Passive scalar fluctuations in intermittent turbulence with applications to wave propagation

H. G. E. Hentschel and Itamar Procaccia
Chemical Physics Department, Weizmann Institute of Science, Rehovot 76100, Israel
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The effects of the fractal nature of intermittent fully developed turbulence on the fluctuations of passive scalars are assessed. Significant intermittency corrections to the "$\frac{2}{3}$ (-power) law" for the length-scale dependence of the structure function and other laws concerning fluctuation spectra are found. The implications of these findings on phenomena found in the context of wave propagation through turbulent media are investigated. Intermittency effect on scintillation of light sources, scattering of sound or electromagnetic waves, and fluctuations in phases and amplitudes of these waves are studied theoretically and compared to experiments.

I. INTRODUCTION

The aim of this paper is to study the effect of intermittency[1–3] in fully developed turbulence on the spectra of fluctuations of "passive scalars." By the term "passive scalars" we mean[4–7] parameters of the fluid that hardly influence its turbulent flow. Examples of such passive scalars are temperature, humidity, refractive index, concentration of contaminants, etc.

The motivation for our study is twofold. From the theoretical point of view, which is our main concern, we want to show that intermittency, which is manifested as spottiness in the turbulent activity on smaller scales, has interesting effects on the fluctuations of passive scalars. Conversely, studying these fluctuations, one can learn a considerable amount about intermittent turbulence. From the practical point of view we want to argue that many processes of technological importance are appreciably influenced by the intermittent nature of turbulence. In a previous paper[8] we concentrated on the effects found in turbulent diffusion and suggested modifications to the classical Richardson's "$\frac{4}{3}$ (-power) law" of the diffusivity as function of the length scale. In this paper we shall concentrate on phenomena involving wave propagation through turbulent media.[9]

The existence of intermittency in fully developed turbulence calls for appreciable modifications of the original Kolmogorov theory[10] for the inertial range of turbulence.[3,11–14] This is unfortunate, because that theory enjoys ingenious simplicity. In trying to modify that theory one is faced with two competing alternatives. The first is to go along the lines of the Soviet school in correcting the statistical assumptions concerning turbulence. This would lead to something of the nature of the so-called "log-normal model."[3,11] Some authors, including the present ones, do not find this approach appealing. It sacrifices the simplicity without having a convincing *raison d'être*. A second approach which is originally due to Mandelbrot is geometrical.[12] It views fully developed turbulence as a fractal. The present authors prefer this approach because it allows one to use scaling concepts combined with fractal statistics to develop an approach which retains much of the simplicity of the original theory of Kolmogorov.

The simplest model of turbulence which does justice to the phenomenon of intermittency is that of "fractally homogeneous turbulence."[12,14] Here one is allowing for the increasing spottiness at small scales by asserting that the turbulent activity is concentrated on a fractal whose dimension is less than 3. By assumption, the activity is homogeneous in regions where it exists. Certainly this is a simplified model. It is just the next step towards realistic description after Taylor's concept of homogeneous turbulence. As has been found experimentally, this model agrees with experiments on the statistics of the higher-velocity correlation functions at least as well (if not better) as the "log-normal" model.[15,16] Owing to a paper by Frisch, Sulem, and Nelkin[17] this approach is sometimes called the "β model." This model is however one way of visualizing "absolute curdling"[12] and is not compelling. Another physical view of fractally homogeneous turbulence, which also led to an estimate of the fractal dimension has been presented by the authors.[18] Throughout this paper we adhere to this model of fully developed turbulence.

The scaling theory for passive scalar fluctuations, which is presented in Sec. II is quite general. The applications considered in this paper, i.e., to wave propagation, will be more restrictive. The reason is that we are not interested in the problem of wave propagation *per se*, but rather as a mean to investigate the effect of intermittency on passive scalars. We shall therefore work within the range of validity of the Rytov approximation.[19] This approximation limits the discussion to shorter propagation paths as discussed in Sec. III. The experiments we shall compare with are all within the range of validity of this approximation.

The structure of the paper is as follows. In Sec. II we discuss the concepts behind the model of fractally homogeneous turbulence and the way to get various predictions on its basis. In particular, we shall obtain formulas for the fluctuations of passive scalars. In Sec. III we discuss wave propagation in intermittent turbulence and obtain predictions concerning the phenomena of scintillation of light sources, scattering of sound and electromagnetic waves, and fluctuations in the phase and amplitude of sound and electromagnetic waves. In parallel to the theoretical considerations we compare the results to the experiments. Section IV offers concluding remarks.

II. PASSIVE SCALARS IN FRACTALLY HOMOGENEOUS TURBULENCE

A. Homogeneous fractals

Consider a fractal object of size l_0. From the definition of the fractal dimension we know that upon a change of length scale from l_0 to l, where $l<l_0$, there result N objects that contain a piece of the fractal.[14] The relation is

$$N=\left[\frac{l_0}{l}\right]^D , \tag{2.1}$$

where D is the fractal dimension. If all the objects are similar (i.e., the fractal is uniform) we refer to the fractal as homogeneous. For homogeneous fractals it is straightforward to calculate the probability that a volume of size l^d (where d is the Euclidean dimension) belongs to the fractal; this probability will be extremely useful in what follows.

