Vacuum polarization corrections and spin-orbit splitting in antiprotonic atoms

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Precise calculations of the energy levels and fine structure of antiprotonic atoms, in which the normal and anomalous magnetic moments of the antiproton are treated on an equal basis and vacuum polarization and relativistic recoil corrections are included, are presented. The results are applied to measurements of the antiproton mass and anomalous magnetic moment with the use of antiprotonic atoms.

Future precise measurements of the energies of x-ray transitions in antiprotonic atoms will require similarly precise calculations of all electromagnetic effects in order to be able to analyze the data for measurements of the antiproton's mass and magnetic moment or in order to be able to extract accurate hadronic shifts and widths. For this reason, it is useful to reexamine the question of the most appropriate two-body equation to use for the description of the energy levels of antiprotonic atoms and to make sure that the most important radiative corrections (in this case vacuum polarization) are included to sufficient accuracy. For antiprotonic atoms, it is necessary to include electron vacuum polarization to all orders, since this gives rise to effective potentials having a range of the order of an electron Compton wavelength. Muonic or hadronic vacuum polarization is of much shorter range and will be unimportant except at distances for which the strong interaction dominates. This is also true for the electromagnetic self-energy. It is desirable, from the point of view of consistency, to treat the normal and anomalous magnetic moments of the antiproton on an equal footing, since these are comparable in magnitude.

Keeping these requirements in mind, it turns out that different lowest-order equations are appropriate, depending on the ratio of the antiproton mass to the mass of the nucleus. Both equations are straightforward generalizations of well-known methods. If the nuclear mass is significantly larger than the antiproton mass ($A>12$) it is best to treat the antiproton relativistically and the nucleus nonrelativistically, as is done for the case of muonic atoms.[1] In this case, one obtains a Dirac equation with reduced mass for the antiproton. Additional corrections are treated as perturbations and need to be considered only to linear order in the mass ratio $m_{\bar{p}}/m_N$. As we shall see below, it is easy to include a Pauli term in the unperturbed problem. If the nuclear mass is not much larger than $m_{\bar{p}}$, the two particles should be treated symmetrically. All spin effects can be treated as perturbations to sufficient accuracy, since they are of relative order $(\alpha Z)^2 \ll 1$, compared with the Schrödinger energies. Keeping terms of order $(v/c)^2$, one obtains the Schrödinger equation, with recoil- and spin-dependent corrections which are treated as perturbations. The derivation of such an equation, to the required level of accuracy, is a textbook exercise.[2] A derivation for the case in which the nucleus also has spin $\frac{1}{2}$, and for the case of arbitrary masses and magnetic moments, has been given in Refs. 3 and 4. The analogous equation for the case of nuclear spin zero will be given below. Further relativistic corrections are negligibly small, compared with the expected experimental accuracy,[5] and it is not clear that it is worth the effort to make a systematic calculation of all of them. Regardless of which equation one uses, one should include vacuum polarization, or at least the lowest-order Ühling-Serber[6] contribution, in the unperturbed Hamiltonian. This, of course, precludes an analytic solution of the wave equation, but since numerical methods for solving the radial equation are so well developed, this presents no special problem. In fact, it may be more efficient from the standpoint of computer time to solve the radial equation directly, rather than to evaluate complicated analytical expressions involving special functions, which are obtained when treating vacuum polarization in perturbation theory. Furthermore, it is straightforward to include higher-order vacuum polarization and to take into account finite nuclear size. The latter has an effect on the vacuum polarization potential and also removes the problems of dealing with an r^{-3} potential. All relevant corrections are easily included, regardless of their spin dependence.

For the case of heavy nuclei, the appropriate equation to use is the Dirac equation with reduced mass, including a Pauli term

$$E_0\psi=(m_R-B_0)\psi=H_0\psi$$
$$=\left[\vec{\alpha}\cdot\vec{p}+\beta m_R+V(r)+\frac{i\kappa}{2m}\frac{dV}{dr}\vec{\gamma}\cdot\hat{r}\right]\psi\ . \tag{1}$$

Here m_R is the nonrelativistic reduced mass, $\vec{\alpha}$ and β are the usual Dirac matrices, $\kappa\simeq-1.793$ is the antiproton anomalous moment. A derivation similar to that given for muonic atoms (without the Pauli term) in Ref. 1 (we employ the same notation as used there) gives the recoil corrections in the form

$$H=H_0+\beta(m-m_R)+(p^2+\{V,\vec{\alpha}\cdot\vec{p}\})/2m_N$$
$$-[\vec{\alpha}\cdot\vec{p},[p^2,w]]/2m_N-\frac{\kappa}{2mm_N}\frac{1}{r}\frac{dv}{dr}\beta\vec{\Sigma}\cdot\vec{L}\ , \tag{2}$$

