Threshold for electron heating by two electromagnetic waves

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The interaction of an electron with two electromagnetic waves propagating in an isotropic plasma is studied with the use of a super-Hamiltonian formalism. Large-scale stochasticity thresholds are analytically estimated. The results may be relevant to laser-plasma interaction.

I. INTRODUCTION

The transition from regular to stochastic behavior in deterministic systems is a relevant problem in plasma physics. A number of important effects such as the appearance of magnetic islands in toroidal configurations,[1,2] ion heating by lower-hybrid waves,[3,4] wave instability saturation via harmonic generation,[5] or the formation of suprathermal electron tails in laser-plasma interaction[6] can be interpreted in terms of that transition.

In this work we study the electron motion in the field of two electromagnetic waves propagating in the same direction in a relativistic plasma. We estimate analytically the large-scale stochasticity threshold for the electron heating. For wave amplitudes above the threshold the electron is expected to explore a wide range of velocity space. The electron motion is described by a nonintegrable Hamiltonian. We will use an autonomous super-Hamiltonian form[7] instead of the usual time-dependent relativistic Hamiltonian. The role of the collisions is assumed to be negligible because we are dealing with high-frequency fields.

The format of the paper is the following. In Sec. II we derive the equations of motion for an electron in the presence of two transverse electromagnetic waves using a super-Hamiltonian formalism. In Sec. III we discuss the particular situations for which the super-Hamiltonian becomes integrable. In Sec. IV we estimate analytically the large-scale stochasticity threshold valid for the nonintegrable situations and for motion in a medium with a refractive index greater than unity, $N>1$. As particular cases we derive the threshold for nonrelativistic motion[6] and for electrostatic waves.[8] In Sec. V we estimate the threshold for a relativistic plasma and for vacuum, $N\leq 1$. A simplified version of this threshold is obtained in Sec. VI and compared with the results of a direct numerical integration of the equations of motion. Our conclusions are given in Sec. VII.

II. SUPER-HAMILTONIAN OF THE MOTION

Let us consider two transverse electromagnetic waves propagating in an isotropic medium. They can be described by the vector potential:

$$\vec{A}(\vec{r},t)=\sum_{j=1}^{2}\vec{A}(j)\cos(\vec{k}_j\cdot\vec{r}-\omega_j t)\ . \tag{2.1}$$

The propagation of these two waves is taken along Ox in order to reduce the problem to the simple case of a two-dimensional Hamiltonian. In more than two dimensions a new phenomenon takes place, the Ar'nold diffusion,[9] which will not be discussed here.

The electron motion in the presence of the potential (2.1) can be described by the following Hamiltonian:

$$H(\vec{r},\vec{p},t)=\{m^2c^4+[\vec{p}+e\vec{A}(\vec{r},t)]^2c^2\}^{1/2} \tag{2.2}$$

which is a constant of motion. Let us now define the three four-vectors position $\hat{r}$, momentum $\hat{p}$, and potential $\hat{A}$ by

$$\hat{r}=(ct,\vec{r})\ ,\quad \hat{p}=(H(\hat{r})/c\ ,\vec{p})\ ,\quad \hat{A}=(0,\vec{A}(\hat{r}))\ . \tag{2.3}$$

With the aid of these four-vectors we can define a super-Hamiltonian,[7] which is formally analogous to the nonrelativistic Hamiltonian:

$$\hat{H}(\hat{r},\hat{p})=\frac{1}{2m}[g^{\alpha\beta}(p_\alpha+eA_\alpha)(p_\beta+eA_\beta)]\ . \tag{2.4}$$

In this expression $g^{\alpha\beta}$ is the Minkowski metric tensor, p_α and A_α are the covariant components of $\hat{p}$ and $\hat{A}$. Taking the square of (2.2) and rearranging terms it is easy to prove that (2.4) is a constant of motion, equal to $-mc^2/2$. The equations of motion can then be written in canonical form as

$$\frac{dr^\alpha}{d\tau}=\frac{\partial\hat{H}}{\partial p_\alpha}\ ,\quad \frac{dp_\beta}{d\tau}=-\frac{\partial\hat{H}}{\partial r^\beta}\ , \tag{2.5}$$

the time variable τ is the electron proper time:

$$d\tau=\left[1-\frac{1}{c^2}\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right]^{1/2}dt\equiv\frac{dt}{\Gamma}\ . \tag{2.6}$$

