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Integrable families of Hénon-Heiles–type Hamiltonians and a new duality

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Integrable Hamiltonians of type $H=\frac{1}{2}p_x^2+\frac{1}{2}p_y^2+Cx^{\alpha+2}+x^\alpha y^2$ are discussed. Using the parameters appearing in the Painlevé analysis we find that the integrable potentials for $\alpha=1$ (Hénon-Heiles) and $\alpha=-\frac{2}{3}$ (Holt) are in one-to-one correspondence. Simple relations exist between the corresponding coefficients C, and also the dimensions of the second invariants are found to be related. Quantum integrability is also discussed.

In this paper we present an interesting duality between the Hénon-Heiles and Holt families of integrable potentials, supporting the significance of Painlevé[1–4] property in such systems.

We consider here the generalization of the Hénon-Heiles system, given by the Hamiltonian

$$H=\tfrac{1}{2}p_x^2+\tfrac{1}{2}p_y^2+Cx^{\alpha+2}+y^2x^\alpha \; , \tag{1}$$

where now both C and α are free parameters. A search[5] for those cases which are integrable and possess a second invariant I which is polynomial in p of order 4 or less revealed the following (see Table I): (1) the Hénon-Heiles types [$\alpha=1$, $C=\frac{1}{3}$,[6] $C=2$,[2] and $C=\frac{16}{3}$ (Refs. 3 and 7)]; (2) the Holt types [$\alpha=-\frac{2}{3}$, $C=\frac{3}{4}$,[8] and $C=\frac{9}{2}$ (new)]; and (3) special cases of separable potentials ($\alpha=0$, C free; $\alpha=-4$, C free; $\alpha=-6$, $C=\frac{1}{4}$).

To put some order into these results it is useful to repeat the Painlevé analysis of Chang, Tabor, and Weiss[2] for the present case (1). Assuming the leading behavior $x=At^\mu+\cdots$, $y=Bt^\nu+\cdots$ and substituting into Newton's equations

$$\begin{aligned}\ddot{x}&=-C(\alpha+2)x^{\alpha+1}-\alpha y^2x^{\alpha-1}\;,\\ \ddot{y}&=-2yx^\alpha\;,\end{aligned} \tag{2}$$

TABLE I. Second invariants for the integrable Hamiltonians of type (1) arranged into families. See text for explanations.

$\mu(\mu-1)$	μ	α	$\nu(\nu-1)$	C	$[I]$	Second invariant
6	−2	1	12	$\frac{1}{3}$	2	$p_xp_y+x^2y+\frac{1}{3}y^3$
6	−2	1	2	2	$\frac{8}{3}$	$yp_xp_y-xp_y^2+x^2y^2+\frac{1}{4}y^4$
6	−2	1	$\frac{3}{4}$	$\frac{16}{3}$	4	$p_y^4+4xy^2p_y^2-\frac{4}{3}y^3p_xp_y-\frac{4}{3}x^2y^4-\frac{2}{9}y^6$
6	3	$-\frac{2}{3}$	12	$\frac{3}{4}$	3	$p_y^3+\frac{3}{2}p_yp_x^2+(-\frac{9}{2}x^{4/3}+3x^{-2/3}y^2-\frac{5}{24}\hbar^2x^{-2})p_y+9x^{1/3}yp_x$
6	3	$-\frac{2}{3}$	2	$\frac{9}{2}$	4	$p_y^4+2p_y^2p_x^2+(4x^{-2/3}y^2-\frac{5}{18}\hbar^2x^{-2})p_y^2$ $+24x^{1/3}yp_yp_x+72x^{2/3}y^2-2\hbar^2x^{-2/3}$
6	3	$-\frac{2}{3}$	$\frac{3}{4}$	12	6	$p_y^6+3p_y^4p_x^2+(18x^{4/3}+6x^{-2/3}y^2-\frac{5}{12}\hbar^2x^{-2})p_y^4$ $+72x^{1/3}yp_y^3p_x+(648x^{2/3}y^2-18\hbar^2x^{-2/3})p_y^2+648y^4$
∞	∞	0	−	free	2	$p_y^2+2y^2$
$-\frac{1}{4}$	$\frac{1}{2}$	−4	−	free	0	$(p_xy-p_yx)^2+2(C+1)x^{-2}y^2+2x^{-4}y^4$
$-\frac{2}{9}$	$\frac{1}{3}$	−6	$\frac{4}{9}$	$\frac{1}{4}$	$\frac{3}{2}$	$yp_x^2-xp_xp_y+x^{-4}y+2x^{-6}y^3$

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gives

$$A^{\alpha}=-\nu(\nu-1)/2 \ , \tag{3}$$

$$\mu=-2/\alpha \ , \tag{4}$$

and depending on whether $\nu=\mu$ or $\nu>\mu$ we have the usual[2] two cases.

