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Quantum fluctuations in the two-photon laser

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A two-photon laser is modeled as N two-level atoms interacting via a two-photon transition with a single resonant-cavity field mode. The quadratic form of this interaction makes such a system a candidate to exhibit reduced quantum fluctuations (squeezing) in one quadrature of the output field. A Fokker-Planck equation containing all leading quantum noise terms is derived following the method of Haken. A linearized fluctuation analysis reveals that there is no squeezing in a two-photon laser with injected signal. A small amount of squeezing may be present in two-photon optical bistability. Our calculations confirm the results of Lugiato and Strini, who adopted a factorization of moment equations.

I. INTRODUCTION

Minimum uncertainty states of the single-mode radiation field which show unequal uncertainties in the two components of the complex field amplitude are known as squeezed states.[1–9] These states are potentially useful in optical communication systems[10–12] and also in detection systems for very weak forces, such as gravitational waves.[13] As yet, however, there has been no experimental realization of a squeezed state, although several optical devices have been proposed which may produce them. Theoretically, a parametric amplifier with a classical pump field will produce ideal squeezed states.[14–18] However, recent work on parametric oscillators with the pump mode quantized shows that the amount of squeezing is considerably reduced.[19–21] Squeezing has also been predicted to occur in resonance fluorescence[22] and the free-electron laser.[23]

Since a squeezed state is generated via a quadratic interaction of the electromagnetic field, it has been proposed that a two-photon laser may generate squeezed states.[7,8] A first experimental realization of a two-photon laser has recently been reported.[24] Previous analyses[25–31] of the two-photon laser either have assumed thermal fluctuations to be dominant over quantum fluctuations or the derivation has been in terms of a photon number distribution. Thus, they have not been able to address the question as to whether or not squeezing is present. Squeezing, like photon antibunching, is a manifestation of the quantum nature of the electromagnet field and for this reason a theoretical treatment of the two-photon laser investigating squeezing must treat quantum noise effects fully. Recently, a quantum-statistical analysis of the two-photon emission process has shown that a squeezed state is not produced.[32] Nonclassical photon correlations in a two-mode, two-photon laser have been studied by Zubairy.[33]

In this paper we derive a quantum-mechanical master equation for the two-photon laser which is transformed to an approximate Fokker-Planck equation following the method of Haken.[34] A linearized fluctuation analysis allows us to test for squeezing in the two-photon laser. The equations also contain the results concerning the quantum fluctuations in two-photon optical bistability. This phenomenon has recently been experimentally[35] observed. Most previous theoretical analyses have either been deterministic[36–39] or treated only thermal fluctuations.[31]

Recently, an analysis of the quantum fluctuations in a two-photon laser and two-photon optical bistability has been given by Lugiato and Strini,[40] who used a method of Gaussian factorization of the moment equations. The method which we use relies on the linearization of the stochastic differential equations corresponding to the Fokker-Planck equation, is entirely equivalent to the Gaussian factorization of moments and gives results which are in exact agreement with those obtained by Lugiato and Strini.

II. FOKKER-PLANCK EQUATION

The two-photon laser is modeled as N two-level atoms in an optical cavity interacting with a single resonant-cavity field mode, whose spatial variation is neglected. The two-photon interaction with the atoms is represented by an effective Hamiltonian where a summation over intermediate states is implicit. In the rotating wave and electric-dipole approximations we write the total Hamiltonian as

$$
\begin{aligned}
H &= \sum_{j=1}^{5} H_j \,, \\
H_1 &= \hbar\omega a^\dagger a + \hbar\omega \sum_{\mu=1}^{N} \sigma_\mu^z \,, \\
H_2 &= i\hbar \sum_{\mu=1}^{N} \left(g a^{\dagger 2} \sigma_\mu^- e^{-i\vec{k}\cdot\vec{r}_\mu} - g a^2 \sigma_\mu^+ e^{-i\vec{k}\cdot\vec{r}_\mu} \right) , \\
H_3 &= i\hbar (\mathscr{E} a^\dagger e^{-i\omega t} - \mathscr{E}^+ a e^{i\omega t}) \,, \\
H_4 &= \Gamma_F a^\dagger + \Gamma_F^+ a \,, \\
H_5 &= \sum_{\mu=1}^{N} (\Gamma_p \sigma_\mu^z + \Gamma_a \sigma_\mu^+ + \Gamma_a^+ \sigma_\mu^-) \,.
\end{aligned}
\tag{1}
$$

The operators $a,a^\dagger$ are the boson field operators while $\sigma_\mu^z,\sigma_\mu^+,\sigma_\mu^-$ are the Pauli spin operators for the μ^{th} atom. The term $\mathscr{E}$ describes an external resonant driving field, if present, and ω is the cavity and atomic resonance frequencies. The reservoirs are described by Γ_F for the field mode and Γ_a and Γ_p for the atoms. Γ_p describes phase-damping processes while Γ_a describes radiative decay or spontaneous emission. The coupling parameter g is the dipole matrix element for the two-photon transition with the intermediate states summed over.