Using (2.1) and assuming transverse polarization in the plane Oyz, which means that $\hat{A}\equiv(0,\vec{A})=(0,0,\vec{A}\cdot\vec{e}_y,\vec{A}\cdot\vec{e}_z)$, we can write (2.4) in a more explicit form:

$$\hat{H}(r^\alpha,p_\beta)=\frac{1}{2m}\left[p_1^2-p_0^2+p_\perp^2+e^2\left[\sum_{j=1}^{2}\vec{A}(j)\cos\nu_j\right]^2+2e\sum_{j=1}^{2}\vec{p}_\perp\cdot\vec{A}(j)\cos\nu_j\right]\ , \tag{2.7}$$

where $\vec{p}_\perp=p_2\vec{e}_y+p_3\vec{e}_z$ is the transverse momentum and

$v_j = k_j r^1 - (\omega_j / c) r^0$. We see from this expression that the super-Hamiltonian is not a function of r^2 and r^3, which means that the transverse equations of motion reduce to $\vec{p}_\perp = \text{const}$. We can then use a reduced super-Hamiltonian for the longitudinal motion:

$$\hat{H}(r^0, r^1, p_0, p_1) = \hat{H}(r^\alpha, p_\beta) - \frac{p_\perp^2}{2m} . \tag{2.8}$$

In order to get more specific results we will restrict our analysis to the particular but still important case of two waves having the same frequency, $\omega_1 = \omega_2 = \omega$, and propagating in opposite directions, $k_1 = -k_2 = k$. With the use of new space and time variables,

$$x = kr^1 - \pi/2 , \quad y = kr^0 = (\omega/c) N r^0 , \tag{2.9}$$

where N is the refractive index, we can replace (2.8) by a dimensionless super-Hamiltonian:

$$h(x,y,v,u) = \tfrac{1}{2}(v^2 - u^2) - \alpha_1 \cos\left[2\left[x - \frac{y}{N}\right]\right] - \alpha_2 \cos\left[2\left[x + \frac{y}{N}\right]\right] + \beta_1 \sin\left[x - \frac{y}{N}\right] + \beta_2 \sin\left[x + \frac{y}{N}\right] - \gamma\left[\cos(2x) - \cos\left[\frac{2y}{N}\right]\right] . \tag{2.10}$$

The new parameters introduced in this expression are

$$\alpha_j = \left[\frac{eA(j)}{2mc}\right]^2 , \quad \beta_j = -\frac{e\vec{p}_\perp \cdot \vec{A}(j)}{m^2 c^2} ,$$
$$\gamma = \frac{e^2}{2m^2c^2} \vec{A}(1) \cdot \vec{A}(2) , \tag{2.11}$$

and the variables u and v are given by

$$v = \frac{p^1}{mc} , \quad u = \frac{p^0}{mc} . \tag{2.12}$$

The canonical equations of motion can now be written as

$$\frac{dx}{dz} = v , \quad \frac{dv}{dz} = -\frac{\partial h}{\partial x} ,$$
$$\frac{dy}{dz} = u , \quad \frac{du}{dz} = -\frac{\partial h}{\partial y} , \tag{2.13}$$

where the time variable is now $z = kc\tau$. In the usual canonical equations for the Hamiltonian dynamics we would have $-u$ instead of u in the third equation (2.13). The difference is due to the covariant formulation used here, which implies that $p^0 = -p_0 = mcu$.

III. INTEGRABILITY

If the electron is submitted to the action of a single wave, $\vec{A}(1) \neq \vec{0}$ and $\vec{A}(2) = \vec{0}$, the equations of motion (2.13) become

$$\frac{dv}{dz} = -N \frac{du}{dz} ,$$
$$\frac{du}{dz} = \frac{2}{N} \alpha_1 \sin\left[2\left[x - \frac{y}{N}\right]\right] + \frac{\beta_1}{N} \cos\left[x - \frac{y}{N}\right] . \tag{3.1}$$

In this case we have two constants of motion. One is the super-Hamiltonian itself, $h_0 = h(\alpha_2 = 0, \beta_2 = 0, \gamma = 0)$. The other is the quantity I_0 derived from (3.1):

$$I_0 = v + Nu . \tag{3.2}$$

These constants of motion are in involution and h_0 is integrable. The orbit for which the phase in the oscillatory terms of h_0 is stationary is called a resonance. We can see from (2.10) that the position of the resonance of h_0 in phase space is $v = v_1$ and $u = Nv_1$, where v_1 obeys the equation

$$h_0 = \frac{v_1^2}{2}(1 - N^2) . \tag{3.3}$$

Using (2.8) and (2.10) and recalling that $\hat{H} = -mc^2/2$, we get the position of the resonance:

$$v_1 = \left[\frac{1 + (p_\perp/mc)^2}{N^2 - 1}\right]^{1/2} , \quad u_1 = Nv_1 . \tag{3.4}$$

