Adiabatic drag and initial slip in random processes

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We describe a method for the solution of initial-value problems for random processes arising through adiabatic approximations from Markov processes of higher dimension. In applications to overdamped Brownian motion and to the single-mode laser we calculate correlation functions and discuss initial slips.

I. INTRODUCTION

The adiabatic elimination of fast variables from multidimensional stochastic processes is a twofold problem. One part is the construction of a closed set of equations of motion for the slow variables. The second, independent part is the identification of the correct initial values (or probability distributions) for the slow variables; this identification requires a nontrivial calculation when the original initial-value problem is stated, as is often the case in practice, for the complete set of slow and fast variables.

While the first of the two problems mentioned has received considerable attention in the literature,[1–4] discussions of the second one are rather sparse.[5,1(b)] There is an obvious reason for this disparity of interest. The most commonly studied random systems are Markovian in character. It is intuitive to assume that the slow subsystem of a Markovian system again has Markov properties, and precisely that assumption is made almost invariably. It is thus simply taken for granted that initial-value problems for the slow subsystem can be formulated without regard to the fast remainder.

We here propose to show that the Markov property of a random system does indeed carry over to a slow subsystem, but only to lowest order in τ_f/τ_s, the ratio of the time scales for fast and slow variables. If expectation values and correlation functions for the slow variables are calculated through power series in the time-scale ratio, however, non-Markovian effects do, in general, show up in the next-to-leading order.

The non-Markovian corrections in question can, even though surviving at times of the order τ_s after the initial preparation of the system, be identified as arising during the first few τ_f. They manifest themselves as, e.g., differences between initial mean values imposed on the slow variables at $t=0$ and the corresponding means some τ_f later, when all fast transients have died out while the slow ones are still practically unchanged. Obviously, inasmuch as such "initial slips" must be taken into account, the subset of slow variables cannot be said to constitute a Markovian system by itself.

Even though non-Markovian initial slips formally appear as "corrections" of (at least) first order in the small parameter τ_f/τ_s they need not at all be small effects. They may even be arbitrarily large, provided the initial values of the slow and the fast variables are sufficiently different from certain adiabatic equilibrium values. The distinctive property of such adiabatic equilibria is that they relax towards the absolute equilibrium through slow transients only. (Arbitrary initial states go over into adiabatic equilibria within a few τ_f.)

Stationary multitime correlation functions also involve initial-slip effects for every coincidence of two-time arguments. However, extreme nonequilibrium configurations of the slow and the fast variables tend to enter stationary correlation functions with only small statistical weight. It is therefore not unreasonable to expect the corresponding non-Markovian effects to be truly small corrections of nonqualitative importance.

As a quite interesting result we find that even the latter expectation can be upset. In a single-mode laser operated far above threshold, for instance, the field amplitude can be much slower in its motion than the degrees of freedom of the active atoms. It turns out that all correlation functions of the field amplitude are deterministic in character, to zeroth order in τ_f/τ_s. Fluctuations show up, together with and inseparable from initial slips, in first order. Even though small in relative magnitude, initial slips are part of a qualitatively interesting phenomenon here.

Both our notation and the method to be employed are similar to those of Refs. 1(g) and 2(b). We describe the complete system by a (quasi-) probability density $P(t)$ and a generator of infinitesimal time translation L such that the time evolution is given by

$$\dot{P}(t)=LP(t) \ . \tag{1.1}$$

We assume that there is a unique stationary distribution $\bar{P}$ which is reached from arbitrary initial distributions, $\bar{P}=\lim_{t\to 0} e^{Lt}P(0)$. We need not assume $\bar{P}$ to be known though.

With the reduced quasiprobability of the slow variables,

$$\rho(t)=\int_{\text{fast}} P(t) \ , \tag{1.2}$$

we can associate a generator $l(t)$ of infinitesimal time translations too. This "reduced" generator distinguishes the initial time; it carries an explicit time dependence and also depends on the initial condition imposes on the complete system. However, as a function of t the reduced generator contains only fast transients and thus settles at some limiting operator $l(\infty)\equiv l$ which is independent of the initial distribution $P(0)$. On the time scale characteristic of the slow variables we thus have the asymptotic equation of motion

$$\dot{\rho}(t)=l\rho(t), \quad t \gg \tau_f \ . \tag{1.3}$$

In sufficiently simple cases the asymptotic generator l can be calculated rigorously.[1(e),1(g),5(a)] Most often, however, the construction of l must, if possible at all, be based on expansions in powers of small parameters such as the time-scale ratio τ_f/τ_s.[1-4]

In Sec. II we shall show how the effective initial distribution $\rho_{\text{eff}}(0)$, defined by

$$\rho(t)=\int_{\text{fast}} e^{Lt}P(0)\equiv e^{lt}\rho_{\text{eff}}(0), \quad t\gg\tau_f \tag{1.4}$$

can be calculated. The difference

$$\rho_{\text{eff}}(0)-\int_{\text{fast}} P(0)=\rho_{\text{eff}}(0)-\rho(0) \tag{1.5}$$

measures the initial slip mentioned above.