Since there are N objects of size l^d, their volume is

$$Nl^d=\left[\frac{l_0}{l}\right]^D l^d \tag{2.2}$$

and thus their probability of occurrence is their volume divided by the total volume l_0^d, or

$$\mathscr{P}(l)=\left[\frac{l}{l_0}\right]^{d-D} , \tag{2.3}$$

where $\mathscr{P}(l)$ is the probability that a volume of size l^d belongs to the homogeneous fractal. Since $d>D$ we see that when the length scale decreases it becomes more difficult to find a piece of the fractal. This is precisely the notion of spottiness or intermittency.

B. Fractally homogeneous turbulence

One of the most commonly quoted manifestations of the intermittent nature of turbulence is the long tail in the correlation function of the viscous dissipation, $\epsilon(\vec{r})$.[20] Experimentally one finds

$$\langle\epsilon(\vec{r})\epsilon(\vec{r}+\vec{l})\rangle=\langle\epsilon\rangle^2\left[\frac{l_0}{l}\right]^{\mu} , \tag{2.4}$$

where l is in the inertial range, $l_d \ll l \ll l_0$, and l_0, l_d are the stirring and dissipation length scales, respectively.[3] $\langle\epsilon\rangle$ is the mean energy input per unit mass per unit time. Within the model of fractally homogeneous turbulence one derives this result with the additional output that $\mu=d-D$. This is done as follows.[21]

The average dissipation $\langle\epsilon(\vec{r})\rangle$ will be caused by viscosity at the length scale l_d. We therefore write

$$\langle\epsilon(\vec{r})\rangle\sim\left[\frac{l_d}{l_0}\right]^{d-D}\nu\frac{v_d^2}{l_d^2} , \tag{2.5}$$

where ν is the viscosity, v_d is the velocity difference across an active region of size l_d, and the factor $(l_d/l_0)^{d-D}$ weights the probability that the volume l_d^d belongs to the active, fractal region. Contributions to the correlation function $\langle\epsilon(\vec{r})\rangle\epsilon(\vec{r}+\vec{l})\rangle$ can come only from length scales of size l or larger. The probability that both points $\vec{r}$ and $\vec{r}+\vec{l}$ belong to activity of size $l\le l_n<l_0$ is $\sim(l_n/l_0)^{d-D}$. In addition, when we know that both points $\vec{r}$ and $\vec{r}+\vec{l}$ belong to the active region of size l_n, we have to weight the probability that each point separately belongs to activity on size l_d. This would give rise to the scaling equation

$$\langle\epsilon(\vec{r})\epsilon(\vec{r}+\vec{l})\rangle \sim\sum_{l\le l_n<l_0}\left[\frac{l_n}{l_0}\right]^{d-D}\left[\left[\frac{l_d}{l_n}\right]^{d-D}\nu\frac{v_d^2}{l_d^2}\right]^2 . \tag{2.6}$$

The largest term in the sum is the one for which $l_n=l$. Taking this term,[22] and using Eq. (2.5), we find

$$\langle \epsilon(\vec{r})\epsilon(\vec{r}+\vec{l})\rangle = \langle \epsilon \rangle^2 \left[\frac{l_0}{l} \right]^{d-D} . \qquad (2.7)$$

Comparing with Eq. (2.4) we see that $\mu = d - D$. Experimentally[20] and theoretically[18] one estimates $0.25 < \mu < 0.5$. The same exponent μ will enter all our formulas below.

C. Fluctuations of passive scalars

In the next section we discuss a variety of wave-propagation phenomena in turbulent media. All these processes are sensitive to the nature of the fluctuations of passive scalars, in particular of the refractive index and of the temperature. As is argued in Sec. III, most experiments are analyzed in terms of the structure functions of passive scalars and their Fourier transforms. Denoting the value of a passive scalar θ at a point $\vec{r}$ by $\theta(\vec{r})$, the structure function is written as

$$D_\theta(l) = \langle [\theta(\vec{r}) - \theta(\vec{r}+\vec{l})]^2 \rangle . \qquad (2.8)$$

The corresponding spectral density $\Phi_\theta(\kappa)$ is defined by

$$D_\theta(l) = 8\pi \int_0^\infty \left[1 - \frac{\sin\kappa l}{\kappa l} \right] \Phi_\theta(\kappa)\kappa^2 d\kappa . \qquad (2.9)$$

Thus the analysis of experiments calls for a knowledge of the scaling behavior of the structure functions. When intermittency is not taken into account, one finds the classical result for the structure function $D_\theta(l) \sim l^{2/3}$ [the "$\frac{2}{3}$ (-power) law"[3]]. We shall see now that this result is changed appreciably when the effect of intermittency is included.

To find the scaling behavior in fully developed, fractally homogeneous turbulence, we want to consider fluctuations of linear size l, θ_l. The main idea is that inhomogeneities of size l appear as the result of fluctuations in the fluid velocity across *active* regions of size l, v_l.[23] There are no inhomogeneities created in the inactive regions of the turbulent medium. With this in mind we can write down immediately that the rate of creation of $\langle \theta_l^2 \rangle$ must scale according to

$$\left[\frac{d}{dt} \langle \theta^2 \rangle \right]_l \sim \left[\frac{l}{l_0} \right]^\mu \frac{{\theta_l}^2 v_l}{l} , \qquad (2.10)$$

where again $(l/l_0)^\mu$ weights the probability to belong to an active region.