These values of v_1 and u_1 can be obtained in a different way. The starting point is the physical idea of a resonance: The velocity of a resonant electron, dr^1/dt, must be equal to the phase velocity of the wave, ω/k, that is,

$$\frac{dr^1}{dt} \equiv \Gamma \frac{dr^1}{d\tau} \equiv \Gamma c v_1 = \frac{\omega}{k} . \tag{3.5}$$

As Γ is given by

$$\Gamma = \left[\frac{1 + (p_\perp/mc)^2}{1 - (\omega/kc)^2}\right]^{1/2} , \tag{3.6}$$

we recover (3.4).

When a second wave is present in the medium, $\vec{A}(1) \neq \vec{0}$ and $\vec{A}(2) \neq \vec{0}$, h_0 must be replaced by the whole super-Hamiltonian (2.10) and three new resonances are present. Their position is defined by

$$v_2 = -v_1 , \quad u_2 = u_1 = Nv_1 ,$$
$$v_3 = 0 , \quad u_3 = u_1(1 - N^{-2})^{1/2} , \tag{3.7}$$
$$v_4 = iu_3 , \quad u_4 = 0 .$$

In general, the super-Hamiltonian h is nonintegrable, as can be shown by the numerical integration of the equations of motion (see below). However, integrability of the motion can be shown to exist if the two waves propagate in vacuum, $N = 1$, with orthogonal polarization, $\vec{A}(1) \cdot \vec{A}(2) = \gamma = 0$. Using the new variables

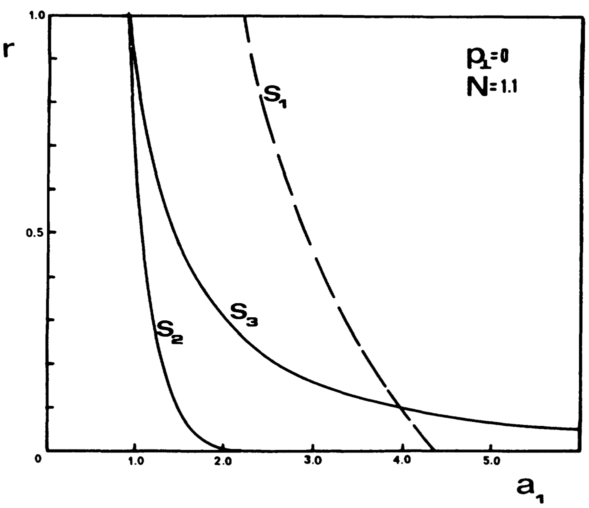

FIG. 1. Stochasticity thresholds for $N=1.1$ and $p_\perp=0$, given by the three overlap criteria (4.2). a_1 is the dimensionless amplitude of the first wave and $r=a_2/a_1$ is the ratio between the two wave amplitudes. For each criterion, the motion is stochastic above the corresponding curve.

$$x_{1,2}=\tfrac{1}{2}(x\mp y)\ ,\quad v_{1,2}=\tfrac{1}{2}(v\mp u)\ , \tag{3.8}$$

we can reduce the equations of motion (2.13) to the separable form ($N=1,\gamma=0$):

$$\begin{aligned}\frac{dx_i}{dz}&=v_i\ ,\quad i=1,2\\ \frac{dv_i}{dz}&=-2\alpha_i\sin(2x_i)-\beta_i\sin x_i\ .\end{aligned} \tag{3.9}$$

In this case the super-Hamiltonian $h'=h(N=1,\gamma=0)$ belongs to the family of the integrable super-Hamiltonian:

$$H(x,y,v,u)=\tfrac{1}{2}(v^2-u^2)+f_1(x-y)+f_2(x+y)\ , \tag{3.10}$$

where f_1 and f_2 are arbitrary analytical functions in phase space.