With the eventual goal of calculating multitime correlation functions of the slow variables in mind, we describe, in Sec. III, the motion of the complete system on the time scale τ_s. Both the slow and the fast variables move slowly in that regime, the fast ones being locked into an adiabatic equilibrium with the slow ones at any instant, once the early-stage rapid transients have died out. Our method also allows us to determine the full stationary distribution through a power series in the time-scale ratio.

In Sec. IV we treat one-dimensional Brownian motion in an arbitrary potential, assuming a heavily overdamped case. The momentum of the Brownian particle is then fast compared to the displacement and can therefore be eliminated adiabatically. We calculate the stationary two-time correlation function of the displacement. Non-Markovian initial-slip effects turn up in second order in the transverse friction coefficient. It is quite interesting to see that for a harmonic potential the asymptotic generator l is of the Fokker-Planck form to all orders in the inverse friction coefficient; nonetheless, the displacement undergoes a Markov process to within corrections of second order only.

We finally treat the single-mode laser in Sec. V. For the near-critical regime we recover the well-known result that the light amplitude undergoes a Markov process to within corrections of the order of the small time-scale ratio. Far above threshold, on the other hand, we show non-Markovian effects to be of qualitative importance in the sense already mentioned above.

II. THE EFFECTIVE INITIAL DISTRIBUTION

In order to establish the effective initial distribution for the slow variables we must follow their early-stage evolution, i.e., evaluate their distribution (1.2) for a time at which all fast transients have just died out. By then extrapolating the remaining slow motion back to $t=0$ we obtain the effective initial distribution $\rho_{\text{eff}}(0)$.

While the quantity $\rho_{\text{eff}}(0)$ assigns the correct weight to all slow transients it need not be positive; it is even possible that the twofold limiting procedure described above yields a $\rho_{\text{eff}}(0)$ which is non-normalizable.[6] In such cases meaningful results for expectation values and correlation functions of the slow variables can still be secured by performing the corresponding ensemble averages before extrapolating the time back to $t=0$.

In order to implement the procedure just outlined we must, in general, split the generator L into two additive pieces,

$$L=L_0+L_1 \; . \tag{2.1}$$

The first piece, L_0, describes the dynamics of the fast variables, it may contain the slow variables parametrically but does not contribute to the time rate of change of any of the slow variables. The rapid motion described by L_0 should be probability conserving in character according to

$$\int_{\text{fast}} L_0 X=0 \tag{2.2}$$

for arbitrary X.

If we neglected the remainder in L, L_1, we would take the slow variables as entirely still parameters. Their distribution $\rho(t)$ would, according to Eqs. (1.2) and (2.2), forever coincide with $\rho(0)=\int_{\text{fast}}P(0)$. To zeroth order in L_1 there is thus no difference between $\rho(0)$ and $\rho_{\text{eff}}(0)$, i.e., between the true and the effective initial distributions.

We may calculate the difference between $\rho(0)$ and $\rho_{\text{eff}}(0)$ by first expanding the exponential operator in Eq. (1.2) in powers of L_1,

$$\rho(t)=\rho(0)+\int_0^t dt' \int_{\text{fast}} L_1 e^{L_0 t'}P(0)+\cdots \; . \tag{2.3}$$

We now assume that L_0 has a unique stationary eigenfunction R,

$$L_0 R=0, \quad \int_{\text{fast}} R=1 \tag{2.4}$$

and separate from the exponential $\exp L_0 t$ the time-independent part of its spectral decomposition as

$$e^{L_0 t}=R\int_{\text{fast}}+\left[e^{L_0 t}-R\int_{\text{fast}}\right] . \tag{2.5}$$

By using the identity (2.5) in the perturbation series (2.3) we obtain, to first order,

$$\begin{aligned}\rho(t)=\rho(0)&+t\int_{\text{fast}} L_1 R\rho(0)\\ &+\int_0^t dt' \int_{\text{fast}} L_1\left[e^{L_0 t'}-R\int_{\text{fast}}\right]P(0)\\ &+\cdots \; .\end{aligned} \tag{2.6}$$

In order to purge $\rho(t)$ of fast transients we simply push the upper limit in the time integral to infinity. On the time scale characteristic of the slow motion we thus obtain

$$\begin{aligned}\rho(t)=\rho(0)&+\int_0^\infty dt' \int_{\text{fast}} L_1\left[e^{L_0 t'}-R\int_{\text{fast}}\right]P(0)\\ &+t\int_{\text{fast}} L_1 R\rho(0)+\cdots \; .\end{aligned} \tag{2.7}$$

The last term in (2.7) involves the first-order part of the asymptotic evolution operator $\exp(lt)$, since $\int_{\text{fast}} L_1 R$ is the first-order contribution to l.[1(g)] Therefore, the effective initial distribution finally results if we extrapolate the right-hand side in Eq. (2.7) back to $t=0$,

$$\begin{aligned}\rho_{\text{eff}}(0)=\rho(0)&+\int_0^\infty dt \int_{\text{fast}} L_1\left[e^{L_0 t}-R\int_{\text{fast}}\right]P(0)\\ &+\cdots \; .\end{aligned} \tag{2.8}$$

There is no difficulty in extending the calculation to higher orders.

