Dynamic symmetries in quantum electronics

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We show that two distinct realizations of the SU(3) symmetry—the Gell-Mann SU(3) symmetry in quark physics and the Elliott SU(3) symmetry in nuclear physics—can be applied to the same dynamical system in intense-field electrodynamics under different experimental conditions. We also present a set of simultaneous soliton solutions in the Elliott SU(3)-symmetry scheme which has not been derived previously.

I. INTRODUCTION

Laser physics and quantum electronics have benefited from unexpected overlaps with other relatively remote fields of science. This is particularly true in the development of new approaches to theoretical problems. One example is the thermodynamic phase-transition analog that is now well known. It has relevance to laser threshold and questions of optical bistability.[1]

In this paper we will discuss some features of a new and unexpected overlap with high-energy quark physics and nuclear physics.

The principle of unitary invariance has developed side by side historically with other forms of invariance: rotational, translational, reflectional, etc., into group-theoretical treatments of physical theories. The supermultiplet theory of Wigner,[2] Franzini and Radicati,[3] Elliott's SU(3) theory[4] of nuclear spectra, and the introduction by Gell-Mann[5] of SU(3) to be the central organizing principle of elementary particles, were some of the outstanding successes. A point not generally appreciated, however, is that not only are group considerations powerful tools in theoretical physics, but that *different realizations of the same group* can be employed for the same dynamical systems under different experimental conditions. We show in this paper that our SU(3) considerations have enabled us to display two distinct realizations of the SU(3) symmetry—the Gell-Mann SU(3) symmetry[5] in quark physics and the Elliott SU(3) symmetry[4] in nuclear physics—to be applied to the same dynamical system in intense-field electrodynamics.

In Sec. II we show that the dynamics of an important special example of two-photon laser-atom interacting system occurring at two-photon resonance exhibits the Gell-Mann SU(3) symmetry, and in Sec. III we show that under a different experimental condition when the two incident laser fields are equally detuned, the dynamics of the system exhibits the Elliott SU(3) symmetry. A set of simultaneous soliton solutions in the Elliott SU(3)-symmetry scheme is also presented. A brief summary is given in Sec. IV.

II. THREE-LEVEL QUANTUM SYSTEMS AT TWO-PHOTON RESONANCE

The dynamical evolution of an N-level atomic system can be expressed in terms of its density matrix $\hat{\rho}$ which satisfies the Liouville equation

$$i\hbar\frac{\partial\hat{\rho}}{\partial t}=[\hat{H},\hat{\rho}]\ . \tag{2.1}$$

The density matrix $\hat{\rho}(t)$ and the Hamiltonian $\hat{H}(t)$ can be expressed in terms of N^2-1 generators $\hat{s}_j$ of the SU(N) algebra by[6–9]

$$\hat{\rho}(t)=N^{-1}\hat{i}+\tfrac{1}{2}\sum_{j=1}^{N^2-1}S_j(t)\hat{s}_j\ , \tag{2.2}$$

$$\hat{H}(t)=\tfrac{1}{2}\hbar\left[\frac{2}{N}\left[\sum_{k=1}^{N}\omega_k\right]\hat{i}+\sum_{j=1}^{N^2-1}\Gamma_j(t)\hat{s}_j\right]\ , \tag{2.3}$$

where $\hbar\omega_k$ is the energy of level k, and $\hat{i}$ is a unit operator. The coefficients $S_j(t)$ and $\Gamma_j(t)$ are given by

$$S_j(t)=\mathrm{Tr}[\hat{\rho}(t)\hat{s}_j]\ , \tag{2.4}$$

$$\hbar\Gamma_j(t)=\mathrm{Tr}[\hat{H}(t)\hat{s}_j]\ , \tag{2.5}$$

if the generators $\hat{s}_j$ are chosen to satisfy

$$\mathrm{Tr}(\hat{s}_j\hat{s}_k)=2\delta_{jk}\ . \tag{2.6}$$

The evolution of the density matrix can be expressed in terms of the evolution of an (N^2-1)-dimensional real coherence vector $\vec{S}=(S_1,S_2,\ldots,S_{N^2-1})$ by

$$\frac{dS_j(t)}{dt}=\sum_{k=1}^{N^2-1}\Lambda_{jk}(t)S_k(t),\ \ j=1,2,\ldots,N^2-1 \tag{2.7}$$

where

$$\Lambda_{jk}=-\frac{1}{2i\hbar}\mathrm{Tr}(\hat{H}[\hat{s}_j,\hat{s}_k])\ . \tag{2.8}$$

We now consider a three-level atomic system in which nonzero dipole moments exist only between levels 1 and 2, and 2 and 3. Let there be two electromagnetic waves incident on the atom and the total electric field given by

$$\vec{E}(z,t)=\vec{\mathscr{E}}_{12}(t)e^{i(\nu_1t-k_1z)}+\vec{\mathscr{E}}_{23}(t)e^{i(\nu_2t-k_2z)}+\mathrm{c.c.} \tag{2.9}$$