IV. THRESHOLD CRITERIA

In the general nonintegrable situation ($N\neq 1,\gamma\neq 0$) the orbits in the four-dimensional phase space do not necessarily belong to a two-dimensional manifold. This means that in a Poincaré surface of section (x,v) the sequence of points representing a given orbit can be randomly distributed over a significant region. In this surface of section, the resonances $v=v_i$ for $i=1,2,3$ define three domains of attraction surrounded by chaotic separatrices. The fourth resonance in (3.7) is imaginary and is disregarded. We say that large-scale stochasticity exists when orbits starting in the domain of attraction of one given resonance can fall in the domain of attraction of a different resonance. The threshold criterion for the onset of large-scale stochasticity can be obtained analytically, comparing the resonance widths with the distance between resonances. The nonperturbed half-widths of the three resonances $v=v_i$ are given by

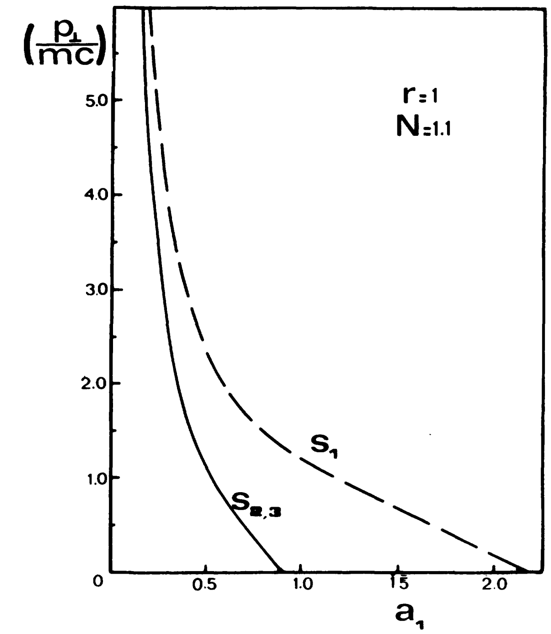

FIG. 2. Stochasticity thresholds for $N=1.1$ and $a_2=a_1$ (or $r=1$), given by the three criteria (4.2). a_1 is the dimensionless wave amplitude and $(p_\perp/mc)$ is the normalized electron momentum perpendicular to the direction of wave propagation.

$$D_i=\begin{cases}2\alpha_i+\beta_i/2\alpha_i\ , & \beta_i<4\alpha_i^2\\ 2\sqrt{\beta_i}\ , & \beta_i\geq 4\alpha_i^2\end{cases} \tag{4.1}$$
$$D_3=2\sqrt{\gamma}$$

for $i=1,2$. Using (3.4), (3.7), and (4.1) we can construct three resonance overlap criteria[9] of the form

$$\begin{aligned}s_1&\equiv D_1+D_2>2v_1\ ,\\ s_2&\equiv D_1+2\sqrt{\gamma}>v_1\ ,\\ s_3&\equiv D_2+2\sqrt{\gamma}>v_1\ .\end{aligned} \tag{4.2}$$

For a given value of the dimensionless wave amplitude $a_1=2\sqrt{\alpha_1}$ we can deduce from (4.2) the threshold value of the ratio between the two wave amplitudes $r=a_2/a_1=\sqrt{\alpha_2/\alpha_1}$ above which large-scale stochasticity exists. If the second wave is due to the reflection of the first wave somewhere in the plasma, the parameter r has the meaning of a reflection coefficient. The results are plotted in Fig. 1 for $p_\perp=0$ and in Fig. 2 for $r=1$, when $N=1.1$. The first criterion in (4.2), given by the curve S_1, is somewhat meaningless because it involves two resonances which are not nearby ones. It can only be retained when the resonance width D_3 is much smaller than the other two and the resonance $v_3=0$ can be assumed as a secondary one. It then gives the threshold for the stochasticity in the whole region of the surface of section between v_1 and $v_2=-v_1$. The other two criteria in (4.2), represented by the curves S_2 and S_3, are of the same order of magnitude and give the threshold for the stochasticity in the two different regions of the surface of section between v_1 and $v_3=0$ and $v_3=0$ and v_2, respectively. The