The most important qualitative feature of the effective initial distribution is already visible in the first-order term in Eq. (2.8). The explicit evaluation of the $\rho_{\text{eff}}(0)-\rho(0)$ obviously requires knowledge of the initial distribution $P(0)$ of both the slow and the fast variables. A nonzero difference $\rho_{\text{eff}}(0)-\rho(0)$ is thus the manifestation of a non-Markovian effect in the dynamics of the slow variables.

III. THE SLOW MOTION OF THE FAST VARIABLES

After a preparation of the complete system corresponding to an initial distribution $P(0)$ all variables will undergo rapid transients. Once these transients have died out the resulting distribution $P(t)$ may be characterized as describing an adiabatic equilibrium of the fast variables, i.e., an equilibrium contingent, at any instant, on the current values of the slow variables. During the subsequent relaxation towards the absolute equilibrium the fast variables are dragged along by the slow ones and thus move no faster than these.

We need to know the large-time behavior of $P(t)$ just described if we want to calculate multiple-time correlation functions of the slow variables. Even if such a correlation function is sought with respect to a time scale on which only slow transients can be noticed, we must determine the weight of these slow transients through effective initial conditions for every coincidence of two-time arguments of the correlation function in question; the effective initial condition at such a coincidence involves, as we have seen in Sec. II, the complete distribution P at the corresponding initial time.

In calculating the large-time behavior of $P(t)$ we have to be somewhat more careful than in Sec. II since we now want to account for the whole course of the slow transients rather than for their initiation only. It is convenient to describe the adiabatic regime with the help of an identity due to Zwanzig,[7]

$$P(t)=e^{(1-\mathscr{P})Lt}(1-\mathscr{P})P(0)+R\rho(t)+\int_0^t dt' e^{(1-\mathscr{P})Lt'}(1-\mathscr{P})LR\rho(t-t') , \quad (3.1)$$

in which $\mathscr{P}$ denotes the projector

$$\mathscr{P}=R\int_{\text{fast}} . \quad (3.2)$$

The identity (3.1) is useful for our purpose since it represents the complete distribution $P(t)$ as dragged by the reduced distribution $\rho(t)$ of the slow variables.

Now we again exploit the smallness of the time-scale ratio by decomposing L as in Eq. (2.1) and expanding all exponential operators in powers of L_1. The first term in Eq. (3.1) is then immediately seen to decay to zero through fast transients while the remaining terms give rise to a perturbative description of the adiabatic drag as

$$P(t)\to R\rho(t)+\int_0^t dt' e^{L_0 t'}(1-\mathscr{P})L_1 R\rho(t-t')+\cdots . \quad (3.3)$$

In zeroth order the adiabatic regime is represented, of course, by the product of the conditional equilibrium distribution R of the fast variables with the distribution

$$\rho(t)=e^{lt}\rho_{\text{eff}}(0) \quad (3.4)$$

of the slow variables. The first-order correction to this zero-order distribution is also displayed in Eq. (3.3). The higher-order terms are easily found as well.

If stationary multitime correlation functions of slow variables are to be calculated we must know the equilibrium distribution $\bar{P}$. It is only in fortunate special cases[8] that an exact result is known for $\bar{P}$. Otherwise the expansion (3.3) can be used to construct $\bar{P}$ perturbatively as

$$\bar{P}=R\bar{\rho}+\int_0^\infty dt\, e^{L_0 t}(1-\mathscr{P})L_1 R\bar{\rho}+\cdots . \quad (3.5)$$

IV. BROWNIAN MOTION

We here consider one-dimensional Brownian motion of a particle in an external potential $\phi(q)$. The corresponding Fokker-Planck equation[9,10] for the probability density $P(q,p,t)$ of the displacement q and the momentum p, $\dot{P}=LP$, is specified by the generator

$$L=-\frac{\partial}{\partial q}p+\frac{\partial}{\partial p}[\gamma p+\phi'(q)]+\frac{\partial^2}{\partial p^2}\gamma d , \quad (4.1)$$

where γ and γd denote the damping and the diffusion constants, respectively.