We define $\alpha(t)$ and $\beta(t)$ in terms of the half-Rabi frequencies $\Omega_{jk}(t)$ by

$$\alpha(t)=\Omega_{12}(t)=\frac{\vec{\mathrm{d}}_{12}\cdot\vec{\mathscr{E}}_{12}(t)}{\hbar}\,,$$
$$\beta(t)=\Omega_{23}(t)=\frac{\vec{\mathrm{d}}_{23}\cdot\vec{\mathscr{E}}_{23}(t)}{\hbar}\,, \tag{2.10}$$

where $\vec{\mathrm{d}}_{jk}$ is the atomic dipole moment between levels j and k. The detunings Δ_{jk} are defined as usual by

$$\Delta_{jk}=\nu_{jk}-\omega_{jk}\,, \tag{2.11}$$

where $\omega_{jk}=\omega_j-\omega_k$, so that $\Delta_{12}=\omega_{21}-\nu_1$ and $\Delta_{23}=\omega_{32}-\nu_2$ for the level configuration shown in Fig. 1(a), $\Delta_{12}=\nu_1-\omega_{12}$ and $\Delta_{23}=\omega_{32}-\nu_2$ for the level configuration shown in Fig. 1(b), and $\Delta_{12}=\omega_{21}-\nu_1$ and $\Delta_{23}=\nu_2-\omega_{23}$ for the level configuration shown in Fig. 1(c). The convention (2.11) is designed so that the Bloch equations would be the same for any level configuration. Such three-level systems are of central importance to a number of problems of current interest including two-photon lasers,[10] trilevel echoes,[11] Raman beats,[12] multiphoton ionization,[13] and many others.[14]

We consider an important special example of two-photon laser-atom interactions which occurs at two-photon resonance when

$$|\omega_{12}|+|\omega_{23}|=|\nu_1|+|\nu_2|\,,$$

or

$$\Delta_{12}=-\Delta_{23}\equiv\Delta\,, \tag{2.12}$$

as shown in Fig. 2. We assume that $\alpha(t)$ and $\beta(t)$ have the same time dependence but possibly different amplitudes

$$\alpha(t)=a\Omega_0(t)\,,$$
$$\beta(t)=b\Omega_0(t)\,, \tag{2.13}$$

where a and b are arbitrary constants. From the physical point of view, the two-photon resonance condition puts levels 1 and 3 on "equal footing," strongly reminiscent of the isodoublet of the quark triplet, and hence suggests the use of Gell-Mann's SU(3) generators given by

$$\hat{A}_1,\hat{A}_2,\hat{A}_3 \tag{2.14a}$$

the isospin components,

$$\hat{C} \tag{2.14b}$$

the hypercharge, and

$$\hat{B}_1,\hat{B}_2,\hat{B}_3,\hat{B}_4 \tag{2.14c}$$

the operators which mix states of different strangeness for

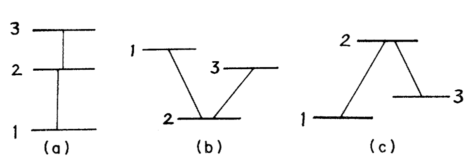

FIG. 1. Three types of three-level atoms having nonzero dipole matrix elements between levels 1 and 2, and 2 and 3. Our analysis applies to all three types.

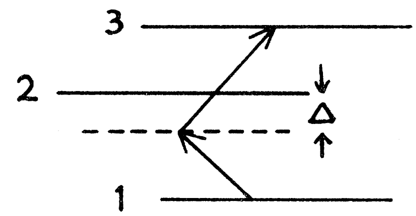

FIG. 2. Three-level system at two-photon resonance.

the problem. The commutation relations among these operators can be symbolically represented by

$$[\hat{A},\hat{A}]=\hat{A},\ \ [\hat{A},\hat{B}]=\hat{B},\ \ [\hat{A},\hat{C}]=0\,,$$
$$[\hat{B},\hat{B}]=\hat{A}+\hat{C},\ \ [\hat{B},\hat{C}]=\hat{B}\,, \tag{2.15}$$
$$[\hat{C},\hat{C}]=0\,,$$

where, for example, $[\hat{A},\hat{A}]=\hat{A}$ states that the commutator of two different members of group A is equal to a member of group A (possibly multiplied by a constant).

A possible choice of the Gell-Mann SU(3) generators which satisfy (2.15) is given in Eq. (2.17) below where we identify

$$\hat{u}_{12},\hat{v}_{12},\hat{w}_1 \text{ with } \hat{A}_1,\hat{A}_2,\hat{A}_3\,,$$
$$\hat{w}_2 \text{ with } \hat{C}\,, \tag{2.16}$$
$$\hat{u}_{23},\hat{u}_{13},\hat{v}_{13},\hat{v}_{23}, \text{ with } \hat{B}_1,\hat{B}_2,\hat{B}_3,\hat{B}_4\,,$$

$$\hat{u}_{12}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\quad \hat{u}_{23}=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},$$
$$\hat{u}_{13}=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\quad \hat{v}_{12}=\begin{bmatrix}0&i&0\\-i&0&0\\0&0&0\end{bmatrix},$$
$$\hat{v}_{23}=\begin{bmatrix}0&0&0\\0&0&i\\0&-i&0\end{bmatrix},\quad \hat{v}_{13}=\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix}, \tag{2.17}$$
$$\hat{w}_1=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad \hat{w}_2=\frac{1}{\sqrt{3}}\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&2\end{bmatrix}.$$