where $m=m_{\bar{p}}$, m_N is the nuclear mass, and $w(r)=(Z\alpha/2)\int\rho_N(r_N)\,|\vec{r}-\vec{r}_N|\,d^3r_N$. Terms of order m^2/m_N^2 have been neglected, in keeping with the assumption that the nucleus is nonrelativistic. The recoil corrections to H_0 are all explicitly of order m/m_N. We have chosen to use the nonrelativistic reduced mass in the unperturbed Eq. (1) for computational convenience. The difference between this and the proper generalization of the reduced mass as given by Todorov[7] is then treated as a perturbation. This gives rise to the contribution $B_0^2/2m_N$ discussed below. After some manipulation, with the use of hypervirial theorems analogous to those given by Friar and Negele,[8] the recoil corrections may be written in the form (expectation values)

$$\Delta E=-\frac{B_0^2}{2m_N}+\frac{\langle h(r)+2B_0P(r)\rangle}{2m_N}+\frac{\kappa}{4mm_N}\langle\beta\nabla^2V\rangle-\frac{\kappa^2}{8m^2m_N}\langle V'^2\rangle$$
$$-\frac{i\kappa}{2mm_N}\langle(B_0-2w'/r)\vec{\gamma}\cdot\vec{\nabla}V+(w'/r-B_0)\nabla^2V\vec{\gamma}\cdot\vec{r}\rangle$$
$$\simeq-B_0^2/2m_N-(Z\alpha)^2\langle r_N^2\rangle\langle r^{-4}\rangle/6m_N-(Z\alpha)^2\kappa^2\langle r^{-4}\rangle/8m^2m_N$$
$$-\frac{i\kappa}{2mm_N}\alpha Z\langle(B_0-\alpha Z/r)\vec{\gamma}\cdot\vec{r}/r^3\rangle \tag{3}$$

with $h(r)=2w'V'-V^2$, $w'(r)=r^{-2}\int_0^r u^2V(u)du$ and $P(r)=-V(r)-\vec{r}\cdot\vec{\nabla}V=\alpha Z\int_r^\infty\rho(u)u\,du$. The second line of Eq. (3) is valid for states having such large angular momentum that the strong interaction is negligible. Only the term $-B_0^2/2m_N$, which is of essentially kinematic origin[7] is numerically significant. All other terms contribute less than 0.25 eV for the $n=10$, $l=9$ state of antiprotonic lead, for example. The unperturbed Dirac equation can be solved in a manner similar to that used for muonic atoms. One obtains the radial equations (see Ref. 1 for notation)

$$(E_0-V-m_R)G=-\frac{dF}{dr}+\frac{\tilde{\kappa}}{r}F+\frac{\kappa}{2m}\frac{dV}{dr}F\ ,$$
$$(E_0-V+m_R)F=\frac{dG}{dr}+\frac{\tilde{\kappa}}{r}G+\frac{\kappa}{2m}\frac{dV}{dr}G\ . \tag{4}$$

Here $\tilde{\kappa}=\pm(j+\frac{1}{2})$ is the eigenvalue of $K=-\beta(1+\vec{\Sigma}\cdot\vec{L})$. One sees that the standard radial equations are modified by the simple substitution

$$\frac{\tilde{\kappa}}{r}\rightarrow\frac{\tilde{\kappa}}{r}+\frac{\kappa}{2m}\frac{dV}{dr}\ .$$

The new terms will give rise to additional spin-orbit splitting, as well as a shift in the centroid of the level. These radial equations were solved using a modified version of the program MUON.[9] The nuclear charge density was taken to be given by a Fermi distribution. This should be accurate enough since the only effect on orbits having large orbital angular momentum is due to the modification of the Ühling and Källen-Sabry potentials, which is sensitive to $\langle r_N^2\rangle$ (for Pb, the rms radius is 5.50 fm). Some typical results are given in the tables. They can be summarized as follows. The order $\alpha Z\alpha$ vacuum polarization potential must be included in the unperturbed problem. The use of perturbation theory gives rise to an error of 13—20 eV in the binding energy of the state with $n=10$, $l=9$. The influence of vacuum

TABLE I. Theoretical and experimental transition energies (in eV) for $\bar{p}$-Pb. E_R is the relativistic recoil correction. E_{sc} is the electron screening correction.

Transition		Point Coulomb	Vacuum polarization (VP) $\alpha Z\alpha+\alpha^2 Z\alpha$	Vacuum polarization (VP) $\alpha(Z\alpha)^{3,5,7}$	Pauli moment	E_R	E_{sc}	E (Theor.)	E (Expt.)
13-12	$j=1+\frac{1}{2}$	171 931	860	-19 ± 2	-149	1	-23 ± 1	$172\,601\pm2$	$172\,607\pm15$[a]
	$j=1-\frac{1}{2}$	172 019	866	-19 ± 2	165	1	-23 ± 1	$173\,009\pm2$	
12-11	$j=1+\frac{1}{2}$	220 971	1220	-26 ± 2	-248	2	-19 ± 1	$221\,900\pm2$	$221\,876\pm17$[a] $221\,909\pm60$[b]
	$j=1-\frac{1}{2}$	221 118	1228	-26 ± 2	277	2	-19 ± 1	$222\,680\pm2$	
11-10	$j=1+\frac{1}{2}$	290 608	1773	-30 ± 3	-431	2	-16 ± 1	$291\,906\pm3$	$291\,873\pm21$[a] $291\,890\pm61$[b]
	$j=1-\frac{1}{2}$	290 865	1790	-30 ± 3	490	2	-16 ± 1	$293\,101\pm3$	

[a]Reference 12.
[b]Reference 11.