In the limit of large damping,

$$\gamma \gg 1 , \quad (4.2)$$

the momentum becomes a fast variable and can be eliminated adiabatically. The slow motion of the reduced probability density $\rho(q,t)$, i.e., the motion on a time scale

$$t\gg\gamma^{-1} , \quad (4.3)$$

can be described by an equation of motion of the form

$$\dot{\rho}(q,t)=l\rho(q,t) , \quad (4.4)$$

where the time-independent generator l is a differential operator with respect to q. The perturbative construction of l can be based on the choice[1(f),1(g)]

$$L_1=-\frac{\partial}{\partial q}p+\frac{\partial}{\partial p}\phi'(q) \quad (4.5)$$

and then yields l as a series in powers of $1/\gamma$,

$$l=\frac{\partial}{\partial q}\left[\frac{1}{\gamma}+\frac{1}{\gamma^3}\phi''+O\left[\frac{1}{\gamma^5}\right]\right]\left[\frac{\partial}{\partial q}d+\phi'\right] . \quad (4.6)$$

It is appropriate to point out that the generator l turns out to be a Fokker-Planck operator up to the order $1/\gamma^3$. It is only in fifth order in the inverse friction expansion that third-order derivatives with respect to q may appear. For a harmonic potential $\phi\sim q^2$, the expansion (4.6) can also be derived from the exact result,

$$l=[\gamma/2-(\gamma^2/4-1)^{1/2}]\frac{\partial}{\partial q}\left[\frac{\partial}{\partial q}d+q\right] , \quad (4.7)$$

which is of the Fokker-Planck form in all orders in $1/\gamma$.[1(g),1(e)]

One might be tempted to surmise that the displacement undergoes a Markov process inasmuch as the generator l is a Fokker-Planck differential operator. It is easy to see, however, that an effective Markov process results in lowest order in $1/\gamma$ only.

In order to illustrate this statement we consider the stationary two-time correlation function of the displacement,

$$\langle q(t)q(0)\rangle=\int dq\int dp\, qe^{Lt}q\bar{P}(q,p) . \quad (4.8)$$

In the limit (4.3) this function is stripped of fast transients and can be determined from

$$\langle q(t)q(0)\rangle=\int dq\, qe^{lt}\rho_{\text{eff}}(q,0) . \quad (4.9)$$

The effective initial distribution $\rho_{\mathrm{eff}}(q,0)$ to be used here is given by the series (2.8) with the initial distribution

$$P(q,p,0)=q\bar{P}(q,p) \sim q\exp\left[-\frac{1}{d}\left[\phi(q)+\frac{p^2}{2}\right]\right]. \tag{4.10}$$

Actually, the first-order term in the series (2.8) vanishes for the present case because the stationary distribution $\bar{P}(q,p)$ is even in the momentum p. Up to corrections of fourth order in $1/\gamma$ we easily obtain

$$\rho_{\mathrm{eff}}(q,0)=\left[q+\frac{1}{\gamma^2}\phi'(q)\right]\bar{\rho}(q) \sim \left[q+\frac{1}{\gamma^2}\phi'(q)\right]e^{-\phi(q)/d}. \tag{4.11}$$

The correlation function (2.8) thus reads

$$\langle q(t)q(0)\rangle=\int dq\, q e^{lt}\left[q+\frac{1}{\gamma^2}\phi'(q)\right]\bar{\rho}. \tag{4.12}$$

Clearly, the term proportional to γ^{-2} constitutes a deviation from the regression theorem valid for Markov processes.

For a purely harmonic potential the result (4.12) can be further evaluated. Up to corrections of fourth order for both the decay constant and the effective initial distribution we obtain

$$\langle q(t)q(0)\rangle=d\left[1+\frac{1}{\gamma^2}\right]\exp\left[-t\left[\frac{1}{\gamma}+\frac{1}{\gamma^3}\right]\right]. \tag{4.13}$$

Again, we conclude that non-Markovian effects show up in second order in $1/\gamma$.

Another point of general interest can be discussed for the harmonic case. It is not only the asymptotic generator l and thus the decay constants of all slow transients which can be calculated rigorously for that benign potential but also the effective initial condition $\rho_{\mathrm{eff}}(q,0)$. The exact time-dependent distribution of the displacement originating from sharp initial values for both q and p, $P(q,t\mid q_0p_0)$, is presented in Chandrasekhar's classic paper.[6] If we erase from this result the fast transients (which decay like $e^{-\gamma t}$) and then extrapolate the remaining slow transients (which decay like $e^{-t/\gamma}$) back to $t=0$ we obtain a non-normalizable distribution $\rho_{\mathrm{eff}}(q,0)$. This is interesting, but not really surprising. The effective initial distribution is a formal device for generating the weights with which the slow transients appear at large times; since it assigns zero weight to all fast transients it does not represent the state of the particle at early times and is therefore not bound to obey the requirement of normalizability.