Case 1: $\nu=\mu$, C undetermined,

$$(B/A)^2=C(\mu-1)-\mu \ . \tag{5}$$

Case 2: $\nu>\mu$,

$$C=\frac{\mu^2}{\nu(\nu-1)} \ . \tag{6}$$

Continuing now to the next-to-leading behavior with case 1 we write

$$x=At^{\mu}+Et^{\mu+\rho}+\cdots, \quad y=Bt^{\mu}+Ft^{\mu+\rho}+\cdots \ , \tag{7}$$

and substituting this to (2) and using (3)–(6) we find that E and F can be nonzero only if

$$\det\begin{vmatrix}\theta+V^0_{xx} & V^0_{xy}\\ V^0_{xy} & \theta+V^0_{yy}\end{vmatrix}=0 \ , \tag{8}$$

where

$$\theta=(\rho+\mu)(\rho+\mu-1) \ , \tag{9}$$

and

$$\begin{aligned}V^0_{xx}&=(\mu-1)[\mu+2-2C(\mu-1)] \ ,\\ V^0_{xy}&=2(\mu-1)[C(\mu-1)-\mu]^{1/2} \ ,\\ V^0_{yy}&=-\mu(\mu-1) \ .\end{aligned} \tag{10}$$

We now find in analogy with Chang, Tabor, and Weiss[2] two solutions for θ:

$$\begin{aligned}\theta_1&=(\mu-1)(\mu-2) \ ,\\ \theta_2&=2C(\mu-1)^2-\mu(\mu-1) \ .\end{aligned} \tag{11}$$

Since C has the dependence (6) we find that θ_2 is a function of $\nu(\nu-1)$ and $\mu(\mu-1)$ only. Let us now see what we can get by focusing on this property. We define two Hamiltonians (1) with parameters α_1 and α_2 to be *dual* if $\mu_1(\mu_1-1)=\mu_2(\mu_2-1)$, i.e.,

$$\mu_1+\mu_2=1 \tag{12}$$

or

$$\alpha_1+\alpha_2+\tfrac{1}{2}\alpha_1\alpha_2=0 \ . \tag{12'}$$

Our claim now is that to each integrable system of type (1) there should correspond a dual system, with relation (12). For the Hénon-Heiles system $\alpha_1=1$, $\mu_1=-2$; therefore its dual has $\mu_2=3$, $\alpha_2=-\frac{2}{3}$. Indeed for this value of α we do have integrable potentials.

Next let us consider the $\nu(\nu-1)$ dependence. If we have an integrable potential with some α and C, then from (6) and (4) we can compute $\nu(\nu-1)$ for it. For its dual system we predict integrability with the same $\nu(\nu-1)$, i.e., the coefficients C_{μ} for different μ's should be related according to (6), where now $\nu(\nu-1)$ is fixed. Accordingly, corresponding to $C_{-2}=\frac{1}{3}$ we should get $C_3=\frac{3}{4}$, for $C_{-2}=2$ we get $C_3=\frac{9}{2}$, and for $C_{-2}=\frac{16}{3}$, $C_3=12$. Indeed two cases found in our search had $C_3=\frac{3}{4}$ and $\frac{9}{2}$, and now we also predict one more integrable case $C_3=12$. We have found that this last case has a sixth-order invariant. Our results are summarized in Table I.

What does duality say for the remaining separable potentials? The cases $\alpha=0$ and -4 turn out to be special in the sense that they are *self-dual* according to (12). The dual case to $\alpha=-6$, $C=\frac{1}{4}$ has $\alpha=-3$, $C=1$, but then θ_1 and θ_2 coincide and such systems with a double pole are generally expected to be nonintegrable.

Still another relation between the dual families can be found as follows: Let us fix the scaling dimension of p as $[p]=1$ and correspondingly $[x]=2/(\alpha+2)$. Now the Hamiltonian is of dimension 2, and as the last item in the table we have indicated the dimension of the second invariant.[1] It turns out that also these dimensions are related; their ratio is equal to the ratio of the $|\mu|$'s. Note that Painlevé analysis makes no predictions about the order of the second invariant.

What about quantum integrability? It can also be discussed with c-number functions if the Poisson bracket is replaced by the Moyal bracket.[9] In a previous search[5] we found that the $\alpha=1$ potentials are also quantum integrable without any modifications to the second invariant given in Table I. The same holds for the $\alpha=0$, -4, and -6 cases. However, the $\alpha=-\frac{2}{3}$ potentials are not quantum integrable.[9] We have now found, however, that it is possible to *deform* these potentials by an extra order $\hbar^2$ term so that

$$V=Cx^{4/3}+y^2x^{-2/3}-\tfrac{5}{72}\hbar^2x^{-2} \tag{13}$$

is quantum integrable for the three cases $C=\frac{3}{4}$, $\frac{9}{2}$, and 12. It is interesting to note that the additional potential is again unique to a given family. The $\hbar^2$ deformations in the second integral I are given in Table I. It should be noted that the $\hbar^2$ coefficients cannot be scaled away.

In this paper we have found interesting relationships between the Hénon-Heiles and Holt families of integrable potentials. The relationships between the coefficients C were found using simple Painlevé analysis. In addition to this, we also found relationships for the dimensions of the second invariants. Still another family-dependent property was the behavior under quantum integrability: A unique deformation of the Holt potential was needed. Further discussion on quantum integrability will be given elsewhere.[10]

After this work was finished we received the paper of Grammaticos, Dorizzi, and Ramani,[11] where the Holt family was also found using a more thorough Painlevé analysis. They also find a special nonintegrable Holt potential for $C_3=\frac{3}{2}$ in our notation $[\nu(\nu-1)=6]$. Its dual Hénon-Heiles potential has $C_{-2}=\frac{2}{3}$, whose nonintegrability was discussed in Ref. 2. Thus duality connects also these peculiar nonintegrable Painlevé cases.

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