In the inertial range these inhomogeneities then break up to smaller scale structures due to the turbulent cascade to small length scales. This subdivision continues until the inhomogeneities disappear due to molecular dissipation on a length scale l_d'. The rate of disappearance, $\langle N \rangle$, can be always written as

$$\langle N \rangle = D \langle (\vec{\nabla}\theta)^2 \rangle , \qquad (2.11)$$

where D is the appropriate diffusion constant. The length scale l_d' is found by equating the rate of creation (or transfer) to the rate of dissipation at this length scale,

$$\frac{v_{l_d'} \theta_{l_d'}^2}{l_d'} \left[\frac{l_d'}{l_0} \right]^\mu \sim \frac{D\theta_{l_d'}^2}{{l_d'}^2} \left[\frac{l_d'}{l_0} \right]^\mu , \qquad (2.12)$$

or $l_d' \sim D/v_{l_d'}$. The dissipation length scale for the energy cascade is defined similarly by $l_d = \nu/v_{l_d}$. In many applications $D \simeq \nu$ and therefore $l_d' \simeq l_d$. For convenience we shall disregard their difference in the following.

In the steady-state situation we can equate the rate of transfer (or creation) on scale l with the dissipation on scale l_d. Therefore

$$\frac{v_l}{l}\theta_l^2 \left[\frac{l}{l_0} \right]^\mu \simeq \langle N \rangle \qquad (2.13)$$

or

$$\theta_l^2 \sim \frac{\langle N \rangle l}{v_l} \left[\frac{l}{l_0} \right]^{-\mu} . \qquad (2.14)$$

In a similar way we can find an expression for v_l in terms of $\langle \epsilon \rangle$. Equating the energy input on the length scale l_0, $\langle \epsilon \rangle$, to the energy transfer, on length scale l (which only occurs in the active regions), we find

$$\langle \epsilon \rangle \sim \left[\frac{l}{l_0} \right]^\mu \frac{v_l^3}{l} \qquad (2.15)$$

or

$$v_l \sim \langle \epsilon \rangle^{1/3} l^{1/3} \left[\frac{l}{l_0} \right]^{-\mu/3} . \qquad (2.16)$$

Using Eqs. (2.14) and (2.16) we have

$$\theta_l^2 \sim \langle N \rangle \langle \epsilon \rangle^{-1/3} l^{2/3} (l/l_0)^{-2\mu/3} . \qquad (2.17)$$

Remember that θ_l^2 is the square of the passive scalar fluctuations *in an active region.* Next we wish to consider the structure functions $D_\theta(l)$ defined by Eq. (2.8). As before, we find the scaling behavior of $D_\theta(l)$ by weighting θ_l^2 by the probability of finding an active region of size l:

$$D_\theta(l) \sim \left[\frac{l}{l_0} \right]^\mu \theta_l^2 \sim \langle N \rangle \langle \epsilon \rangle^{-1/3} l^{2/3} \left[\frac{l}{l_0} \right]^{\mu/3} . \qquad (2.18)$$

as a final form to be used below we write

$$D_\theta(l)=C_\theta^2 l^{2/3}\left[\frac{l}{l_0}\right]^{\mu/3} . \quad (2.19)$$

We see that the structure function contains a universal correction to the classical "$\frac{2}{3}$ (-power) law" for passive scalars. It is straightforward now to find the intermittency corrections to the spectral density $\Phi_\theta(\kappa)$ defined in Eq. (2.9). Using Eq. (2.19) we immediately see that

$$\Phi_\theta(\kappa)=AC_\theta^2\kappa^{-11/3}(\kappa l_0)^{-\mu/3} , \quad (2.20)$$

where A is a dimensionless constant.

Notice that the correction to the "$\frac{2}{3}$ (-power) law" is very large ($\sim 20\%$). In Eq. (2.20) the correction has been reduced to $\sim 5\%$. It is very unfortunate that all the experiments on wave propagation in turbulent media pertain directly to Eq. (2.20). If one could measure Eq. (2.19) directly one could get a much better probe of the intermittency corrections. We shall see below that all the experiments at our disposal are pretty rough and are not sensitive enough to allow precise determination of μ. Taken as a body of data, however, they provide a strong indication that intermittency corrections are significant for wave-propagation phenomena.

III. WAVE PROPAGATION IN FRACTALLY HOMOGENEOUS TURBULENCE

Random fluctuations of the temperature and humidity of the atmosphere result in corresponding fluctuations in the index of refraction.[9,19] These fluctuations are functions of position and time, so that the index of refraction can be written as

$$n(\vec{r},t)=1+n_1(\vec{r},t) . \quad (3.1)$$

The propagation of an electric field $\vec{E}$ in the turbulent atmosphere is governed by the equation

$$\nabla^2\vec{E}+k^2(1+n_1)^2\vec{E}=0 , \quad (3.2)$$

where k is the wave vector. Writing

$$E=\exp(\psi)\equiv\exp(\chi+iS) , \quad (3.3)$$

and substituting in (3.2), one finds

$$\nabla^2\psi+(\vec{\nabla}\psi)^2+k^2(1+n_1)^2=0 . \quad (3.4)$$

In the Rytov approximation, one writes

$$\psi=\psi_0+\psi_1 \quad (3.5)$$

and neglects $\vec{\nabla}\psi_1$ compared to $\vec{\nabla}\psi_0$, to get the two following equations:

$$\nabla^2\psi_0+(\vec{\nabla}\psi_0)^2+k^2=0 , \quad (3.6)$$

$$\nabla^2\psi_1+2\vec{\nabla}\psi_0\cdot\vec{\nabla}\psi_1+2k^2n_1\simeq 0 . \quad (3.7)$$

ψ_0 is obviously independent of the fluctuations $n_1(\vec{r},t)$. On the other hand ψ_1 is proportional to n_1 through the solution

$$\psi_1(\vec{r})=\frac{k}{2\pi E_0(\vec{r})}\int d^3r' n_1(\vec{r}\,')E_0(\vec{r}\,') \times\frac{\exp(ik\,|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|} , \quad (3.8)$$

where $E_0=\exp\psi_0$.