In terms of these, $\hat{\rho}(t)$ and $\hat{H}(t)$ can be expressed, from Eqs. (2.2) and (2.3), as

$$\hat{\rho}(t)=\tfrac{1}{3}\hat{i}+\tfrac{1}{2}(u_{12}\hat{u}_{12}+u_{23}\hat{u}_{23}+u_{13}\hat{u}_{13}$$
$$+v_{12}\hat{v}_{12}+v_{23}\hat{v}_{23}+v_{13}\hat{v}_{13}$$
$$+w_1\hat{w}_1+w_2\hat{w}_2)\,, \tag{2.18}$$

$$\hat{H}(t)=\tfrac{1}{2}\hbar\left[\tfrac{2}{3}\left[\sum_{j=1}^{3}\omega_j\right]\hat{i}-2\alpha\hat{u}_{12}-2\beta\hat{u}_{23}\right.$$
$$\left.-\Delta\hat{w}_1+\frac{1}{\sqrt{3}}\Delta\hat{w}_2\right], \tag{2.19}$$

where

$$u_{jk}=\rho_{jk}+\rho_{kj}\ ,$$
$$v_{jk}=-i(\rho_{jk}-\rho_{kj}),\quad 1\le j<k\le 3 \tag{2.20}$$
$$w_l=-[2/l(l+1)]^{1/2}(\rho_{11}+\rho_{22}+\cdots+\rho_{ll}-l\rho_{l+1,l+1})\ ,\quad 1\le l\le 2$$

following from Eqs. (2.4) and (2.5). We find that the equations of motion for the coherence vector

$$\vec{S}=(u_{12},u_{23},u_{13},v_{12},v_{23},v_{13},w_1,w_2)$$

which follow from Eqs. (2.7) and (2.8), do not exhibit any special symmetry properties which we may have hoped for, namely, the dynamical space represented by the matrix $\underline{\Lambda}$ does not factorize into smaller independent subspaces.

Instead, we introduce the following set of eight SU(3) generators:

$$\hat{U}=\epsilon^{-1}(\alpha\hat{u}_{12}+\beta\hat{u}_{23})\ ,$$
$$\hat{V}=\epsilon^{-1}(-\alpha\hat{v}_{12}+\beta\hat{v}_{23})\ ,$$
$$\hat{W}=(\tfrac{1}{2}\epsilon^{-2})[-(2\alpha^2+\beta^2)\hat{w}_1+\sqrt{3}\beta^2\hat{w}_2+2\alpha\beta\hat{u}_{13}]\ ,$$
$$\hat{v}_{13}=\hat{v}_{13}\ , \tag{2.21}$$
$$\hat{\mathscr{U}}=\epsilon^{-1}(\beta\hat{u}_{12}-\alpha\hat{u}_{23})\ ,$$
$$\hat{\mathscr{V}}=\epsilon^{-1}(\beta\hat{v}_{12}+\alpha\hat{v}_{23})\ ,$$
$$\hat{\mathscr{W}}=\epsilon^{-2}[-\alpha\beta\hat{w}_1-\sqrt{3}\alpha\beta\hat{w}_2-(\alpha^2-\beta^2)\hat{u}_{13}]\ ,$$
$$\hat{\mathscr{Y}}=(\tfrac{1}{2}\epsilon^{-2})[-\sqrt{3}\beta^2\hat{w}_1+(2\alpha^2-\beta^2)\hat{w}_2-2\sqrt{3}\alpha\beta\hat{u}_{13}]\ ,$$

where

$$\epsilon=(\alpha^2+\beta^2)^{1/2}\ . \tag{2.22}$$

Expressed in matrix form, we have

$$\hat{U}=\frac{1}{\epsilon}\begin{bmatrix}0&\alpha&0\\ \alpha&0&\beta\\ 0&\beta&0\end{bmatrix},\quad \hat{V}=\frac{i}{\epsilon}\begin{bmatrix}0&-\alpha&0\\ \alpha&0&\beta\\ 0&-\beta&0\end{bmatrix},\quad \hat{W}=\frac{1}{\epsilon^2}\begin{bmatrix}\alpha^2&0&\alpha\beta\\ 0&-\epsilon^2&0\\ \alpha\beta&0&\beta^2\end{bmatrix},$$

$$\hat{v}_{13}=i\begin{bmatrix}0&0&1\\ 0&0&0\\ -1&0&0\end{bmatrix},\quad \hat{\mathscr{U}}=\frac{1}{\epsilon}\begin{bmatrix}0&\beta&0\\ \beta&0&-\alpha\\ 0&-\alpha&0\end{bmatrix},\quad \hat{\mathscr{V}}=\frac{i}{\epsilon}\begin{bmatrix}0&\beta&0\\ -\beta&0&\alpha\\ 0&-\alpha&0\end{bmatrix},\quad \hat{\mathscr{W}}=\frac{1}{\epsilon^2}\begin{bmatrix}2\alpha\beta&0&\beta^2-\alpha^2\\ 0&0&0\\ \beta^2-\alpha^2&0&-2\alpha\beta\end{bmatrix}, \tag{2.23}$$

$$\hat{\mathscr{Y}}=\frac{1}{\sqrt{3}\epsilon^2}\begin{bmatrix}2\beta^2-\alpha^2&0&-3\alpha\beta\\ 0&-\epsilon^2&0\\ -3\alpha\beta&0&2\alpha^2-\beta^2\end{bmatrix}.$$

In terms of these generators, $\hat{\rho}(t)$ and $\hat{H}(t)$ are given by

$$\hat{\rho}(t)=\tfrac{1}{3}\hat{i}+\tfrac{1}{2}\{U\hat{U}+V\hat{V}+W\hat{W}+v_{13}\hat{v}_{13}+\mathscr{U}\hat{\mathscr{U}}+\mathscr{V}\hat{\mathscr{V}}+\mathscr{W}\hat{\mathscr{W}}+\mathscr{Y}\hat{\mathscr{Y}}\}\ , \tag{2.24}$$

$$\hat{H}(t)=\tfrac{1}{2}\hbar\left[\tfrac{2}{3}\left[\sum_{j=1}^{3}\omega_j\right]\hat{i}-2\epsilon\hat{U}+\Delta\hat{W}+\frac{1}{\sqrt{3}}\Delta\hat{\mathscr{Y}}\right]. \tag{2.25}$$

