domain I. Consequently we first introduce the scaling assumption for the time-dependent functions

$$y=C^{1/2}Y,\ \ x=C^{-1/2}X,\ \ s=C^{-1/2}S\ , \tag{A1}$$

and Y, X, S, $\delta\sim O(1)$. The scaled Maxwell-Bloch equations then become ($\epsilon=C^{-1}$)

$$\epsilon\dot{X}=-\epsilon X+Y-2S\ , \tag{A2}$$

$$\frac{\kappa}{\gamma_\perp}\dot{S}=-S+X\delta\ , \tag{A3}$$

$$\frac{\kappa}{\gamma_{||}}\dot{\delta}=-\delta+1-\epsilon XS\ . \tag{A4}$$

The linear-stability analysis of the stationary solutions of these equations yields the three roots

$$\lambda_a=-\gamma_{||}/\kappa+O(\epsilon)\ ,$$

$$\lambda_{b,c}=-\frac{\kappa+\gamma_\perp}{2\kappa}\pm i(2\delta\gamma_\perp/\epsilon\kappa)^{1/2}+O(\epsilon^{1/2})\ .$$

It is quite remarkable that the following three properties are common to the domains I and II:

(i) $\delta\sim O(1)$,

(ii) $\text{Im}\lambda=(2\delta\gamma_\perp/\epsilon\kappa)^{1/2}$,

(iii) $\text{Re}\lambda\sim O(1)$, $\text{Im}\lambda\sim O(\epsilon^{-1/2})$. From Eq. (A2) we express S in terms of X and Y:

$$2S=Y-\epsilon(X+\dot{X})\ .$$

Inserting this relation into Eqs. (A2) and (A3) yields two coupled equations for δ and X. The equation for X is precisely the same as the Eq. (3.12) for x whereas for δ we have

$$\frac{\kappa}{\gamma_{||}}\dot{\delta}+\delta-1+\frac{\epsilon}{2}XY-\frac{\epsilon^2}{2}X(X+\dot{X})=0\ . \tag{A5}$$

Since the eigenvalues of the linear-stability analysis have the same scalings as in domain I and since the frequency of the oscillations is the same, it is natural to seek in the long time limit solutions of (A5) and (3.12) depending on the two times t and T defined through (3.21). Using the expansions (3.18) we obtain

$$\delta_0=\delta_0(t)=1+[\delta_0(0)-1]e^{-\gamma_{||}t/\kappa}\ ,$$

$$\delta_1=\delta_1(t)=\delta_1(0)e^{-\gamma_{||}t/\kappa}\ , \tag{A6}$$

$$\delta_2=\delta_2(t),\ \ \frac{\kappa}{\gamma_{||}}\dot{\delta}_2=-\delta_2-\frac{1}{4\delta_0}\left[Y^2+\frac{\kappa}{\gamma_\perp}Y\dot{Y}\right]\ ,$$

$$X_0(t,T)=\frac{1}{2\delta_0}\left[y+\frac{\kappa}{\gamma_\perp}\dot{y}\right]+2\alpha(t)\cos T\ , \tag{A7}$$

where

$$\alpha(t)=[2\gamma_\perp\delta_0(t)/\kappa]^{-1/4}\alpha_0\exp\left[-\frac{\kappa+\gamma_\perp}{2\kappa}t\right]\ . \tag{A8}$$

Here again we have assumed that the external field amplitude Y is a function of t but not of T.

APPENDIX B

In this appendix we carry through the analysis of the upper branch by considering the domain in which the stationary solution scales like

$$y=O(C),\ \ x=O(C)\ ,$$

$$s=O(C^{-}1),\ \ \delta=O(C^{-2})\ .$$

Consequently, we introduce the following new functions:

$$y=YC,\ \ x=XC,\ \ s=SC^{-1},\ \ \delta=C^{-2}\Delta\ , \tag{B1}$$

with Y, X, S, and $\Delta\sim O(1)$ functions in terms of which the Maxwell-Bloch equations become

$$\dot{X}=-2\epsilon S-X+Y\ , \tag{B2}$$

$$\frac{\kappa}{\gamma_\perp}\dot{S}=-S+X\Delta\ , \tag{B3}$$

$$\frac{\kappa}{\gamma_{||}}\epsilon^2\dot{\Delta}=1+\epsilon^2\Delta-XS\ . \tag{B4}$$

The linear stability of the stationary solutions yields the following three eigenvalues:

$$\lambda=-1+O(\epsilon)\ ,$$

$$\lambda=-\frac{\gamma_{||}+\gamma_\perp}{2\kappa}\pm iX\epsilon^{-1}(\gamma_{||}\gamma_\perp/\kappa^2)^{1/2}+O(\epsilon)\ . \tag{B5}$$

Here again the imaginary part of λ gives the Rabi frequency. However, since x scales differently than in the vicinity of the limit point (x_m,y_m), the scaling of the λ is accordingly different:

$$\text{Re}\lambda\sim O(1)\ ,$$

$$\text{Im}\lambda\sim O(\epsilon^{-}1)\ . \tag{B6}$$

Using Eq. (B4) we express S in terms of X and Δ and insert this relation in the remaining two equations; the result is

$$\dot{X}+X-Y+\frac{2}{X}\left[\epsilon-\epsilon^3\left[\Delta+\frac{\kappa}{\gamma_{\parallel}}\dot{\Delta}\right]\right]=0\ , \quad \text{(B7)}$$

$$\frac{\kappa}{\gamma_{\perp}}\epsilon^2 X\left[\dot{\Delta}+\frac{\kappa}{\gamma_{\parallel}}\ddot{\Delta}\right]-\frac{\kappa}{\gamma_{\perp}}\epsilon^2\dot{X}\left[\Delta+\frac{\kappa}{\gamma_{\parallel}}\dot{\Delta}\right]+\epsilon^2 X\left[\Delta+\frac{\kappa}{\gamma_{\parallel}}\dot{\Delta}\right]+\frac{\kappa}{\gamma_{\perp}}\dot{X}-X+X^3\Delta=0\ . \quad \text{(B8)}$$