TABLE II. Fine structure $E_{nlj=1+1/2}-E_{nlj=1-1/2}$ in eV for $\bar{p}$-Pb ($\kappa_{\bar{p}}=-\kappa_p$) with and without vacuum polarization (VP).

n	1	κ exact with VP	κ exact no VP	κ perturb. with VP	κ perturb. no VP
10	9	3092	3052	3092	3052
11	10	1896	1875	1900	1877
12	11	1215	1202	1216	1203
13	12	807	800	808	802

polarization on the wave functions and potential $i\kappa\vec{\gamma}\cdot\vec{\nabla}V/2m$ gives rise to a change of about 1% in the fine-structure splitting. A similar (but not identical) result has been obtained by Pilkuhn and Schlaile.[10] However, they were unable to calculate the full transition energies, since these are dominated by spin-independent contributions. The spin-orbit splitting could have been obtained from perturbation theory, provided vacuum polarization is treated correctly. The calculated transition energies, using $m_{\bar{p}}=m_p$ and $\kappa_{\bar{p}}=-\kappa_p$ are in satisfactory agreement with experiment.[11,12] Similar calculations for other antiprotonic atoms (^{10}B, ^{40}Ca, ^{138}Ba, ^{174}Yb, and ^{238}U) have been performed. Space does not permit a complete presentation of the results, which are available upon request.

For the case of light nuclei, for which the mass ratio is not small, it is better to treat both particles symmetrically. The reduction to the form of the Schrödinger equation plus corrections is standard.[2,4] The treatment was generalized to include nonpointlike charge distributions for the case in which the nucleus has spin $\frac{1}{2}$ in Ref. 3. If the nucleus is spinless, the analogous equation, up to and including terms of order $(v/c)^2$ for both particles is

$$E\psi=H\psi\ ,$$

$$H=p^2/2m_R+V-p^4/8m^3-p^4/8m_N^3+\frac{1}{r}\frac{dV}{dr}\vec{\sigma}\cdot\vec{L}\left[\frac{1}{4m_R^2}+\frac{1}{4m_N^2}+\frac{2\kappa}{4mm_R}\right]+\frac{\nabla^2V(1+2\kappa)}{8m^2}+\frac{1}{2mm_R}\left[p^2V+Vp^2+\nabla^2V+\frac{1}{r}\frac{dV}{dr}L^2\right]. \tag{5}$$

This equation is also valid for the K^--p and K^--^{3}He systems (reversing the roles of m and m_N). The terms in p^4/m^3 and the last term of (5) correspond to the well-known relativistic recoil correction $B^2/2m_N$. A nonrelativistic reduction of Eq. (2) for

TABLE III. Vacuum polarization, recoil, and screening corrections to the binding energy of antiprotonic lead (in eV).

Level n	l	j	Ühling $\alpha Z\alpha$	Ühling (perturb. theory)	Källen-Sabry $\alpha^2 Z\alpha$	$\alpha(Z\alpha)^{3,5,7}$	Recoil $B^2/2m_N$	Screening
10	9	$\frac{19}{2}$	6309	6296	44	−150	7	47
10	9	$\frac{17}{2}$	6349	6328	44	−150	7	47
10	8	$\frac{17}{2}$	6593	6580	47	−150	8	55
10	8	$\frac{15}{2}$	6647	6622	47	−152	8	54
11	10	$\frac{21}{2}$	4550	4540	32	−120	5	63
11	10	$\frac{19}{2}$	4572	4558	31	−120	5	63
12	11	$\frac{23}{2}$	3338	3331	23	−94	3	82
12	11	$\frac{21}{2}$	3350	3341	24	−94	3	82
13	12	$\frac{25}{2}$	2484	2479	17	−77	2	105
13	12	$\frac{23}{2}$	2491	2484	18	−77	2	105

the antiproton gives Eq. (5) without the quadratic terms in $1/m_N$. One can obtain sufficiently accurate binding energies for all experimental needs by solving the Schrödinger equation, including vacuum polarization, and treating all other terms as perturbations. Some results for the $\bar{p}p$ system, treating vacuum polarization as a perturbation, have been presented in Refs. 3 and 13. A program to calculate the energy levels of light antiprotonic atoms is being written. The corrections due to treating vacuum polarization exactly are expected to be of the order of a few meV, comparable to the fine structure in the $3d$ state[3] and somewhat smaller than the expected hadronic effects for the $2p$ states.[14,15]

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