Chaos in piecewise-linear systems

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A new class of physical systems, those which can be described by piecewise-linear equations, are found to exhibit chaotic behavior similar to that found in previously investigated nonlinear dissipative systems. The example of a damped, sinusoidally forced harmonic oscillator with two possible spring-constant values is investigated in detail. The system exhibits period doubling to chaos characterized by Feigenbaum's universal exponents for a certain range of parameters and an iterated map similar to that in the Lorenz equations for another.

There has been much recent work on physical systems which show chaotic behavior and the equations which model them.[1] These equations, such as the Lorenz equation,[2] mode coupling equations,[1] and a variety of anharmonic oscillator equations,[3-6] all share a common feature: they are nonlinear due to the presence of terms which are other than first order in one or more of the time-dependent variables (the term "nonlinear" being reserved for the nonadditive feature of solutions to the equations). In this paper we present a new class of simple differential equations which are first order in the dependent variables but which also have chaotic solutions. Furthermore, the chaotic motion is similar to that produced by conventional nonlinear equations.

The equations of interest here are piecewise-linear ordinary differential equations. These are simply systems of linear equations in which the modification is made that the coefficients can make discontinuous jumps among otherwise constant values depending on the values of the dependent variables. A simple example is the forced harmonic-oscillator equation

$$M\ddot{x}+2B\dot{x}+K(x)x=F\sin(\omega t+\phi)\ , \tag{1}$$

in which $x=x(t)$, $K(x)=K_1$ for $x>0$, and $K(x)=K_2$ for $x<0$. For $K_1=K_2$ the solution is that of the well-known damped harmonic oscillator. For $K_1\neq K_2$, the equation is nonlinear in that the sum of two different solutions is not a solution. As will be shown, there is a variety of possible behaviors for the solutions with $K_1\neq K_2$, including simple periodicity, periodicity with a period equal to multiples of $2\pi/\omega$, and chaotic solutions.

Besides providing a new class of equations which have chaotic solutions, piecewise-linear equations have the following significant attributes. First, they can be solved trivially with use of a digital computer without time integration. Since the explicit form of the solution is known within each piecewise region [i.e., when $K(x)=K_1$ or $K(x)=K_2$ in Eq. (1)] the overall solution is found by matching solutions at the boundaries between regions.[7] Solving the equation reduces to the problem of finding zeros of simple transcendental functions, easily accomplished by use of Newton's method. The ease of solution, combined with the rich chaotic behavior produced, makes piecewise continuous equations ideal test cases for the analytic study of chaotic motion. Second, certain properties of more general nonlinear equations can be fruitfully investigated by studying simpler piecewise-linear modifications. This was done, for example, for the van der Pol equations by Levinson[8] and Levi.[9] Third, certain physical systems may be accurately described as being piecewise linear. An example is the human eardrum which, in a simple model, may be characterized as having an elastic coefficient for inward displacement twice as large as that for outward displacement.[10] This has been conjectured to be the cause of the production of audible subharmonics.[11]

In this paper we report the results of an investigation of the solutions of Eq. (1). It can be written in the dimensionless form

$$\ddot{y}+2b\dot{y}+k(y)y=\sin(2\pi\tau+\phi) \tag{2}$$

by using units of $2\pi/\omega$, $4\pi^2F/M\omega^2$, $M\omega^2/4\pi^2$, and $M\omega/2\pi$ for τ, y, $k(y)$, and b, respectively. The function $k(y)$ is defined similarly to $K(x)$, but in terms of the new units. The solutions to Eq. (2) depend on the three parameters k_1, k_2, and b. Note that the amplitude of the applied force F does not appear explicitly. Thus it serves only to set the scale of the oscillations and has no effect in producing chaos, in contrast to previous anharmonic-oscillator equations.[3-6] The behavior of the solutions varies widely as k_1, k_2, and b are changed. Here, we will illustrate the variety of behavior produced by providing a detailed examination of some of the solutions in two regions of the parameters, k_1, k_2, and b. Each region is characterized by a particular choice of k_1 and k_2 values, while the value of b varies within each region. The selection of parameter values is somewhat arbitrary in that many other choices can be made which produce an equally rich variety of behavior. In the first region the approach to chaos through period doubling is clearly observable and the well-known universal exponents discovered by Feigenbaum[12] are reproduced. Within the second region can be found a chaotic solution characterized by a triangular iterated map similar to that produced by the Lorenz equations.[2]