Writing $\psi_1(\vec{r})=\chi_1(\vec{r})+iS_1(\vec{r})$ where χ_1 and S_1 are the fluctuating logarithmic amplitude and phase, respectively, it becomes clear that their correlation and structure functions can be expressed in terms of the refractive index structure function (2.19) or (equivalently) the three-dimensional spectral density (2.20). Similarly, the formulas for scattering of waves are related to the structure function of the refractive index.

We turn now to the various applications. As mentioned before, most of the intermittency corrections are found to be rather small and most experiments are rather rough. Our aim is to point out that almost invariably one finds at least qualitative agreement with our predictions and as a whole the data seem to support this approach.

A. Scintillation of terrestrial light sources

1. Theory

The scintillation of light sources is caused by fluctuations in the amplitude of light which has propagated through a distance L.[9] In experiments the mean-square fluctuation $\langle\chi_1^2\rangle$ is measured. To find a scaling expression for this quantity, we take L to be in the x direction, and define $B_\chi(\rho)$ by

$$B_\chi(\rho)=\langle\chi_1(L,y_1,z_1)\chi_1(L,y_2,z_2)\rangle , \quad (3.9)$$

where

$$\rho=[(y_1-y_2)^2+(z_1-z_2)^2]^{1/2} .$$

We see that $\langle\chi_1^2\rangle\equiv B_\chi(0)$. The expression for $B_\chi(\rho)$ which follows from Eq. (3.8) has been derived in Ref. 9 with the result that $B_\chi(\rho)$ has the two-dimensional spectral representation

$$B_\chi(\rho)=2\pi\int_0^\infty J_0(\kappa\rho)F_\chi(\kappa)\kappa\, d\kappa \quad (3.10)$$

with

$$F_\chi(\kappa)=\pi k^2L\left[1-\frac{k}{\kappa^2L}\sin\frac{\kappa^2L}{k}\right]\Phi_n(\kappa) . \quad (3.11)$$

Using Eq. (2.20) in Eq. (3.11) we see that in the inertial subrange of turbulence, i.e., when

$$l_d \ll \sqrt{\lambda L} \ll l_0 \ , \tag{3.12}$$

the fluctuation in amplitude is expressed as

$$\langle \chi_1^2 \rangle = B_\chi(0) \sim C_n^2 k^{7/6} L^{11/6} (k l_0)^{-\mu/3} (kL)^{\mu/6} \ . \tag{3.13}$$

This result modifies the expression given by the Kolmogorov theory[9] for which $\mu=0$. There are intermittency corrections both to the k and L dependence of $\langle \chi_1^2 \rangle$.

2. Experiments

An investigation of the scintillation of a terrestrial light source was described in Ref. 23. A portion of flat farmland was selected for its homogeneity of the turbulent activity along the propagation length, and a light source was located at various distances from a fixed point. The fluctuations $\langle \chi_1^2 \rangle$ as a function of distance were measured. The experimental results are reproduced in Table I. If one plots the data on a double-logarithmic plot one can obtain a good fit to the formula $\langle \chi_1^2 \rangle \propto L^n$. Theoretically we expect $n=(11+\mu)/6$. A least-squares fit produced the value $\mu=0.68$. As is seen from the table a best fit with $\mu=0$ gives much worse agreement with experiment. The value $\mu=0.68$ is higher than the usually quoted upper bound on μ, i.e., $\mu \lesssim 0.5$. However, the errors in this early experiment are quite large and one clearly cannot hope to obtain precise information on the value of μ.

In other experiments the dependence of $\langle \chi_1^2 \rangle$ on k was determined.[24] According to Eq. (3.13) we expect a dependence of $k^{(7-\mu)/6}$. In Ref. 24 the values of $\langle \chi_1^2 \rangle$ were determined for three values of k with a fixed propagation distance $L=1.4$ km. These measurements allow a comparison with theory without any free parameter. The reason is that a double-logarithmic plot of $\langle \chi_1^2 \rangle$ for k_1 as a function of $\langle \chi_1^2 \rangle$ for k_2 should yield a straight line with a unit slope and a known intercept:

$$\langle \chi_1^2 \rangle_{k_1} = \left[\frac{k_1}{k_2} \right]^{(7-\mu)/6} \langle \chi_1^2 \rangle_{k_2} \ . \tag{3.14}$$

In addition, the advantage of this type of measurement is that it avoids the dependence of fluctuations on long-term changes in C_n^2 due to atmospheric conditions (which is a problem that enters all other types of experiments discussed here). In Fig. 1 we show two sets of experimental results and two sets of straight lines. In both cases the upper line pertains to $\mu=0$. The lower line has been calculated with $\mu=0.5$. It seems that the lower line represents the data better in both cases.

TABLE I. Fluctuations in amplitude of the light of a terrestrial source at distance L from the source. The third and fourth columns were calculated by fitting straight lines to a double-logarithmic plot of the experimental data.