The disaster just described does not happen if the initial distribution for q and p is a sufficiently smooth function. Especially if Chandrasekhar's distribution is convoluted with the initial distribution (4.10), we may, in the result, drop all fast transients and then let t go to zero without incurring any existence problems. More difficult cases may require that t be kept finite until after expectation values are evaluated.

V. THE SINGLE-MODE LASER

Among the parameters characterizing a single-mode laser the following six are the most important ones.[2(a)] First, there are damping constants $\gamma_\perp$ and $\gamma_\parallel$ for the atomic polarization and inversion pertinent to the active level pair in each atom. A damping constant κ measures the rate at which diffraction and reflection losses tend to attenuate the mode amplitude. A coupling constant g (proportional to the dipole matrix element for the active level pair, normalized to have the dimension of a frequency) describes the effective strength of the interaction of the active atoms and the field mode. Finally, the number of active atoms N and the unsaturated inversion $2\sigma_0$ per atom characterize the pump mechanism providing the energy on which the laser feeds.

We shall now consider cases for which the atomic damping constants are much larger than the other parameters of equal dimension,

$$\gamma_\perp,\gamma_\parallel \gg g,\kappa. \tag{5.1}$$

The atomic polarization and inversion then tend to follow the field mode amplitude adiabatically.

As a first step in exploiting the adiabatic limit (5.1) we may, starting from a master equation describing the dynamics of both the active atoms and the field,[2(a),11] eliminate the atomic degrees of freedom. In this way we have recently derived an equation of motion for the reduced density operator of the field mode.[2(b)] If we represent this density operator by a mixture of coherent states[12] as

$$\hat{\rho}(t)=\int_0^\infty dr\int_0^{2\pi}d\phi\,\rho(r,\phi,t)\,|re^{i\phi}\rangle\langle re^{i\phi}|, \tag{5.2}$$

the weight function $\rho(r,\phi,t)$ has a generator l of infinitesimal time translations[2(b)] which is of the Fokker-Planck form and reads

$$l=\frac{\partial}{\partial r}D_r(r)+\frac{\partial^2}{\partial r^2}D_{rr}(r)+\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}(r). \tag{5.3}$$

The construction of l and, especially, of the drift coefficient D_r and the diffusion coefficients D_{rr} and $D_{\phi\phi}$ in Ref. 2(b) is based on a certain splitting $L=L_0+L_1$ of the generator L for the density operator of the atoms and the field. The main piece L_0 includes all parts of L which directly contribute to the time rate of change of atomic observables: it depends on the complex field amplitude $re^{i\phi}$ parametrically, and thus describes a relaxation of the atoms on the time scale given by the atomic damping constants $\gamma_\perp$ and $\gamma_\parallel$, towards a conditional equilibrium contingent on $re^{i\phi}$; the field amplitude can be said to be still on the time scale of the atomic relaxation. The result (5.3) is then obtained in second order in the remainder L_1 and holds for times larger than the atomic decay times. The diffusion coefficients turn out to arise in second order and read

$$D_{rr}(r)=\frac{Ng^2/4\gamma_\perp}{(1+r^2/n_s)^3}\times\left\{1+2\sigma_0+2\left[1-2\sigma_0^2\left[1+\frac{\gamma_\perp}{\gamma_\parallel}\right]\right]\frac{r^2}{n_s}+(1-2\sigma_0)\frac{r^4}{n_s^2}\right\} \tag{5.4}$$

and

$$D_{\phi\phi}(r)=\frac{Ng^2}{4\gamma_\perp}\frac{1+2\sigma_0+r^2/n_s}{r^2(1+r^2/n_s)} ,$$

where the so-called saturation photon number,

$$n_s=\gamma_\perp\gamma_\parallel/4g^2 , \tag{5.5}$$

appears as the natural scale on which the diffusion coefficients depend on r^2. The radial drift coefficient has a first-order contribution

$$D_r^{(1)}=\left[\kappa-\frac{\sigma_0 Ng^2/\gamma_\perp}{1+r^2/n_s}\right]r \tag{5.6a}$$

and a second-order contribution which can be represented by

$$D_r^{(2)}=-rD_{\phi\phi}+\left\{\frac{2Ng^2\sigma_0\kappa/\gamma_\perp^2}{(1+r^2/n_s)^3}\left[\left[1+\frac{2\gamma_\perp}{\gamma_\parallel}\right]\frac{r^2}{n_s}-1\right]+\frac{4Ng^4\sigma_0^2/\gamma_\perp^3}{(1+r^2/n_s)^4}\left[N\left[1-\frac{r^2}{n_s}\right]-2\frac{r^2}{n_s}\right]\right.$$
$$\left.+\frac{Ng^4/\gamma_\perp^2\gamma_\parallel}{(1+r^2/n_s)^4}\left[-8\sigma_0^2\frac{Nr^2}{n_s}+4\sigma_0^2\left[1-\frac{r^2}{n_s}\right]+8\sigma_0\left[1+\frac{r^2}{n_s}\right]+\left[3+\frac{r^2}{n_s}\right]\left[1+\frac{r^2}{n_s}\right]^2\right]\right\}r . \tag{5.6b}$$