For the first region, let k_1 and k_2 be chosen such that $\tau_1\equiv 2\pi/\sqrt{k_1}=1.1$ and $\tau_2\equiv 2\pi/\sqrt{k_2}=4.5$. The damping time, $\tau_b\equiv 1/b$, is chosen as the parameter which controls period doubling. For values of τ_b less than 0.5889, there is a stable limit cycle with period equal to that of the applied force. Table I lists the values of τ_b at which period doubling occurs. Table I also demonstrates the convergence of the period-doubling values of τ_b, T_n, to the geometric series predicted by Feigenbaum. It is apparent that the values of δ_n, defined as

$$\delta_n=(T_n-T_{n-1})/(T_{n+1}-T_n)\ , \tag{3}$$

are consistent with the predicted limiting value $\delta=4.6692$. The value of τ_b at which an infinite period is expected is close to $\tau_b^*=0.657\,466$. It is interesting to note that for

TABLE I. T_n, period-doubling values of τ_b, and δ_n, convergence rate exponents, for $\tau_1=1.1$, $\tau_2=4.5$.

n	T_n	δ_n
1	0.5889	
2	0.6479	7.80
3	0.655 46	4.79
4	0.657 038	4.69
5	0.657 374 8	4.68
6	0.657 446 80	

these values of τ_b a harmonic oscillator with the above value for τ_1 is underdamped, whereas a harmonic oscillator with τ_2 is overdamped (note that τ_1 and τ_2 should be compared with $2\pi\tau_b$).

For values of τ_b slightly larger than τ_b^*, the chaotic behavior of the solution causes a large enough sampling of phase space so that an iterated map can be displayed. It is most convenient in this case to calculate the y values, y_i which are separated in time by the fundamental period $2\pi/\omega$. A plot of y_{i+1} vs y_i for $\tau_b=0.66$ (not shown) clearly exhibits the parabolic maximum implied by the observed values of δ_n.

For the τ_b values $\tau_b^* < \tau_b < 1.477$ there exists a complicated pattern of periodic and chaotic behavior. All the stable periodic solutions which extend over τ_b interval widths on the order of 0.01 or more have periods which are multiples of three times the fundamental period. Small intervals containing period 13 ($\tau_b \cong 0.688$), period 11 ($\tau_b \cong 0.690$), and period 10 ($\tau_b \cong 0.830$) limit cycles, among others, were also found. At least two additional period-doubling sequences were identified: a 9×2^n sequence ($0.678 < \tau_b < 0.687$) and a 3×2^n sequence ($1.07 < \tau_b < 1.213$). The 3×2^n sequence occupies a τ_b interval larger than the 2^n sequence just described. Thus it is not the familiar 3×2^n sequence normally associated with a parabolic iterated map,[1] but an additional one.

The iterated maps are very useful in sorting out the behavior in this region. Since many of the periods found are multiples of three, it is appropriate to examine the iterated map y_{i+m} vs y_i, where m is an integer multiple of three. For example, Fig. 1 shows a plot of y_{i+9} vs y_i for $\tau_b=0.675$, a value of τ_b just below the 9×2^n period-doubling sequence. It shows that the system is on the verge of undergoing a tangent bifurcation.[1] A plot of $y(\tau)$ exhibits intermittent chaos and periodicity characteristic of a tangent bifurcation.[13]

As Fig. 1 demonstrates, the iterated maps for these values of τ_1 and τ_2 may be quite complex. In fact, maps which consist of several sheets are common. For example, Fig. 2 displays a portion of the y_{i+3} vs y_i iterated map for $\tau_b=1.305$. There are two other portions to this map which are very similar to the one shown centered near $y_{i+3}=y_i=-1.2$ and -1.7. After each single iteration the system goes from one of the three portions to the other in an invariant order. The chaotic behavior can be analyzed by fixing attention on just one of the three portions. The portion shown in Fig. 2 consists of four parts, labeled 1 through 4. The system evolves in such a way as to visit the four parts in the labeled order. Except between parts 4 and 1, it progresses from one part to the next after each triple

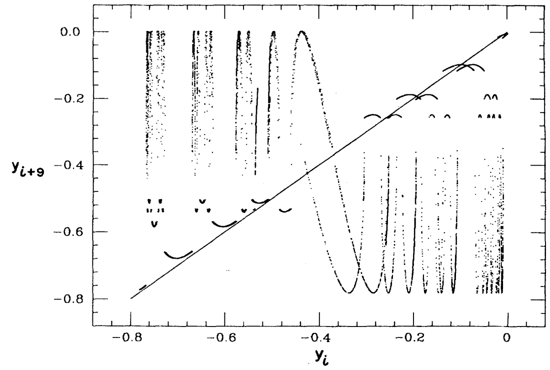

FIG. 1. Iterated map, y_{i+9} vs y_i, for $\tau_1=1.1$, $\tau_2=4.5$, and $\tau_b=0.675$. The straight line is $y_{i+9}=y_i$.

iteration. Once the system reaches part 4 it may remain within it for an apparently indefinite number of iteractions depending on exactly where on part 4 it first arrives. It then progresses on to part 1 again. This chaotic behavior can be deduced with the simple technique involving the $y_{i+3}=y_i$ line[1] and the iterated map shown.