L (m)	$\langle \chi_1^2 \rangle_{\rm expt}$	$\langle \chi_1^2 \rangle_{\mu=0}$	$\langle \chi_1^2 \rangle_{\mu=0.68}$	$\frac{\langle \chi_1^2 \rangle_{\mu=0}}{\langle \chi_1^2 \rangle_{\rm expt}}$	$\frac{\langle \chi_1^2 \rangle_{\mu=0.68}}{\langle \chi_1^2 \rangle_{\rm expt}}$
2000	0.420	0.389	0.419	1.07	1.05
1000	0.128	0.110	0.113	0.86	0.90
500	0.027	0.031	0.030	1.13	1.07
250	0.0078	0.0087	0.0080	1.12	0.99

B. Phase fluctuations and their frequency spectra

1. Theory

The quantity considered here[9] is the time structure function of the phase fluctuations,

$$\langle [S_1(t+\tau) - S_1(t)]^2 \rangle \equiv H_s(\tau) \ , \tag{3.15}$$

and its experimentally measurable frequency spectrum $W_s(\omega)$, which are related by

$$H_s(\tau) = \frac{1}{\pi} \int_0^\infty (1-\cos\omega\tau) W_s(\omega) d\omega \ . \tag{3.16}$$

These time-dependent quantities can be related to the fluctuations of the refractive index by invoking the "Taylor hypothesis." The basis of this hypothesis is the assumption that the time dependence of the index of refraction fluctuations is determined by their space dependence as they are convected by the velocity field. In the context of phase fluctuations it is argued[9] that motions transverse to the propagation directions are most important. The reason is that the effective "width" of a diffracted, propagating wave is much smaller than its propagation distance ($\sqrt{\lambda L} \ll L$). Consequently, in a short time τ there can be drastic changes in the transverse inhomogeneities but not in the longitudinal ones. These considerations lead to an approximate expression for $H_s(\tau)$ in terms of the structure function $D_s(\rho)$, which is

$$D_s(\rho) = \langle [S_1(L,y_1,z_1) - S_1(L,y_2,z_2)]^2 \rangle \ , \tag{3.17}$$

where again

$$\rho = [(y_1-y_2)^2 + (z_1-z_2)^2]^{1/2} \ .$$

The relation is

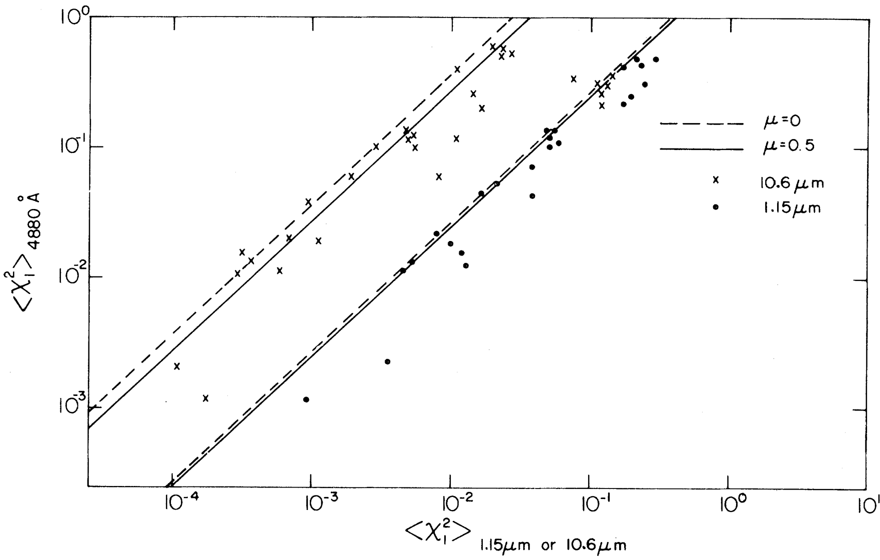

FIG. 1. Scintillation of terrestrial light sources. The logarithmic amplitude variances $\langle\chi_1^2\rangle_{\lambda_1}$ for $\lambda_1=4880$ Å is plotted against $\langle\chi_1^2\rangle_{\lambda_2}$ for $\lambda_2=10.6$ μm and 1.15 μm (Ref. 24). Experimentally the points $\times$ represent $\lambda_2=10.6$ μm while $\bullet$ represent $\lambda_2=1.15$ μm. Theoretically the upper two straight lines are the theory for $\lambda_2=10.6$ μm. The dashed line corresponds to classical theory $\mu=0$, while the solid line is that for $\mu=0.5$. Similarly the lower two lines correspond to $\lambda_2=1.15$ μm.

$$H_s(\tau)=D_s(v_T\tau)\ , \tag{3.18}$$

where v_T is the average wind velocity transverse to the propagation direction. Within the Rytov approximation we can relate $D_s(\rho)$ to the three-dimensional spectrum of the refractive index fluctuations for which we have an expression including intermittency corrections. $D_s(\rho)$ has the two-dimensional spectral representation

$$D_s(\rho)=4\pi\int_0^\infty[1-J_0(\kappa\rho)]F_s(\kappa)\kappa\,d\kappa\ , \tag{3.19}$$

where

$$F_s(\kappa)=\pi k^2L\left[1+\frac{k}{\kappa^2L}\sin\frac{\kappa^2L}{k}\right]\Phi_n(\kappa)\ . \tag{3.20}$$

Using Eq. (2.20) we find, in the inertial subrange of turbulence (i.e., $l_d \ll \sqrt{\lambda L} \ll l_0$),

$$D_s(\rho)\propto C_n^2k^2L\rho^{5/3}\left[\frac{\rho}{l_0}\right]^{\mu/3}\ . \tag{3.21}$$

From this relation we see that

$$H_s(\tau)\sim\tau^{(5+\mu)/3}\ , \tag{3.22}$$

and, consequently, using this result in Eq. (3.16), we find the scaling relation

$$W_s(\omega)\sim\omega^{-(8+\mu)/3}\ . \tag{3.23}$$

2. Comparison with experiments

The phase spectra $W_s(\omega)$ were reported, for example, in Refs. 25 and 26. A clear scaling behavior seems to be found over a wide range of frequencies. Comparing with Figs. 83 and 84 of Ref. 9 and Fig. 2 of Ref. 26 one convinces oneself that a slope higher than $\frac{8}{3}$ is needed to fit the data. In fact, $\mu\sim0.5$ would do justice to both sets of data. However, although one gets a clear feeling that intermittency corrections are found there, one has to admit that the accuracy of the data is not sufficient to draw firm conclusions.