The generators given by (2.23) can be verified to satisfy (2.15) [and (2.6)] if we make the following identification:

$$\{\hat{U},\hat{V},\hat{W}\}=\{\hat{A}_1,\hat{A}_2,\hat{A}_3\}\ ,$$
$$\{\hat{v}_{13},\hat{\mathscr{U}},\hat{\mathscr{V}},\hat{\mathscr{W}}\}=\{\hat{B}_1,\hat{B}_2,\hat{B}_3,\hat{B}_4\}\ , \tag{2.26}$$
$$\{\hat{\mathscr{Y}}\}=\{\hat{C}\}\ .$$

The complete set of commutation relations among these generators, which we represented symbolically by Eq. (2.15), are given in Appendix A. The Hamiltonian $\hat{H}(t)$ of the system (2.25), if now written in terms of the $\hat{A}$, $\hat{B}$, and $\hat{C}$ operators defined by (2.26), becomes

$$\hat{H}(t)=\tfrac{1}{2}\hbar\left[\text{const}-2\epsilon\hat{A}_1+\Delta\hat{A}_3+\frac{1}{\sqrt{3}}\Delta\hat{C}\right]. \tag{2.27}$$

Notice the absence of any $\hat{B}$ operators in (2.27), as opposed to Eq. (2.19) where $\hat{u}_{23}$ is a $\hat{B}$ operator when we used the representation (2.17). From Eq. (2.27), and Eqs. (2.15) and (2.6)—(2.8), it is easy to see that the matrix elements Λ_{jk} in Eq. (2.7) for the evolution of the coherence vector $\vec{S}=(A_1,A_2,A_3,B_1,B_2,B_3,B_4,C)$, where the components of $\vec{S}$ are related to the density matrix $\hat{\rho}$ by Eq. (2.2), would be nonzero only if j and k belong to the same group. Therefore, matrix $\underline{\Lambda}$ in the equations of motion

$$\frac{d\vec{S}}{dt}=\underline{\Lambda}\vec{S} \tag{2.28}$$

is a block-diagonal matrix of dimensions 3, 4, and 1

$$\underline{\Lambda}=\begin{bmatrix}\underline{\Lambda}_3&\underline{0}&\underline{0}\\ \underline{0}&\underline{\Lambda}_4&\underline{0}\\ \underline{0}&\underline{0}&\underline{\Lambda}_1\end{bmatrix}. \tag{2.29}$$

The nonzero elements can be found from Eqs. (2.27) and (2.28), giving

$$\underline{\Delta}_3=\begin{bmatrix}0 & -\Delta & 0\\ \Delta & 0 & 2\epsilon\\ 0 & -2\epsilon & 0\end{bmatrix},$$
$$\underline{\Delta}_4=\begin{bmatrix}0 & -\epsilon & 0 & 0\\ \epsilon & 0 & \Delta & 0\\ 0 & -\Delta & 0 & -\epsilon\\ 0 & 0 & \epsilon & 0\end{bmatrix},\quad \underline{\Delta}_1=\underline{0}. \tag{2.30}$$

It immediately follows that[7]

$$\begin{aligned}&U^2+V^2+W^2=k_{\text{motion}}\ ,\\ &v_{13}^2+\mathscr{U}^2+\mathscr{V}^2+\mathscr{W}^2=k_{\text{motion}}\ ,\\ &\mathscr{Y}=k_{\text{motion}}\ ,\end{aligned} \tag{2.31}$$

where k_{motion} represents the constant of motion and where

$$\begin{aligned}&U=\epsilon^{-1}(\alpha u_{12}+\beta u_{23})\ ,\\ &V=\epsilon^{-1}(\alpha v_{12}-\beta v_{23})\ ,\\ &W=(\tfrac{1}{2}\epsilon^{-2})[-(2\alpha^2+\beta^2)w_1+\sqrt{3}\beta^2 w_2+2\alpha\beta u_{13}]\ ,\\ &v_{13}=v_{13}\ ,\\ &\mathscr{U}=\epsilon^{-1}(\beta u_{12}-\alpha u_{23})\ ,\\ &\mathscr{V}=-\epsilon^{-1}(\beta v_{12}+\alpha v_{23})\ ,\\ &\mathscr{W}=\epsilon^{-2}[-\alpha\beta w_1-\sqrt{3}\alpha\beta w_2-(\alpha^2-\beta^2)u_{13}]\ ,\\ &\mathscr{Y}=(\tfrac{1}{2}\epsilon^{-2})[-\sqrt{3}\beta^2 w_1+(2\alpha^2-\beta^2)w_2-2\sqrt{3}\alpha\beta u_{13}]\ ,\end{aligned} \tag{2.32}$$

are, respectively, the expectation values of the corresponding operators given in Eq. (2.21). In terms of the evolution of the eight-dimensional coherence vector $\vec{S}$, not only is the length of $\vec{S}$ conserved, but the lengths in the three subspaces are also separately conserved.[7]

It may be noted that the quark Hamiltonian normally assumed for the strong interaction in particle physics shares a common feature with our Hamiltonian (2.27): it does not have any $\hat{B}$ operators. It readily follows that the subspaces of $\hat{A}$, $\hat{B}$, and $\hat{C}$ in which constants of motion exist are analogous to the subspaces of pions (π^+,π^0,π^-), kaons $(K^+,K^0,\bar{K}^+,\bar{K}^0)$, and eon (η^0), respectively.