The phenomenon of hysteresis is also found in this system. Hysteresis, in general, is caused by the existence of more than one stable attractor.[14] For example, in addition to the chaotic attractor just described, a stable period-9 solution also exists for $\tau_b=1.305$. The period-9 solution is first found for $\tau_b \cong 1.26$. The system maintains itself in the period-9 solution as τ_b is increased gradually to 1.305. Decreasing τ_b to 1.305 from above produces the chaotic solution as in Fig. 2. The regions in phase space occupied by the basins of attraction of the two solutions change size with respect to each other in a manner similar to that found in the Lorenz equations.[14] The change in size in this case occurs over a small τ_b interval of width ~ 0.001 about $\tau_b=1.305$.

No stable chaotic solutions were detected for τ_b values above 1.477. A stable period-3 limit cycle appears for $\tau_b=1.477$ and persists at all greater values.

The second region provides other examples of chaotic behavior in this system. In this region we choose $\tau_1=0.157$, $\tau_2=0.628$, and, again, τ_b varies. The situation is

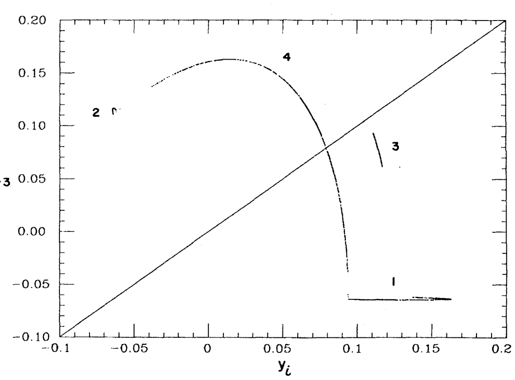

FIG. 2. One of three similar portions of the iterated map, y_{i+3} vs y_i, for $t_1=1.1$, $\tau_2=4.5$, and $\tau_b=1.305$. The straight line is $y_{i+3}=y_i$.

somewhat simpler in this case. For τ_b less than 1.455, the motion is periodic with period 1. For slightly greater values, up to $\tau_b = 1.653$, the period is twice the driving period. Larger values of τ_b produce chaotic behavior, with no stable periodic orbits observed.

The nature of the chaotic behavior is clearly revealed by examining the second iterated map, y_{i+2} vs y_i. Figure 3 shows such a map with $\tau_b = 1.9$. The iterated map is reminiscent of the triangular map first investigated by Lorenz.[2] The implications of the map are the same in both cases: the absence of stable, physically accessible limit cycles is assured for τ_b values which produce such a map.

As τ_b is increased, the second iterated map does continue to evolve, however. Additional features appear which cannot be interpreted in a straightforward manner. Based on a partial survey of τ_b values and initial conditions, there apparently continue to be no stable limit cycles for arbitrarily large τ_b values.

The above two examples for two parameter regions of τ_1, τ_2, and τ_b provide a small glimpse of the rich periodic and chaotic behavior of Eq. (2). In this respect, it is similar to the behavior of previous nonlinear systems that have been studied. However, the much greater ease of solution of Eq. (2), and similar piecewise-linear systems (including higher dimensional equations), imply that piecewise-linear equations provide an ideal test case for studies of chaos in differential equations.

A recent study[15,16] models a driven *RLC* (resistance-inductance-capacitance) electronic circuit by a piecewise-linear equation which has similarities to that of Eq. (2). A Varactor diode is used as a capacitor to supply the nonlinearity. It was noted that including both a nonzero forward bias and a reverse recovery time was necessary for bifurcations to occur. The present paper shows that a simpler method to produce chaos, without a forward bias or a reverse recovery time, is to have the capacitance switch between two constant values.

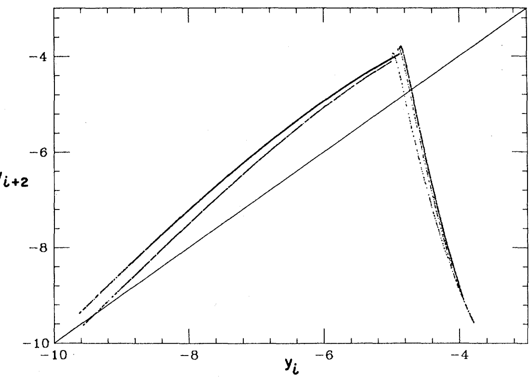

FIG. 3. Iterated map, y_{i+2} vs y_i, for $\tau_1 = 0.157$, $\tau_2 = 0.628$, and $\tau_b = 1.9$. The straight line is $y_{i+2} = y_i$.

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