Finally, we have the amusing result of Ref. 27 which studied experimentally the fluctuations of the functions

$$\Delta_1(\vec{r},\tau)=S(\vec{r},t+\tau)-S(\vec{r},t)$$

and

$$\Delta_2(\vec{r},\tau)=\Delta_1(\vec{r},t+\tau)-\Delta_1(\vec{r},t)\ .$$

Their fluctuations can be expressed in terms of the frequency spectrum by

$$\langle \Delta_1^2 \rangle = \frac{1}{\pi} \int_0^\infty (1-\cos\omega\tau) W_s(\omega) d\omega \ ,$$
$$\langle \Delta_2^2 \rangle = \frac{1}{\pi} \int_0^\infty [3-4\cos(\omega\tau) + \cos(2\omega\tau)] W_s(\omega) d\omega \ . \tag{3.24}$$

If the phase frequency spectrum scales as $W_s(\omega) \sim \omega^{-m}$, then the ratio $\langle \Delta_2^2 \rangle / \langle \Delta_1^2 \rangle$ is a number which depends only on m,

$$\frac{\langle \Delta_2^2 \rangle}{\langle \Delta_1^2 \rangle} = 4(1-2^{m-3}) \ . \tag{3.25}$$

Experimentally, this ratio was measured giving $m=2.87$. As

$$m_{\text{theor}} = (8+\mu)/3$$

we find $\mu=0.61$, which, although somewhat high, agrees reasonably with other estimates.

C. Scattering of electromagnetic waves

1. Theory

Suppose a plane monochromatic electromagnetic wave $\vec{E}_0 e^{i\vec{k}\cdot\vec{r}-i\omega t}$ is incident on a volume V of a turbulent medium. This wave will be scattered by the fluctuations of the refractive index $n(\vec{r})$. If we assume that the fluctuations are stationary and homogeneous, we can use Maxwell's equations and perturbation theory to show (within the Rytov approximation) that the differential cross section $d\sigma/d\Omega$ for scattering through an angle θ is given by[9]

$$\frac{d\sigma}{d\Omega} = 2\pi k^4 V (\sin^2\eta) \Phi_n [2k \sin(\theta/2)] \ , \tag{3.26}$$

where η is the angle between $\vec{E}_0$ and $\vec{r}$. Substituting Eq. (3.20) we have

$$\frac{d\sigma}{d\Omega} \propto C_n^2 k^{1/3} (kl_0)^{-\mu/3} (\sin^2\eta) [\sin(\theta/2)]^{-(11+\mu)/3} \ ,$$

where

$$2\pi/l_0 \ll 2k \sin(\theta/2) \ll 2\pi/l_d \ . \tag{3.27}$$

Thus there are clear intermittency corrections to classical theory. The change in the $\frac{11}{3}$ exponent is small ($\sim 4\%$). However, the change in the differential cross section with wave number at fixed angle of scattering θ is large. The exponent is reduced from a $\frac{1}{3}$ by up to 50% and should be easily observable.

2. Comparison with experiments

Experimentally the ratio of the received scattered power to the power ideally received in a vacuum $P_s/P_{fs} \sim d^{-8/3} \, d\sigma/d\Omega$ is normally measured, where d is the distance of propagation. This ratio as a function of distance d has been measured in long-range tropospheric propagation of ratio waves.[9] Unfortunately this distance is probably outside the range of validity of the Rytov approximation. However, lacking better data we shall use these results in a comparison with theory. The distance d is related to the scattering angle θ by $\theta \simeq d/a$ where a is the radius of the earth. Thus using Eq. (3.27) we see that the normalized scattered power scales like

$$\frac{P_s}{P_{fs}} \sim \lambda^{-(1-\mu)/3} d^{-(19+\mu)/3} \ . \tag{3.28}$$

A plot of $\log_{10}(P_s/P_{fs})$ vs $\log_{10} d$ can be seen in Fig. 2. The gradient is clearly greater than the $\frac{19}{3}$ expected if $\mu=0$. However, the experiment is clearly not accurate enough to allow clear-cut statements.

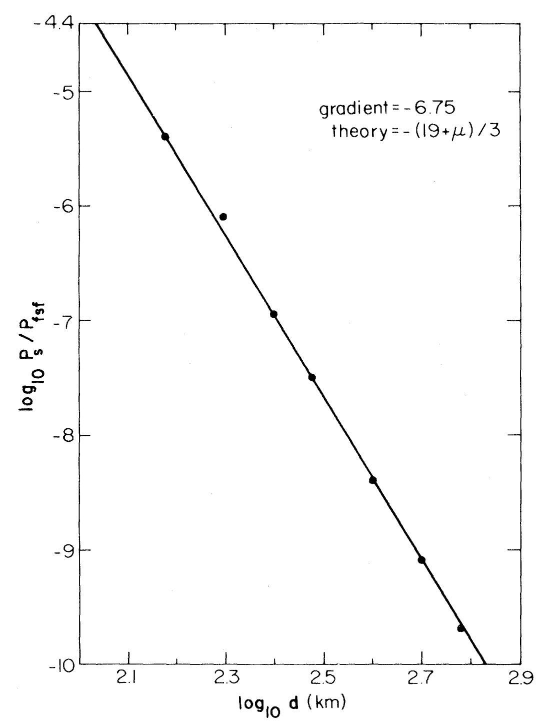

FIG. 2. A double-logarithmic plot of the ratio of the received scattered power to the power ideally received in a vacuum, with distance of propagation (Ref. 9). A least-squares fit gives a slope of -6.75. Theory gives a slope of $-(19+\mu)/3$. Thus for $\mu=0.5$ we expect a slope of -6.5.