The block-factored form of Eq. (2.28) can be shown to also provide a unifying approach to the theory of simultaneous soliton propagation[15–20] and population trapping.[6,21,22] If we combine the equation of motion for the "pion" group with the Maxwell equation, then the soliton solution of Stroud and Cardimona[18] follows. If we combine the equation of motion for the "kaon" group with the Maxwell equation, then we get in case (i) $\Delta=0$, the soliton solution of Konopnicki and Eberly[15] and in case (ii) $\Delta\gg\alpha,\beta$, the two-photon soliton solution of Tan-no *et al.*[20] The constant of motion represented by the eon gives the amount of population trapped.

If the laser-atom interactions occur away from the two-photon resonance, or if decays are taken into consideration, then the Hamiltonian of the system would involve generally the $\hat{B}$ operators as well, and the dynamical subspaces obtained previously are no longer completely independent. The Gell-Mann SU(3) symmetry of the system is said to be "broken." If the symmetry-breaking effects are small enough, then the effects can sometimes be treated by perturbation theory.

III. THREE-LEVEL SYSTEMS IN THE CASES $\Delta_{12}=\Delta_{23}$ AND $\alpha=\beta$

To appreciate the use in Sec. II of Gell-Mann SU(3) symmetry, and to see that this is not the only type of symmetry which can be exploited, consider the following case. Assume the two incident laser fields are equally detuned, i.e.,

$$\Delta_{12}=\Delta_{23}\equiv\Delta\ , \tag{3.1}$$

as shown in Fig. 3, and assume also that

$$\alpha=\beta\ . \tag{3.2}$$

Such a system is a special case of a more general type of problem studied by Cook and Shore.[23] The three-level system becomes equivalent to a spin-1 particle. Another way of looking at it is that the system has a three-dimensional rotational symmetry. This symmetry is Elliott's SU(3) symmetry[4] in nuclear physics. The eight SU(3) generators for this problem, unlike (2.14), consist of the following two parts:

$$\hat{L}_x,\hat{L}_y,\hat{L}_z \tag{3.3a}$$

the angular momentum (or spin-1) operators and

$$\hat{Q}_1,\hat{Q}_2,\hat{Q}_3,\hat{Q}_4,\hat{Q}_5 \tag{3.3b}$$

the quadrupole tensors. For a specific realization of these operators, one can use, for example, the matrix representation given by Morris[24]

$$\begin{aligned}&\hat{U}_0=(\tfrac{2}{3})^{1/2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \hat{U}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix},\quad \hat{U}_2=\frac{i}{\sqrt{2}}\begin{bmatrix}0&-1&0\\1&0&-1\\0&1&0\end{bmatrix},\quad \hat{U}_3=\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix},\\ &\hat{U}_4=\frac{1}{\sqrt{3}}\begin{bmatrix}1&0&0\\0&-2&0\\0&0&1\end{bmatrix},\quad \hat{U}_5=\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&-1\\0&-1&0\end{bmatrix},\quad \hat{U}_6=\frac{i}{\sqrt{2}}\begin{bmatrix}0&1&0\\-1&0&-1\\0&1&0\end{bmatrix},\quad \hat{U}_7=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\\ &\hat{U}_8=i\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\end{aligned} \tag{3.4}$$

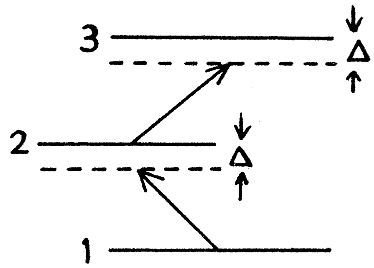

FIG. 3. Three-level system in which the incident laser fields are equally detuned from the atomic transition frequencies.

with the following identification:

$$\{\hat{U}_1,\hat{U}_2,\hat{U}_3\}=\{\hat{L}_x,\hat{L}_y,\hat{L}_z\}$$

and (3.5)

$$\{\hat{U}_4-\hat{U}_8\}=\{\hat{Q}_1-\hat{Q}_5\} .$$

The generators $\hat{U}_0-\hat{U}_8$ can be verified to satisfy Eq. (2.6), and the complete set of commutation relations among them is given in Appendix B. These commutation relations can be represented symbolically by

$$[\hat{L},\hat{L}]=\hat{L},\ [\hat{L},\hat{Q}]=\hat{Q},\ [\hat{Q},\hat{Q}]=\hat{L} . \quad (3.6)$$

Notice the structural dissimilarity, compared with (2.15).[25] From Eqs. (3.7), (2.8), and (2.7), and by noting that the Hamiltonian of the system can be expressed, in this case, as

$$\hat{H}(t)=\hbar(\text{const}-\sqrt{2}\alpha\hat{L}_x+\Delta\hat{L}_z) , \quad (3.7)$$

we find that the matrix $\underline{\Lambda}$ for the evolution of the coherence vector

$$\vec{S}=(L_x,L_y,L_z,Q_1,Q_2,Q_3,Q_4,Q_5)$$

is a block-diagonal matrix of dimensions 3 and 5

$$\underline{\Lambda}=\begin{bmatrix}\underline{\mathscr{L}}_3 & \underline{0}\\ \underline{0} & \underline{\mathscr{L}}_5\end{bmatrix} , \quad (3.8)$$

where

$$\underline{\mathscr{L}}_3=\begin{bmatrix}0 & -\Delta & 0\\ \Delta & 0 & \sqrt{2}\alpha\\ 0 & -\sqrt{2}\alpha & 0\end{bmatrix} ,$$

$$\underline{\mathscr{L}}_5=\begin{bmatrix}0 & 0 & \sqrt{6}\alpha & 0 & 0\\ 0 & 0 & \Delta & 0 & \sqrt{2}\alpha\\ -\sqrt{6}\alpha & -\Delta & 0 & -\sqrt{2}\alpha & 0\\ 0 & 0 & \sqrt{2}\alpha & 0 & 2\Delta\\ 0 & -\sqrt{2}\alpha & 0 & -2\Delta & 0\end{bmatrix} . \quad (3.9)$$