It is interesting to note that the results (5.4) and (5.6a) have been obtained independently by Lugiato *et al.*[13] As an important benefit of the adiabatic limit we immediately have the static quasiprobability for the field amplitude as the stationary eigenfunction of l,

$$\bar{p}\sim\exp\left[-\int dr\frac{D_r(r)+D'_{rr}(r)}{D_{rr}(r)}\right] . \tag{5.7}$$

This distribution function describes the static photon statistics of the laser.

We may also inquire into the special properties of the laser output as represented by the stationary correlation function for the light amplitude

$$G_1(t)=\int_0^\infty dr\int_0^{2\pi}d\phi\,\mathrm{tr}_A re^{-i\phi}e^{Lt}re^{i\phi}\bar{P} ,$$
$$G_2(t)=\int_0^\infty dr\int_0^{2\pi}d\phi\,\mathrm{tr}_A r^2e^{Lt}r^2\bar{P} . \tag{5.8}$$

In these formal expressions tr_A means the trace operator for the atoms, L the generator of infinitesimal time translations in the complete atom-field master equation, and $\bar{P}$ the corresponding stationary solution which is a density operator for the atoms and a quasiprobability for the field amplitude in the sense of Eq. (5.2).

In order to evaluate the correlation functions $G_i(t)$ we can introduce the reduced generator l as

$$G_2(t)=\int dr\int d\phi\, r^2e^{lt}\rho_{\text{eff}}(0)$$

[and similarly for $G_1(t)$] and determine the effective initial distribution $\rho_{\text{eff}}(0)$ from Eq. (2.8) with $P(0)=r^2\bar{P}$. The stationary state $\bar{P}$ can then be obtained through Eq. (3.10). We shall skip the lengthy but elementary algebra which yields the expressions

$$G_1(t)=\int dr\int d\phi\, re^{-i\phi}e^{lt}\left[1+\frac{\partial}{\partial r}rs(r)\right]re^{i\phi}\bar{p} ,$$
$$G_2(t)=\int dr\int d\phi\, r^2e^{lt}\left[1+\frac{\partial}{\partial r}rs(r)\right]r^2\bar{p} , \tag{5.9}$$

in which the function

$$s(r)=\frac{Ng^2\sigma_0}{\gamma_\perp^2}\frac{(1+r^2)/n_s+2\gamma_\perp/\gamma_\parallel}{(1+r^2/n_s)^2}$$

represents the non-Markovian initial slip to lowest order in L_1.

Before we proceed to discuss the static distribution (5.7) and the correlation functions (5.9) it is appropriate to comment on the nature of our perturbative approximations. The expansions in powers of L_1 obviously involve the ratios $\kappa/\gamma_{\perp,\parallel}$ as expansion parameters. It is more difficult to find out in which sense the ratios $g/\gamma_{\perp,\parallel}$ are expansion parameters. One difficulty lies in the fact that part of the atom-field interaction is included in L_0. This inclusion may be understood as a partial resummation of a "bare" expansion in powers of $g/\gamma_{\perp,\parallel}$. It is undoubtedly necessary to use such a partially summed expansion since the adiabatic equilibrium of the fast atomic variables in a laser supporting a strong field is very different from the absolute equilibrium in a vanishing field.

The qualitative argument just given cannot, of course, explain for which values of κ, g, $\gamma_\perp$, and $\gamma_\parallel$ the field damping part of L_1 should be treated as of the same order as the atom-field interaction part. Unfortunately, such an explanation can only be given in the self-consistent way to be described now, together with the explicit evaluation of the general results (5.7) and (5.9).

For a laser operated near threshold, i.e., for $\sigma_0\approx\sigma_{\text{thr}}=\kappa\gamma_\perp/2Ng^2$, both the stationary moments $\langle r^{2n}\rangle$ and the corresponding cumulants will turn out to be of nth order in $n_{\text{thr}}\approx\sqrt{N\gamma_\parallel/k}$, the photon number at threshold. If this number is smaller than the saturation photon number n_s it is natural to scale the amplitude r as $\tilde{r}^2=r^2/n_{\text{thr}}$ and to use the small parameter n_{thr}/n_s to simplify all of the general formulas given above. Up to corrections of relative order n_{thr}/n_s and $\kappa/\gamma_{\perp,\parallel}$ the generator (5.3) then takes the form