For the dependence on wavelength we have the surprising result that our correction is in the right direction but *too small*. Experimentally one finds[9] a behavior of the form $\lambda^{-\alpha}$ where α is always less than $\frac{1}{3}$, but in fact in half of the cases $\alpha \sim -1$ was found. This may be due to a breakdown in the Rytov approximation. In any case, it clearly would be worthwhile to do shorter-range multiwavelength experiments, especially to check the large correction to the $\lambda^{-1/3}$ behavior predicted by the classical theory.

D. Amplitude and phase fluctuations of sound waves

1. Theory

As in the case of electromagnetic waves, there also occur fluctuations in the phase and amplitude of sound waves as they propagate through a distance L of a turbulent medium. However, here the fluctuations will mainly be due to velocity and temperature inhomogeneities. This is due to the fact that the speed of sound c depends on the air temperature T through the relation $c=\sqrt{\gamma RT}$ with $\gamma=C_p/C_v$, while in addition the sound waves are transported by the air motion and therefore the turbulent velocity fields cause additional fluctuations. The theory in this case follows essentially the same lines as for electromagnetic waves except that the constant C_n^2 in the structure function of the refractive index fluctuation is replaced by the constant $C_{n'}^2$ for the "effective refractive index" of the sound waves[9]

$$C_{n'}^2=\frac{C_T^2}{T_0^2}+\frac{4C_v^2}{c_0^2}\ , \tag{3.29}$$

where C_T^2 and C_v^2 refer to the temperature and velocity structure functions, respectively, while T_0 and c_0 are the average temperature and speed of sound. Experimentally the standard deviations $\sigma_s=\sqrt{D_s(\sigma_T\tau)}$ and $\sigma_\chi=\langle \chi_1^2\rangle^{1/2}$ are normally measured, and thus theoretically we would expect the scaling behavior

$$\sigma_s \propto C_{n'}kL^{1/2}(\tau v_T)^{5/6}\left[\frac{\tau v_T}{l_0}\right]^{\mu/6}, \tag{3.30}$$

$$\sigma_\chi \propto C_{n'}k^{7/12}L^{11/12}(kl_0)^{-\mu/6}(kL)^{\mu/12}\ . \tag{3.31}$$

2. Experiment

In Refs. 9 and 28 there is a description of a series of 28 experiments on the logarithmic amplitude fluctuations σ_χ at frequencies from 3 to 76 kHz. Writing $\sigma_\chi \propto L^\alpha$ an average $\alpha=1.1$ for acoustic waves and $\alpha=0.95$ for ultrasonic frequencies (30–76 kHz) were found. As $\alpha_{\rm theor}(\mu)=(11+\mu)/12$ we see that $\alpha_{\rm theor}(\mu=0)=0.92$ while $\alpha_{\rm theor}(\mu=0.4)=0.95$. Thus the intermittency corrections are in the right direction, and in the case of ultrasonic sound waves they are also quantitatively correct. Again this result should not be considered by itself but with the body of other data.

E. Scattering of sound waves

1. Theory

The turbulent velocity and temperature fields will not only cause fluctuations of sound waves—they will also cause scattering. The propagation of sound waves is described by the equations of hydrodynamics. Specifically the reduced acoustic pressure $\Pi=p_a/\gamma p_0$ (where $p=p_0+p_a$ are the constant external and acoustic pressures, respectively) obeys[9]

$$\nabla^2\Pi+k^2\Pi=-\partial_i\left[\frac{T'}{T_0}\partial_i\Pi\right] -\frac{2}{i\omega}\frac{\partial^2}{\partial x_i\partial x_k}\left[U_i\frac{\partial\Pi}{\partial x_k}\right], \tag{3.32}$$

where the right-hand side is correct to first order in the temperature and velocity fluctuations T' and U, respectively, and $\omega=c_0k$. Writing $\Pi=\Pi_0+\Pi_s$ where Π_0 is the incident field and Π_s the scattered field we have, on substituting in (3.32),

$$\nabla^2\Pi_0+k^2\Pi_0=0\ , \tag{3.33}$$

$$\nabla^2\Pi_s+k^2\Pi_s=-\frac{\partial}{\partial x_i}\left[\frac{T'}{T_0}\frac{\partial\Pi_0}{\partial x_i}\right] -\frac{2}{i\omega}\frac{\partial^2}{\partial x_i\partial x_j}\left[U_i\frac{\partial\Pi_0}{\partial x_j}\right]. \tag{3.34}$$

The solution of (3.34) is

$$\Pi_s(\vec r)=\frac{1}{4\pi}\int_v d\vec r'\frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|}\frac{\partial}{\partial x_i'}\left[\frac{T'(\vec r')}{T_0}\frac{\partial\Pi_0(\vec r')}{\partial x_i'}+\frac{2}{i\omega}\frac{\partial}{\partial x_j'}\left[U_i(\vec r')\frac{\partial\Pi_0(\vec r')}{\partial x_j'}\right]\right]. \tag{3.35}$$