The lengths in the two subspaces of $\vec{S}$ are thus separately conserved

$$L_x^2+L_y^2+L_z^2=k_{\text{motion}} ,$$
$$Q_1^2+Q_2^2+Q_3^2+Q_4^2+Q_5^2=k_{\text{motion}} , \quad (3.10)$$

where in terms of the density-matrix elements, we have

$$L_x=\frac{1}{\sqrt{2}}(u_{12}+u_{23}),\ L_y=\frac{1}{\sqrt{2}}(v_{12}+v_{23}) ,$$
$$L_z=-\tfrac{1}{2}(w_1+\sqrt{3}w_2) , \quad (3.11)$$
$$Q_1=\tfrac{1}{2}(-\sqrt{3}w_1+w_2),\ Q_2=\frac{1}{\sqrt{2}}(u_{12}-u_{23}) ,$$
$$Q_3=-\frac{1}{\sqrt{2}}(v_{12}-v_{23}),\ Q_4=u_{13},\ Q_5=-v_{13} .$$

The "octet" in this case can be characterized by a one-dimensional diagram based on simple spin-1 addition, in contrast to the famous hexagon-shaped octet of Gell-Mann symmetry.

It is interesting to note that the conditions for the two special cases we considered, Figs. 2 and 3, become identical when $\alpha=\beta$, $\Delta=0$, and, in this case, considerations of Gell-Mann SU(3) symmetry still predict population trapping which considerations of Elliott SU(3) symmetry fail to predict.

A generalization of Elliott's type of SU(3) symmetry to an N-level problem for which the angular momentum operators are of spin $(N-1)/2$, is actually the model of Cook and Shore,[23] although they did not exploit the underlying symmetry structure. Let us assume a plane-wave incident electric field $\vec{E}(z,t)$ with N-1 distinct frequency components, in which the frequencies $|\nu_j|$ are chosen to be nearly resonant with the successive transition frequencies $|\omega_{j,j+1}|$ in a chain of N dipole-connected energy levels in an atomic or molecular system. The conditions which must be satisfied for the system to possess the generalized Elliott type of SU(3) symmetry [i.e., the generalization of Eqs. (3.1) and (3.2)] are the following.

(i) The N-1 laser fields must be equally detuned from the respective atomic transitions, i.e.,

$$\Delta_{12}=\Delta_{23}=\cdots=\Delta_{N-1,N}\equiv\Delta . \quad (3.12)$$

(ii) The respective Rabi frequencies must satisfy the relation

$$\Omega_{j,j+1}=[j(N-j)]^{1/2}\Omega_0,\ j=1,2,\ldots,N-1 , \quad (3.13)$$

where Ω_0 can be an arbitrary function of time and position.

Then we find that in the absence of decays, not only the length of the coherence vector $\vec{S}$ is conserved, but also

$$L_x^2+L_y^2+L_z^2=k_{\text{motion}} , \quad (3.14)$$

where

$$L_x=\tfrac{1}{2}\sum_{j=1}^{N-1}[j(N-j)]^{1/2}u_{j,j+1}(t) ,$$
$$L_y=\tfrac{1}{2}\sum_{j=1}^{N-1}[j(N-j)]^{1/2}v_{j,j+1}(t) , \quad (3.15)$$
$$L_z=-\tfrac{1}{2}\sum_{j=1}^{N-1}[\tfrac{1}{2}j(j+1)]^{1/2}w_j(t) .$$

The u's, v's, and w's are given in Eq. (2.20) with $1\le j<k\le N$, $1\le l\le N-1$. The equations of motion in the dynamical subspace involving L_x, L_y, and L_z are

$$\frac{d}{dt}\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & 2\Omega_0 \\ 0 & -2\Omega_0 & 0 \end{bmatrix} \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} . \tag{3.16}$$

Considerations of Eq. (3.16) with the reduced Maxwell equations

$$\left[\frac{\partial}{\partial z}+\frac{\partial}{\partial (ct)}\right]\Omega_{j,j+1}(z,t)=\frac{2\pi\mathscr{N}}{\hbar c}\nu_j d_{j,j+1}^2\langle v_{j,j+1}\rangle , \tag{3.17}$$

where $\mathscr{N}$ is the atomic density, further lead us to a set of simultaneous solitary pulses not previously discovered.[16,20] First, consistency requirements with the Maxwell's equations impose the following additional conditions, besides (3.12) and (3.13), for the existence of these solitary pulses.

(i) The chainwise dipole-connected level configuration must be of the cascade type, i.e., the atomic energy levels are such that $E_N > E_{N-1} > \cdots > E_1$.