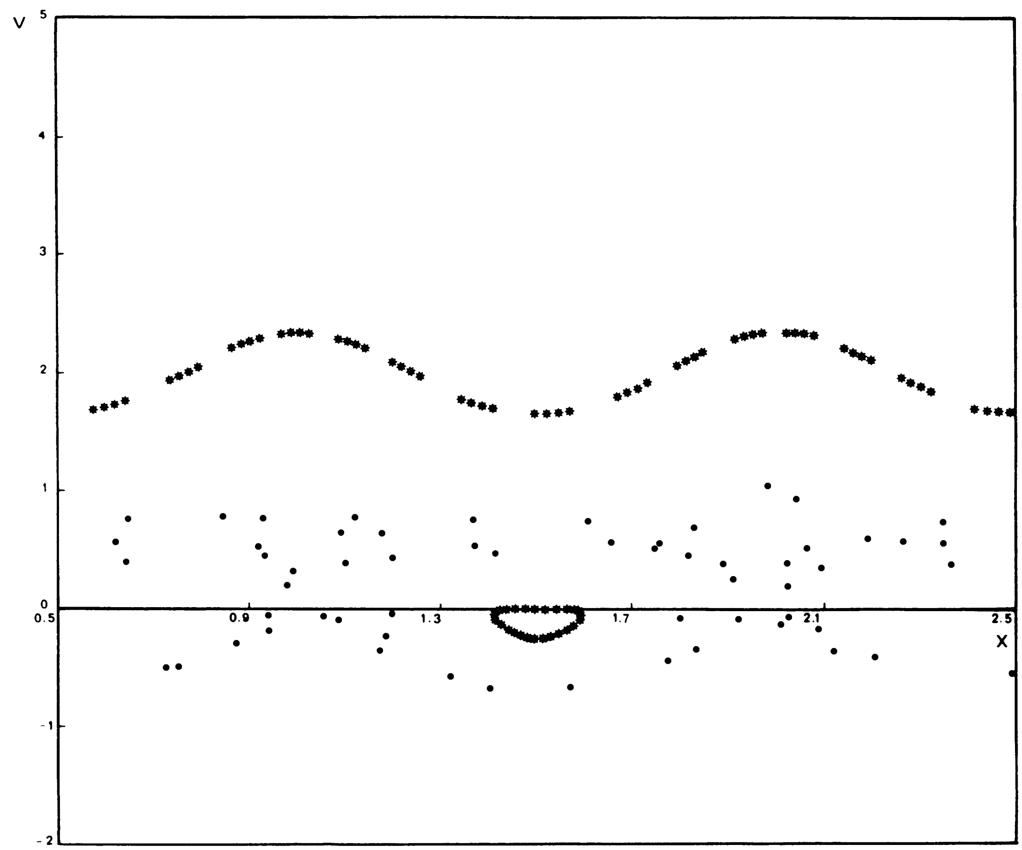

FIG. 3. Poincaré surface of section (x,v) for $N=1$, $p_{\perp}=0$, $a_1=0.35$, and $r=0.5$, resulting from the numerical integration of (2.13). Three orbits have initial conditions: $x_1=1.5$, $v_1=0$; $x_2=1.85$, $v_2=0$; $x_3=0.85$, $v_3=2.182$. Second orbit is stochastic.

evolution of the threshold as a function of the transverse momentum $p_{\perp}$ is particularly interesting, as shown in Fig. 2. We see that when $p_{\perp}$ grows the threshold is significantly reduced and becomes of the order of $(p_{\perp}/mc)^{-1}$ when $p_{\perp} \gg mc$. Another interesting feature of the criteria (4.2) is that the threshold grows with N and becomes infinite when $N=1$. This is not in agreement with the numerical results, which show that stochasticity remains even for vacuum or plasmas ($N<1$) as will be discussed in Sec. V. Nevertheless, Eq. (4.2) can be useful in the plasma case, if we take the nonrelativistic limit ($v_1 \to N^{-1}$). The threshold is no longer dependent on N and we get the new criteria

$$\begin{aligned} s'_1 &\equiv D'_1 + D'_2 > 1 , \\ s'_2 &\equiv D'_1 + 2\sqrt{\gamma'} > \tfrac{1}{2} , \\ s'_3 &\equiv D'_2 + 2\sqrt{\gamma'} > \tfrac{1}{2} . \end{aligned} \tag{4.3}$$

Here, the new half-widths D'_i ($i=1,2$) are obtained replacing in (4.1) the wave parameters α_i and β_i by $\alpha'_i = N^2\alpha_i/4$ and $\beta'_i = N^2\beta_i/4$. We have also used $\gamma' = N^2\gamma/4$. The new criteria (4.3) are coincident with those obtained directly from the nonrelativistic Hamiltonian and were shown to be in good agreement with the numerical integration of the equations of motion.[6] More sophisticated criteria than those used here, in particular the criteria obtained from a renormalization theory,[10] were also shown to be in qualitative agreement with (4.3).[6] If we make a further simplification on (4.3) and take $\gamma'=0$ we get a single criterion $s'_1 > 1$, which is the usual overlap criterion for the electron motion in the field of two electrostatic waves,[8] provided we conveniently replace the vector potentials $\vec{A}(j)$ by scalar potentials.