$$l_{\text{vdP}}=\frac{2Ng^2}{\gamma_\perp}\left[\frac{\partial}{\partial\tilde{r}}\tilde{r}\left[\sigma_{\text{thr}}-\sigma_0+\frac{\sigma_0}{\sigma_{\text{thr}}}\tilde{r}^2\right]+\frac{1+2\sigma_0}{4}\left[\frac{\partial^2}{\partial\tilde{r}^2}+\frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial\phi^2}\right]\right] , \tag{5.10}$$

well known to describe a noisy van der Pol oscillator.[2(a),14] Similarly, the non-Markovian slip terms in (5.9) turn out to correct the leading terms in order $\kappa/\gamma_\perp$ and must therefore be neglected in the van der Pol limit. All of this is self-consistent since the static moments $\langle\tilde{r}^{2n}\rangle$ as well as the corresponding cumulants, derived from the static eigenfunction of l_{vdP}, to indeed prove to be of order unity.[2(a),14] We, therefore, do not have anything to add to Risken's classic discussion of a near-critical single-mode laser.[14]

We now think of the pump strength as increased such that the mean photon number becomes comparable to n_s or even enters the deterministic regime determined by $D_r^{(1)}$,

$$n_{\rm det}=\frac{N\gamma_{\|}}{2\kappa}(\sigma_0-\sigma_{\rm thr}) , \tag{5.11}$$

which may be assumed larger yet than n_s. The scaling $r^2 \to \tilde{\tilde{r}} n_s$ is thus intuitive and will indeed turn out to be self-consistent below. That scaling immediately reveals the drift correction $D_r^{(2)}$ as well as the diffusion coefficients to be smaller than $D_r^{(1)}$ by either $\kappa/\gamma_\perp$ or $g^2/\gamma_{\perp,\|}^2$. Therefore, and in contrast to the near-critical behavior, the laser amplitude now moves deterministically to leading order. The physical reasons for the smallness of all fluctuations in a laser well above threshold are, of course, amplitude stabilization and a slowing down of the phase diffusion.[2(a)]

In order to describe the small fluctuations in question we first replace the radial amplitude as an independent variable by its deviation ξ from the most probable value $\hat{r}$ of r in the static distribution (5.7),

$$r=\hat{r}+\sqrt{\epsilon}\sqrt{n_s}\xi . \tag{5.12}$$

This transformation contains, as a notational device, a parameter ϵ which serves to count orders with respect to the effective expansion parameters $\kappa/\gamma_{\perp,\|}$ and $g^2/\gamma_{\perp,\|}^2$. In all final results ϵ may be set equal to one.

By expanding in powers of $\sqrt{\epsilon}$ we obtain, from (5.7), the static moments

$$n_s^{1/2}\langle r\rangle=\left[\frac{\sigma_0-\sigma_{\rm thr}}{\sigma_{\rm thr}}\right]^{1/2}-\epsilon\left[\frac{\hat{D}_r^{(2)}}{\hat{D}_r^{\prime(1)}}+\frac{\hat{D}_{rr}\hat{D}_r^{\prime\prime(2)}}{2[\hat{D}_r^{\prime(1)}]^2}\right] , \tag{5.13}$$

$$\langle r^{2n}\rangle=\langle r\rangle^{2n}+\epsilon n(2n-1)\langle r\rangle^{2n-2}n_s\hat{D}_{rr}/\hat{D}_r^{\prime(1)} , \tag{5.14}$$

where the carets represent evaluation at the most probable radial amplitude,

$$\hat{r}=(n_s/\sigma_{\rm thr})^{1/2}(\sigma_0-\sigma_{\rm thr})^{1/2}+O(\epsilon) . \tag{5.15}$$

The results (5.13), (5.14), and (5.15) show that the photon number does indeed scale as $n_{\rm det}$ for $\sigma_0=O(1)$ and as n_s for $(\sigma_0-\sigma_{\rm thr})/\sigma_{\rm thr}$ not too much larger than unity. Moreover, we may conclude that the moments $\langle r^{2n}\rangle$ define a Gaussian distribution of width $n_s\hat{D}_{rr}/\hat{D}_r^{\prime(1)}$, located at $r=\langle r\rangle$. This is in accord with Haken's early findings.[2(a)]

Similarly, we introduce the transformation (5.12) into the generator (5.3) in order to evaluate the correlation functions $G_i(t)$. In zeroth order in ϵ the generator l describes a linear Gaussian process for which all eigenvalues and eigenfunctions are known. To within corrections of order $\epsilon^{3/2}$ the phase diffusion decouples from the motion of the "radial" coordinate ξ. To obtain the $G_i(t)$ to order ϵ we must calculate the eigenfunctions and eigenvalues of the radial part of l through ordinary perturbation theory. Actually, only the first excited state turns out to contribute. The corresponding eigenvalue is easily found as

$$\Gamma=\hat{D}_r^{\prime(1)}+\epsilon\left[\hat{D}_r^{\prime(2)}+\frac{1}{2}\frac{\hat{D}_{rr}\hat{D}_r^{\prime\prime\prime(1)}}{\hat{D}_r^{\prime(1)}}+2\frac{\hat{D}_{rr}^{\prime}\hat{D}_r^{\prime\prime(1)}}{\hat{D}_r^{\prime(1)}}-\tfrac{3}{2}\hat{D}_{rr}\left[\frac{\hat{D}_r^{\prime\prime(1)}}{\hat{D}_r^{\prime(1)}}\right]^2\right] . \tag{5.16}$$