Note the similarity of (3.35) to (3.8), except that temperature and velocity fluctuations cause the pressure fluctuations instead of refractive index fluctuations which scatter electromagnetic radiation. Thus it is not surprising that the expression for the differential cross section for sound scattering into an angle θ,

$$\sigma_0(\theta)=\frac{\pi}{2}k^4\cos^2\theta\left[\frac{\Phi_T[2k\sin(\theta/2)]}{T_0^2}+\frac{\cos^2(\theta/2)\Phi_v[2k\sin(\theta/2)]}{\Pi_0^2[2k\sin(\theta/2)]}\right], \tag{3.36}$$

where $\Phi_T(\kappa)$ and $\Phi_v(\kappa)$ are the three-dimensional spectral densities for the temperature and velocity fluctuations, should be very similar to (3.26). Substituting in the expressions for the spectral densities valid in the inertial subrange of turbulence $2\pi/l_0 \ll 2k\sin(\theta/2) \ll 2\pi/l_d$,

$$\Phi_v(\kappa)=A'C_v^2\kappa^{-11/3}(\kappa l_0)^{-\mu/3}, \tag{3.37}$$

$$\Phi_T(\kappa)=AC_T^2\kappa^{-11/3}(\kappa l_0)^{-\mu/3}, \tag{3.38}$$

in (3.36) we find

$$\sigma_0(\theta)=k^{1/3}(kl_0)^{-\mu/3}(\cos^2\theta)[\sin(\theta/2)]^{-(11+\mu)/3}\left[\frac{B'C_v^2\cos^2(\theta/2)}{C_0^2}+\frac{BC_T^2}{4T_0^2}\right], \tag{3.39}$$

where B and B' are dimensionless constants. Thus there is no scattering at $\theta=\pi/2$ as $\cos^2\theta=0$, while all scattering at $\theta=\pi$ is by temperature fluctuations as here $\cos^2(\theta/2)=0$. Inspecting Eq. (3.39) we see that as in the case of electromagnetic waves there is a small correction to the angular exponent to a value $(11+\mu)/3$ while the wave number exponent is reduced greatly to $(1-\mu)/3$ and should be easily observable.

2. *Experiment*

There exist some experiments which measure the angular dependence of sound scattering by a turbulent atmosphere.[29] To eliminate the nonstationariness of the meterological conditions the ratios $\sigma_0(\theta)/\sigma_0(25°)$ were measured as a function of angle. We can neglect the effect of temperature fluctuations if θ is not too large,[3] and in this case we have, theoretically,

$$\frac{\sigma_0(\theta)}{\sigma_0(25°)}=\frac{\cos^2\theta\cos^2(\theta/2)\,|\sin(\theta/2)|^{-(11+\mu)/3}}{\cos^2(25°)\cos^2(12.5°)\,|\sin(12.5°)|^{-(11+\mu)/3}}. \tag{3.40}$$

A comparison of this expression with experiment (see Fig. 3) shows that the inclusion of intermittency gives a qualitative improvement of the agreement with experiment, especially for higher scattering angles.

IV. CONCLUDING REMARKS

We have presented a scaling theory of fluctuations of passive scalars in intermittent fully developed fluid turbulence. The theoretical model has been that of "fractally homogeneous turbulence." The main results were the expressions for the structure function $D_\theta(l)$ [Eq. (2.18)] and its spectral representation [Eq. (2.20)]. The expressions contained intermittency corrections to classical theory. In Eq. (2.18) the intermittency correction is very large. Unfortunately most of the experiments that have been performed so far pertained to Eq. (2.20) in which the intermittency correction is much smaller. It would be advisable to design modern experiments that would probe directly the corrections to the "$\frac{2}{3}$ (-power) law."

The comparison of the theory to experiment has been done in the context of phenomena related to the nature of wave propagation through turbulent media. The examples discussed in Sec. III were all in qualitative, and sometimes quantitative, agree-

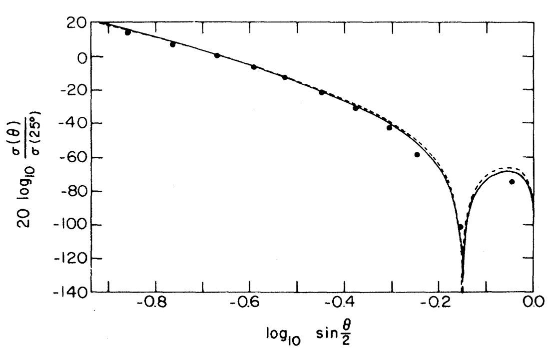

FIG. 3. A double-logarithmic plot of the normalized differential cross section $\sigma(\theta)/\sigma(25°)$ versus $\sin(\theta/2)$ for the angular dependence of sound waves scattered by a turbulent medium. The experimental points ● (Ref. 24) are compared with the dashed curves of classical theory $\mu=0$, and the solid curve which includes an intermittency correction of $\mu=0.5$.

ment with our approach. Taken as a body of data they provide strong support for the notions advanced here. If correct, these intermittency corrections cannot be neglected in technological applications. Together with our previous results on turbulent diffusion they point out the fact that monitoring the behavior of passive scalars might provide insight to intermittent fluid turbulence that is not easily attainable with the widespread methods of looking at higher velocity correlation functions.

ACKNOWLEDGMENTS

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