V. THRESHOLD FOR RELATIVISTIC PLASMAS

Turning back to the relativistic motion, we see from (4.2) that a further step must be done in the theory in order to get analytical criteria which do not diverge. One can be sure that stochasticity remains for $N \leq 1$ by performing a numerical integration of the equations of motion (2.13). The results of such integration are illustrated by the numerically generated surface of section of Fig. 3, for three different orbits and $N=1$. The two wave potentials are assumed parallel to each other, $\vec{A}(1)//\vec{A}(2)$. The standard fourth-order Runge-Kutta method was used in the integration. Stochasticity appears in the region between $v=0$ and $v \simeq 1$. The value of $r=0.5$ is the lowest value for which stochasticity is clearly observed. This gives then a numerical estimation of the threshold for $a_1=0.35$.

Let us see how to construct the analytical threshold for

$N<1$. The resonances v_1 and v_2 are now imaginary and our main attention turns to the third resonance $v_3=0$. The electron motion near this resonance is slightly perturbed by the other two and such perturbation is the reason for the stochasticity to remain. In order to study in more detail the structure of (2.10) around $v_3=0$ we can make a canonical transformation into new variables $(\theta_1,\theta_2,I_1,I_2)$ using the generating function

$$F(x,y,I_1,I_2) = xI_1+yI_2+\frac{\alpha_1}{2(I_1+I_2/N)}\sin\left[2\left[x-\frac{y}{N}\right]\right] + \frac{\beta_1}{(I_1+I_2/N)}\cos\left[x-\frac{y}{N}\right] . \quad (5.1)$$

The resulting super-Hamiltonian is given by

$$h(\theta_1,\theta_2,I_1,I_2)=h_0+h_1 , \quad (5.2)$$

where the unperturbed term associated to the first wave is

$$h_0=\tfrac{1}{2}(I_1^2-I_2^2)+\tfrac{1}{2}\frac{N^2-1}{(NI_1+I_2)^2}[\alpha_1\cos(2\theta)-\beta_1\sin\theta]^2 \quad (5.3)$$

and the perturbed term due to the second wave is

$$h_1=-\alpha_2\cos(2\varphi)+\beta_2\cos\varphi-\gamma\left[\cos(2x)-\cos\frac{2y}{N}\right] . \quad (5.4)$$

Here x and y are functions of θ_1 and θ_2, as well as $\theta=x-y/N$ and $\varphi=x+y/N$. The transformation generated by (5.1) destroys the resonance v_1, which is absent in the new super-Hamiltonian (5.2) if we neglect terms of order $(N^2-1)\alpha_1^2$ in the expression of h_0. This is a valid procedure because such resonance is infinitely far from the actual electron orbits and only acts as a perturbation.

Let us consider the limiting but important case of an underdense plasma, $N\simeq 1$. We then have

$$\theta=\theta_1-\frac{\theta_2}{N} , \quad \varphi=\theta_1+\frac{\theta_2}{N}+(N^2+1)f(\theta_1,\theta_2) ,$$
$$x=\theta_1+N^2 f(\theta_1,\theta_2) , \quad y=\theta_2+Nf(\theta_1,\theta_2) , \quad (5.5)$$

where

$$f(\theta_1,\theta_2)=(NI_1+I_2)^{-2}\left\{\frac{\alpha_1}{2}\sin\left[2\left[\theta_1-\frac{\theta_2}{N}\right]\right] +\beta_1\cos\left[\theta_1-\frac{\theta_2}{N}\right]\right\} . \quad (5.6)$$

As $h(\theta_1,\theta_2,I_1,I_2)$ is periodic in the angular variables θ_1 and θ_2, we can expand it in Fourier series in θ_1 and θ_2/N, using (5.5) and (5.6):

$$h(\theta_1,\theta_2,I_1,I_2) = h_0(I_1,I_2)+\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} R_{np}\cos\left[n\theta_1-p\frac{\theta_2}{N}\right] + Q_{np}\sin\left[n\theta_1-p\frac{\theta_2}{N}\right] , \quad (5.7)$$

where

$$R_{np}=\frac{1}{4}\frac{\alpha_{nm}}{(2\pi)^2}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\left[\frac{\theta_2}{N}\right]F(\theta_1,\theta_2) \times\cos\left[n\theta_1-p\frac{\theta_2}{N}\right] ,$$
$$Q_{np}=\frac{1}{4}\frac{\alpha_{nm}}{(2\pi)^2}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\left[\frac{\theta_2}{N}\right]F(\theta_1,\theta_2) \times\sin\left[n\theta_1-p\frac{\theta_2}{N}\right] , \quad (5.8)$$