The final result for the first-order correlation function also involves the eigenvalue pertaining to the first excited state for the phase diffusion,

$$\Gamma_\phi=\hat{D}_{\phi\phi} , \tag{5.17}$$

and reads (now $\epsilon=1$)

$$G_1(t)=\left[\langle r^2\rangle-n_s\frac{\hat{D}_{rr}}{\hat{D}_r^{\prime(1)}}\right]e^{-\Gamma_\phi t}+\left[n_s\frac{\hat{D}_{rr}}{\hat{D}_r^{\prime(1)}}-\langle r^2\rangle\hat{s}\right]e^{-(\Gamma+\Gamma_\phi)t} . \tag{5.18}$$

The second-order correlation function takes the form

$$G_2(t)=\langle r^2\rangle^2+2\langle r^2\rangle\left[2\frac{\hat{D}_{rr}}{\hat{D}_r^{\prime(1)}}-\langle r^2\rangle\hat{s}\right]e^{-\Gamma t} \tag{5.19}$$

or, in a more common normalization,

$$\frac{G_2(t)}{\langle r^2\rangle^2}=1+\left[4\frac{\hat{D}_{rr}}{\langle r^2\rangle\hat{D}_r^{\prime(1)}}-2\hat{s}\right]e^{-\Gamma t} \equiv 1+\alpha e^{-\Gamma t} . \tag{5.20}$$

The non-Markovian initial slips showing up in these correlation functions may be small effects—they are no smaller, though, than the other corrections to the leading deterministic terms. Interestingly, the initial slip shows up in the normalized second-order correlation function, as the deviation from unity, for large times while the diffusion correction determines that deviation at $t=0$.

It is also noteworthy that the drift correction $D_r^{(2)}$ is present, together with the diffusion constant, in the static moments (5.13) and (5.14) as well as in the eigenvalue Γ.[13]

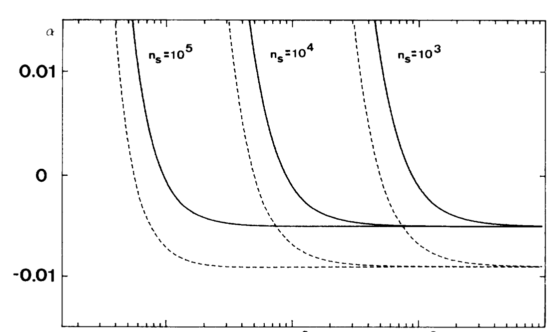

FIG. 1. Correlation coefficient α [see (5.20)] as a function of $\sigma_0/\sigma_{\rm thr}$; for details see text.

The sign of the coefficient α defined in (5.20) determines whether the emitted photons bunch or antibunch in the early stage of the adiabatic regime, $\gamma^{-1} < t < \Gamma^{-1}$. Figure 1 shows that coefficient as a function of $\sigma_0/\sigma_{\text{thr}}$ for $N=10^7$, $\gamma_\perp=\gamma_\parallel$, $\kappa/\gamma_\perp=0.005$ (solid lines) and $\kappa/\gamma_\perp=0.009$ (broken lines), and various values of n_s. Note that there is always antibunching for sufficiently strong pumping. The magnitude of the antibunching, $|\alpha|$, saturates at $\kappa/\gamma_\perp$, that limit being dominanted by the initial slip contribution. Consequently, antibunching will hardly be observable in the extreme adiabatic limit $\kappa/\gamma_\perp \rightarrow 0$. For the largest values of $\kappa/\gamma_\perp$ still compatible with our adiabatic approximation, however, the effect may be quite detectable. At any rate, we conclude that the effect can be much more pronounced at times $\gamma_{\perp,\parallel}^{-1} \ll t < \Gamma^{-1}$ than at $t=0$ where $\langle r^4 \rangle / \langle r^2 \rangle^2 - 1 = O(1/n_s)$.

We should add that all our final results (5.13) through (5.20) remain unchanged when, instead of the representation (5.20), the Wigner function[13] or Glauber's Q function or any other quasiprobability is used to represent the density operator of the field.[2(b),15] This innocent remark is, in fact, an important one since $\hat{D}_{rr}$ may be negative[13,2(b)] in which case the representation (5.2) cannot be used.

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