(ii) The laser field frequencies and the atomic dipole moments must satisfy

$$\nu_j d_{j,j+1}^2 = \nu_1 d_{12}^2,\quad j=2,3,\ldots,N-1 . \tag{3.18}$$

(iii) The initial-level populations must satisfy

$$w_{j,j+1}(0)=w_{12}(0)>0,\quad j=2,3,\ldots,N-1$$

where

$$w_{j,j+1}=\rho_{jj}-\rho_{j+1,j+1} . \tag{3.19}$$

When these conditions are satisfied, incident simultaneous pulses of the shape

$$\Omega_{j,j+1}(\xi)\equiv\frac{\vec{d}_{j,j+1}\cdot\vec{\mathscr{E}}_{j,j+1}(\xi)}{\hbar} = \frac{[j(N-j)]^{1/2}}{\tau}\operatorname{sech}\frac{\xi-\xi_0}{\tau} ,\quad j=1,2,\ldots,N-1 \tag{3.20}$$

and area

$$\Theta(z,t)=\int_{-\infty}^{t}\frac{2}{(N-1)^{1/2}}\Omega_{12}(z,t')dt' = 4\tan^{-1}\left[\exp\frac{\xi-\xi_0}{\tau}\right] , \tag{3.21}$$

where $\xi=t-z/V$, V being the velocity of the pulse, and τ the pulse length, would emerge from the medium unaltered if Θ is an integral multiple of π. The evolutions of the atomic variables during the propagation of the pulses are given by

$$\begin{aligned} u_{j,j+1}(\xi)&=-w_{12}(0)\frac{2[j(N-j)]^{1/2}\Delta\tau}{1+(\Delta\tau)^2}\operatorname{sech}\frac{\xi-\xi_0}{\tau} , \\ v_{j,j+1}(\xi)&=-w_{12}(0)\frac{2[j(N-j)]^{1/2}}{1+(\Delta\tau)^2}\operatorname{sech}\frac{\xi-\xi_0}{\tau} \times\tanh\frac{\xi-\xi_0}{\tau} , \\ w_j(\xi)&=w_{12}(0)\left[\frac{j(j+1)}{2}\right]^{1/2}\times\left[-1+2\operatorname{sech}^2\frac{\xi-\xi_0}{\tau}\right] , \\ j&=1,2,\ldots,N-1 . \end{aligned} \tag{3.22}$$

All other atomic variables are equal to zero. When $\Delta=0$, the above simultaneous solitons solution reduces to the solution given by Konopnicki, Drummond, and Eberly.[17]

IV. SUMMARY

We have shown that a full exploitation of the dynamical-symmetry properties requires appropriate choices of the SU(N) representation. We have shown in particular that two distinct realizations of the SU(3) symmetry (Gell-Mann's and Elliott's) can actually be employed in the same dynamical problem, but under different experimental conditions, in intense-field electrodynamics. With the appropriate choices of representation, multilevel, multiphoton, and symmetry-breaking effects can be grouped and classified accordingly.

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APPENDIX A

The complete set of commutation relations among the generators $(\hat{U},\hat{V},\hat{W},\hat{v}_{13},\hat{\mathscr{U}},\hat{\mathscr{V}},\hat{\mathscr{W}},\hat{\mathscr{Y}})$ given by Eq. (2.23), which we rename $(\hat{A}_1,\hat{A}_2,\hat{A}_3,\hat{B}_1,\hat{B}_2,\hat{B}_3,\hat{B}_4,\hat{C})$, are as follows:

$$\begin{aligned} &[\hat{A}_1,\hat{A}_2]=2i\hat{A}_3,\quad [\hat{A}_2,\hat{A}_3]=2i\hat{A}_1,\quad [\hat{A}_3,\hat{A}_1]=2i\hat{A}_2 , \\ &[\hat{A}_1,\hat{B}_1]=-i\hat{B}_2,\quad [\hat{A}_1,\hat{B}_2]=i\hat{B}_1,\quad [\hat{A}_1,\hat{B}_3]=-i\hat{B}_4,\quad [\hat{A}_1,\hat{B}_4]=i\hat{B}_3 , \\ &[\hat{A}_2,\hat{B}_1]=i\hat{B}_3,\quad [\hat{A}_2,\hat{B}_2]=-i\hat{B}_4,\quad [\hat{A}_2,\hat{B}_3]=-i\hat{B}_1,\quad [\hat{A}_2,\hat{B}_4]=i\hat{B}_2 , \\ &[\hat{A}_3,\hat{B}_1]=-i\hat{B}_4,\quad [\hat{A}_3,\hat{B}_2]=-i\hat{B}_3,\quad [\hat{A}_3,\hat{B}_3]=i\hat{B}_2,\quad [\hat{A}_3,\hat{B}_4]=i\hat{B}_1 , \\ &[\hat{A}_1,\hat{C}]=[\hat{A}_2,\hat{C}]=[\hat{A}_3,\hat{C}]=0 , \end{aligned} \tag{A1}$$

$$[\hat{B}_1,\hat{B}_2]=-i\hat{A}_1,\ [\hat{B}_1,\hat{B}_3]=i\hat{A}_2,\ [\hat{B}_1,\hat{B}_4]=-i(\hat{A}_3-\sqrt{3}\hat{C})\ ,$$
$$[\hat{B}_2,\hat{B}_3]=-i(\hat{A}_3+\sqrt{3}\hat{C}),\ [\hat{B}_2,\hat{B}_4]=-i\hat{A}_2\ ,$$
$$[\hat{B}_3,\hat{B}_4]=-i\hat{A}_1\ ,$$
$$[\hat{B}_1,\hat{C}]=-i\sqrt{3}\hat{B}_4,\ [\hat{B}_2,\hat{C}]=i\sqrt{3}\hat{B}_3,\ [\hat{B}_3,\hat{C}]=-i\sqrt{3}\hat{B}_2,\ [\hat{B}_4,\hat{C}]=i\sqrt{3}\hat{B}_1\ .$$