and $\alpha_{nm}=(1+\delta_{n0})(1+\delta_{p0})$ are defined with the aid of Kroenecker δ symbols. The function $F(\theta_1,\theta_2)$ is given by

$$F(\theta_1,\theta_2)=-\alpha_2\cos(2\varphi)+\beta_2\sin\varphi-\gamma\cos(2x) , \quad (5.9)$$

where φ and x are given by (5.5). The new super-Hamiltonian shows the existence of a double infinity of secondary resonances, corresponding to the condition of stationary phase:

$$\frac{d}{dz}\left[n\theta_1-p\frac{\theta_2}{N}\right]=0 . \quad (5.10)$$

Using the canonical equations of motion in the new coordinates, valid to order $(N^2-1)\alpha_1^2$, we see from (5.10) that the location in phase space of the (n,p)th resonance obeys the following relation:

$$I_1=\frac{p}{n}\frac{I_2}{N} . \quad (5.11)$$

Using the unperturbed term (5.3) to calculate the value of I_2 at resonance,

$$h_0(I_1,I_2)=\frac{I_2^2}{2}\left[\left[\frac{p}{nN}\right]^2-1\right] \simeq-\frac{1}{2}\left[1+\left[\frac{p_\perp}{mc}\right]^2\right] , \quad (5.12)$$

we get the position of the (n,m)th resonance:

$$I_1(n,m)=\left[\frac{1+(p_\perp/mc)^2}{(Nn/p)^2-1}\right]^{1/2} . \quad (5.13)$$

An estimation of the resonance half-width can be made in the standard way and we get

$$D_{np}=2(R_{np}^2+Q_{np}^2)^{1/4} . \quad (5.14)$$

An overlap criterion for nearby resonances (n,p) and (n',p') can then be constructed[11]:

$$D_{np}+D_{n'p'}>I_1(n,p)-I_1(n',p') . \quad (5.15)$$

Such criterion can be used for real resonances, which obey the condition $N>p/n$, as shown by (5.13). It remains valid, of course, for $N>1$, as well as for $N<1$. But in the first case it can only be used as a local criterion for small-scale stochasticity, because the large-scale behavior is governed by (4.2). In the second case (4.2) no longer holds and (5.15) is the only result we have. Nevertheless, a large-scale stochasticity criterion can be found in this case, if we apply (5.15) to the dominant secondary resonances.

VI. APPROXIMATE THRESHOLD

The overlap criteria (5.15) is not easy to handle. A simplified version can be obtained, in order to make qualitative estimations of the threshold. Let us consider that the electron is at rest in the absence of the waves, $\beta_1=\beta_2=p_\perp=0$, and let us assume the small wave amplitude limit, $\alpha_1 \ll 1$. We can then write $x=\theta_1+O(\alpha_1)$ and replace (5.2)–(5.4) by the simpler expression

$$h(\theta_1,\theta_2,I_1,I_2)=h_0(I_1,I_2)-\alpha_2\cos(2\varphi)-\gamma\cos(2\theta_1)\,, \tag{6.1}$$

where

$$\varphi=\theta_1+\frac{\theta_2}{N}+\frac{\alpha_1}{2}\frac{1+N^2}{(NI_1+I_2)^2}\sin\left[2\left[\theta_1-\frac{\theta_2}{N}\right]\right]. \tag{6.2}$$

Making a Fourier expansion in θ_2 we get

$$h(\theta_1,\theta_2,I_1,I_2)\simeq h_0(I_1,I_2)-\alpha_2\sum_{n=-\infty}^{\infty}\frac{2-\delta_{n2}}{2}J_{n-2}(r_1)\cos\left[n\theta_1-(n-4)\frac{\theta_2}{N}\right]-\gamma\cos(2\theta_1)\,. \tag{6.3}$$

Here $J_{n-2}(r_1)$ are the Bessel functions of first kind with argument

$$r_1=\alpha_1\frac{1+N^2}{(NI_1+I_2)^2}\,. \tag{6.4}$$

From (6.3) we see that the main resonance $v_3=0$ is surrounded by an infinity of secondary resonances. The positions of the nth secondary resonances are given by

$$I_1(n)=\left[1-\frac{4}{n}\right]\frac{I_2(n)}{N}=\left[\frac{1+(p_\perp/mc)^2}{[Nn/(n-4)]^2-1}\right]^{1/2}, \tag{6.5}$$

and their half-widths are

$$D_n=2[\alpha_2(1-\delta_{n0}/2)J_{n-2}(r_1)]^{1/2}\,. \tag{6.6}$$

From this infinite set of secondary resonances we choose the nearest to $v=0$, which corresponds to $n=3$. Again using the overlap criterion for nearby resonances we have