On the basis of (B6) we seek solutions of the last two equations depending on t and T where

$$T=\epsilon^{-1}\int^t \sigma(t')dt' =\epsilon^{-1}\int^t[\sigma_0(t')+\epsilon\sigma_1(t')+\cdots]dt'\ .$$

Let

$$X(t)=X(t,T,\epsilon)=\sum_{n:0}^{\infty}\epsilon^n X_n(t,T)\ ,$$

$$\Delta(t)=\Delta(t,T,\epsilon)=\sum_{n:0}^{\infty}\epsilon^n\Delta_n(t,T)\ .$$

We easily obtain

$$X_0=X_0(t),\quad \dot{X}_0=Y-X_0\ , \quad \text{(B9)}$$

$$X_1=X_1(t),\quad \dot{X}_1=-X_1-\frac{2}{X_0}\ , \quad \text{(B10)}$$

$$\sigma_0^2(t)=\gamma_{\parallel}\gamma_{\perp}X_0^2(t)/\kappa^2\ , \quad \text{(B11)}$$

$$\Delta_0(t,T)=\frac{1}{X_0^2}\left[1+\frac{\kappa}{\gamma_{\perp}}-\frac{\kappa}{\gamma_{\perp}}\frac{Y}{X_0}\right]+2\alpha(t)\cos T\ , \quad \text{(B12)}$$

$$\alpha(t)=\alpha(0)\exp\left[-\frac{\gamma_{\parallel}+\gamma_{\perp}}{2\kappa}t\right]\ . \quad \text{(B13)}$$

---

[1]A. Szöke, V. Daneu, J. Goldhar, and N. A. Kurnit, Appl. Phys. Lett. 15, 376 (1969).

[2]S. L. McCall, Phys. Rev. A 9, 1515 (1974).

[3]R. Bonifacio and L. A. Lugiato, Opt. Commun. 19, 172 (1976).

[4]*Optical Bistability,* edited by C. M. Bowden, M. Ciftan, and H. R. Robl (Plenum, New York, 1981).

[5]Apart from the numerous examples presented in Ref. 4 see also (a) H. M. Gibbs, S. L. McCall, and T. N. C. Venkatesan, Phys. Rev. Lett. 36, 1135 (1976); (b) K. G. Weyer, H. Wiedenmann, M. Rateike, W. R. MacGillavray, P. Meystre, and H. Walther, Opt. Commun. 37, 426 (1981).

[6](a) E. Garmire, J. H. Marburger, S. D. Allen, and H. G. Winful, Appl. Phys. Lett. 34, 374 (1979); (b) D.A.B. Miller, S. D. Smith, and A. M. Johnston, *ibid.* 35, 658 (1979); (c) Y. Ogawa, H. Itô, and H. Inaba, Jap. J. Appl. Phys. 20, L646 (1981).

[7]R. Bonifacio and L. A. Lugiato, Phys. Rev. A 18, 1129 (1978).

[8]S. S. Hassan, P. D. Drummond, and D. F. Walls, Opt. Commun. 27, 480 (1978).

[9]P. D. Drummond, IEEE J. Quantum Electron. QE-17, 301 (1981).

[10]M. Gronchi, V. Benza, L. A. Lugiato, P. Meystre, and M. Sargent III, Phys. Rev. A 24, 1419 (1981).

[11]K. Ikeda, Opt. Commun. 30, 257 (1979).

[12](a) E. L. Reiss, SIAM J. Appl. Math. 39, 440 (1980); (b) A. K. Kapila, *ibid.* 39, 21 (1980).

[13](a) R. Haberman, SIAM J. Appl. Math. 37, 69 (1979); (b) A. K. Kapila, *ibid.* 41, 29 (1981).

[14](a) A. H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973); (b) J. D. Cole, *Perturbation Methods in Applied Mathematics* (Ginn, Boston, 1968); (c) M. Van Dycke, *Perturbation Methods in Fluid Mechanics* (Academic, New York, 1964); (d) C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods For Scientists and Engineers* (McGraw-Hill, New York, 1978).

[15](a) H. Haken, *Synergetics, An Introduction* (Springer, Heidelberg, 1978); (b) *Dissipative Systems In Quantum Optics,* edited by R. Bonifacio (Springer, Heidelberg, 1982).

[16]V. Benza and L. A. Lugiato, Lett. Nuovo Cimento 26, 405 (1979).

[17]P. D. Drummond, Opt. Commun. 40, 224 (1982).

[18]R. Bonifacio and P. Meystre, Opt. Commun. 27, 147 (1978).

[19]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).

[20]R. Bonifacio and P. Meystre, Opt. Commun. 29, 131 (1979).

[21]P. Mandel and T. Erneux, Opt. Commun. 42, 362 (1982).