The operators $\hat{A}^2,\hat{B}^2,\hat{C}^2$ defined by

$$\hat{A}^2=\hat{A}_1^2+\hat{A}_2^2+\hat{A}_3^2\ ,\qquad \hat{B}^2=\hat{B}_1^2+\hat{B}_2^2+\hat{B}_3^2+\hat{B}_4^2\ , \tag{A2}$$

or in matrix form

$$\hat{A}^2=\frac{3}{\epsilon^2}\begin{bmatrix}\alpha^2 & 0 & \alpha\beta\\ 0 & \epsilon^2 & 0\\ \alpha\beta & 0 & \beta^2\end{bmatrix},\quad \hat{B}^2=\frac{2}{\epsilon^2}\begin{bmatrix}\alpha^2+2\beta^2 & 0 & -\alpha\beta\\ 0 & \epsilon^2 & 0\\ -\alpha\beta & 0 & 2\alpha^2+\beta^2\end{bmatrix}, \tag{A3}$$

are also of interest because we find that $\hat{A}^2$, $\hat{B}^2$, and $\hat{C}^2$ all commute with $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$, and $\hat{C}$, and hence commute with $\hat{H}(t)$, Eq. (2.28), the Hamiltonian of the laser-atom system under the two-photon resonance condition. This implies that $\langle\hat{A}^2\rangle$, $\langle\hat{B}^2\rangle$, and $\langle\hat{C}^2\rangle$ (or $\langle\hat{C}\rangle$) are all constants of motion. It should be noted, however, that Eqs. (2.31) which are obtained from the block factorization of $\underline{\Lambda}$ in Eq. (2.28), *do not* follow from these results.

Neither $\hat{A}^2$ nor $\hat{B}^2$ commute with any members of the B group. The commutators are given by

$$[\hat{B}_1,\hat{A}^2]=3i\hat{B}_4,\ [\hat{B}_2,\hat{A}^2]=-3i\hat{B}_3,\ [\hat{B}_3,\hat{A}^2]=3i\hat{B}_2,\ [\hat{B}_4,\hat{A}^2]=-3i\hat{B}_1\ ,$$
$$[\hat{B}_1,\hat{B}^2]=-2i\hat{B}_4,\ [\hat{B}_2,\hat{B}^2]=2i\hat{B}_3,\ [\hat{B}_3,\hat{B}^2]=-2i\hat{B}_2,\ [\hat{B}_4,\hat{B}^2]=2i\hat{B}_1\ . \tag{A4}$$

APPENDIX B

The complete set of commutation relations among the generators $(\hat{U}_1,\hat{U}_2,\ldots,\hat{U}_8)$ given by Eq. (3.4), which we rename $(\hat{L}_x,\hat{L}_y,\hat{L}_z,\hat{Q}_1,\ldots,\hat{Q}_5)$, is as follows:

$$[\hat{L}_x,\hat{L}_y]=i\hat{L}_z,\ [\hat{L}_y,\hat{L}_z]=i\hat{L}_x,\ [\hat{L}_z,\hat{L}_x]=i\hat{L}_y\ ,$$
$$[\hat{Q}_1,\hat{Q}_2]=i\sqrt{3}\hat{L}_y,\ [\hat{Q}_1,\hat{Q}_3]=i\sqrt{3}\hat{L}_x,\ [\hat{Q}_1,\hat{Q}_4]=[\hat{Q}_1,\hat{Q}_5]=0\ ,$$
$$[\hat{Q}_2,\hat{Q}_3]=-i\hat{L}_z,\ [\hat{Q}_2,\hat{Q}_4]=i\hat{L}_y,\ [\hat{Q}_2,\hat{Q}_5]=i\hat{L}_x\ ,$$
$$[\hat{Q}_3,\hat{Q}_4]=i\hat{L}_x,\ [\hat{Q}_3,\hat{Q}_5]=i\hat{L}_y\ ,$$
$$[\hat{Q}_4,\hat{Q}_5]=-2i\hat{L}_z\ ,$$
$$[\hat{L}_x,\hat{Q}_1]=i\sqrt{3}\hat{Q}_3,\ [\hat{L}_y,\hat{Q}_1]=i\sqrt{3}\hat{Q}_2,\ [\hat{L}_z,\hat{Q}_1]=0\ ,$$
$$[\hat{L}_x,\hat{Q}_2]=i\hat{Q}_5,\ [\hat{L}_y,\hat{Q}_2]=-i(\sqrt{3}\hat{Q}_1-\hat{Q}_4),\ [\hat{L}_z,\hat{Q}_2]=-i\hat{Q}_3\ ,$$
$$[\hat{L}_x,\hat{Q}_3]=-i(\sqrt{3}\hat{Q}_1+\hat{Q}_4),\ [\hat{L}_y,\hat{Q}_3]=i\hat{Q}_5,\ [\hat{L}_z,\hat{Q}_3]=i\hat{Q}_2\ ,$$
$$[\hat{L}_x,\hat{Q}_4]=i\hat{Q}_3,\ [\hat{L}_y,\hat{Q}_4]=-i\hat{Q}_2,\ [\hat{L}_z,\hat{Q}_4]=-i\hat{Q}_2\ ,$$
$$[\hat{L}_x,\hat{Q}_5]=-i\hat{Q}_2,\ [\hat{L}_y,\hat{Q}_5]=-i\hat{Q}_3,\ [\hat{L}_z,\hat{Q}_5]=2i\hat{Q}_4\ .$$

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[25]While the generators in (3.3) are related to the generators in (2.26) by a unitary transformation, one cannot *separately* associate $\hat{L}_x,\hat{L}_y,\hat{L}_z$ in (3.3) with $\hat{A}_1,\hat{A}_2,\hat{A}_3$ in (2.26), and $\hat{Q}_1-\hat{Q}_5$ with $\hat{B}_1-\hat{B}_4$, $\hat{C}$. This can be seen from Appendix B, for example, that none of the $\hat{Q}$ operators commute with all members of the $\hat{L}$ group. The generators are equivalent but the groupings are different.