$$2\sqrt{\alpha_2 J_1(r_1)}+2\sqrt{\gamma}>I_1(3)\,. \tag{6.7}$$

This expression gives the threshold for large-scale stochasticity in the region of phase space lying between $I_1\simeq v_3=0$ and $I_1=I_1(3)$ and it replaces the third criterion (4.2) for $N<1$ and $\beta_1=0$. An expression replacing the second criterion (4.2) can be found in the same way:

$$2\sqrt{\alpha_1 J_1(r_2)}+2\sqrt{\gamma}>I_1(3)\,, \tag{6.8}$$

where $r_2=r_1(\alpha_1\to\alpha_2)$. It is important to note that the quantities r_i with $i=1,2$ are nearly constant for each orbit, because for $N\simeq 1$ the denominator in (6.4) is nearly equal to I_0^2, which is a constant of motion in the limit $\alpha_i\to 0$. This means that (6.7) and (6.8) can be applied to each orbit. The first orbits to become stochastic when α_1 is given and α_2 is increased from zero are those for which $J_1(r_i)$ attain its maximum value 0.58. We can then obtain the threshold curves S_2 and S_3 of Fig. 4. The condition for which most of the orbits become stochastic corresponds to $J_1(r_i)=0$ and is represented by the curve S_M of Fig. 4 for both (6.7) and (6.8). Comparing these curves with the numerical calculation of Fig. 3 we see that there is a good agreement between the analytical estimate and the numerical observation of stochasticity. We can also see from Fig. 3 that the width of the stochastic region Δv is of the order of the distance between the two resonances used in (6.7): $\Delta v > I_1(3)=1/\sqrt{8}$. In Fig. 4 we have also represented, for comparison, the third nonrelativistic criterion (4.3). We see that the results for the relativistic case are of the same order of magnitude. This is an acceptable result, because the stochastic region lies mainly on the weakly relativistic domain of phase space around $v_3=0$.

VII. CONCLUSION

We have studied the electron motion in the field of two electromagnetic waves, using a super-Hamiltonian description in space-time, and were able to determine analytically the threshold for the electron heating. The results extend to the relativistic case those obtained in a pre-

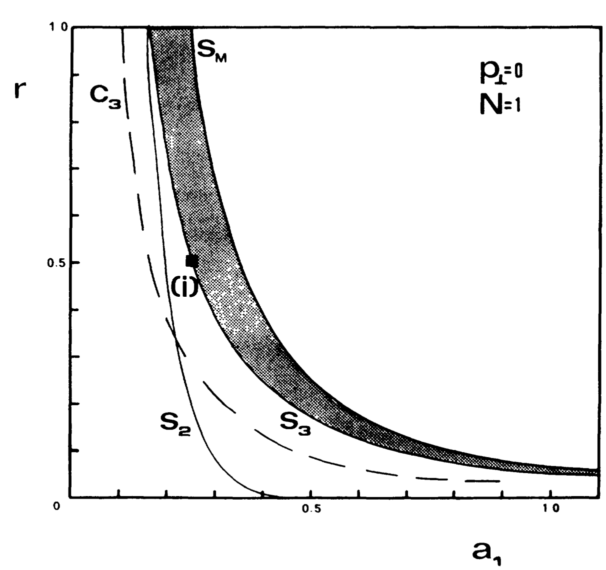

FIG. 4. Stochasticity thresholds for $N_s=1$ and $p_\perp=0$. S_2 is the criterion (6.7) for $J_1(r_1)=0.58$ and S_3 in the criterion (6.8) for $J_1(r_2)=0.58$. S_M corresponds to either (6.7) or (6.8) for $J_1=0$. C_3 is the nonrelativistic criterion of Ref. 6. ■ defines the parameters of Fig. 3.

vious work[6] and are in qualitative agreement with the numerical integration of the equations of motion. An important consequence of this work is the possibility of explaining the high-energy electron tails produced in laser-plasma interaction experiments. In such a case, the first wave is the incident laser beam and the second wave is the reflected beam propagating in the underdense region of the plasma pellet. Taking the root-mean-square momentum to be of the order of mc, we can see that, in the conditions of Fig. 3, the energy of the electron heated by the two waves is about 200 keV. This is compatible with the observed energies.[12] However, more specific calculations are needed to apply these results to the interpretation of a given experiment and to estimate the influence of the collective effects which are not present in our one-particle model.

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