The master equation for the density operator ρ of the system is derived by tracing over the reservoir operators and using the Markovian assumption[34,41]:

$$
\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}[H_1+H_2+H_3,\rho] + \left.\frac{\partial\rho}{\partial t}\right|_A + \left.\frac{\partial\rho}{\partial t}\right|_F \,,
\tag{2}
$$

where the dissipative term for the atoms is

$$
\left.\frac{\partial\rho}{\partial t}\right|_A = \sum_{\mu=1}^{N} \left\{ \frac{\omega_{21}}{2}([\sigma_\mu^-,\rho\sigma_\mu^+]+[\sigma_\mu^-\rho,\sigma_\mu^+]) + \frac{\omega_{12}}{2}([\sigma_\mu^+,\rho\sigma_\mu^-]+[\sigma_\mu^+\rho,\sigma_\mu^-]) + \frac{\gamma_p}{4}([\sigma_\mu^z\rho,\sigma_\mu^z]+[\sigma_\mu^z,\rho\sigma_\mu^z]) \right\}
$$

and, for the field mode,

$$
\left.\frac{\partial\rho}{\partial t}\right|_F = K([a\rho,a^\dagger]+[a,\rho a^\dagger]) + 2Kn_{\text{th}}[[a,\rho],a^\dagger] \,.
$$

The parameter K is the cavity damping rate and n_{th} is the mean thermal photon number for the field reservoir. The term ω_{21} is the transition rate, caused by the atomic reservoirs, from level 2 to 1, while ω_{12} is that from 1 to 2 and allows for the description of incoherent pumping. γ_p is the rate of collision-induced phase decay of the atoms.

To convert the master equation into a Fokker-Planck equation for the distribution function f in atomic and field c-number variables we use a standard technique developed by Haker for normal laser theory.[34,42] A correspondence between complex c numbers and system operators is defined as follows:

$$
\begin{aligned}
&\alpha,\alpha^+ \leftrightarrow a,a^\dagger \,, \\
&V \leftrightarrow S^- = \sum_{\mu=1}^{N} \sigma_\mu^- e^{-i\vec{k}\cdot\vec{r}_\mu} \,, \\
&V^+ \leftrightarrow S^+ = \sum_{\mu=1}^{N} \sigma_\mu^+ e^{i\vec{k}\cdot\vec{r}_\mu} \,, \\
&D \leftrightarrow 2S_z = \sum_{\mu=1}^{N} \sigma_\mu^z \,.
\end{aligned}
\tag{3}
$$

Since the resulting equation would not in general have a positive-definite diffusion matrix when using the standard representation, it is necessary to define a representation in a complex phase space.[43] This means that α and α^+, V and V^+, are not complex conjugate and that D may be complex. Thus we are defining a distribution function on a five-dimensional complex space C^5, not R^5 as in the laser case. The appropriate normally ordered characteristic function is

$$
\chi = \mathrm{Tr}(O\rho) \,,
\tag{4}
$$

where

$$O=e^{i\epsilon^+S^+}e^{i\eta S_z}e^{i\epsilon S^-}e^{i\beta^+a^\dagger}e^{i\beta a}$$

and the distribution function f is the Fourier transform of χ

$$f(\alpha,\alpha^+,V,V^+,D)=\int\cdots\int\exp\left[-i\left[V\epsilon+V^+\epsilon^++\eta\frac{D}{2}+\alpha\beta+\alpha^+\beta^+\right]\right]\chi(\epsilon,\epsilon^+,\eta,\beta,\beta^+)d^2\epsilon\,d\eta\,d^2\beta\,. \tag{5}$$

Briefly, the method[34] involves deriving the equation of motion for χ from the master equation by using operator rules, and then taking the Fourier transform to obtain an equation of motion for f. The resulting equation is written

$$\begin{aligned}
\dot{f}&=Lf\,,\\
L&=L_A+L_F+L_{A-F}\,,\\
L_F&=K\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^+}\alpha^+\right]+2Kn_{\text{th}}\frac{\partial^2}{\partial\alpha\,\partial\alpha^+}+\frac{\partial\mathscr{E}}{\partial\alpha}+\frac{\partial\mathscr{E}^*}{\partial\alpha^+}\,,\\
L_{A-F}&=g\left[\left[\alpha^+-\frac{\partial}{\partial\alpha}\right]^2V-\left[e^{-2(\partial/\partial D)}V^+-\frac{\partial^2}{\partial V^2}V+\frac{\partial}{\partial V}D\right]\alpha^2\right.\\
&\qquad\left.+\left[\alpha-\frac{\partial}{\partial\alpha^+}\right]^2V^+-\left[e^{-2(\partial/\partial D)}V-\frac{\partial^2}{\partial(V^+)^2}V^++\frac{\partial}{\partial V^+}D\right](\alpha^+)^2\right]\,,\\
L_A&=\frac{\omega_{12}}{2}\left[N(e^{-2(\partial/\partial D)}-1)+Ne^{2(\partial/\partial D)}\frac{\partial^4}{\partial V^2\partial(V^+)^2}+2N\frac{\partial^2}{\partial V\partial V^+}+\frac{\partial}{\partial V}\left[\frac{2\partial^2}{\partial V\partial V^+}+2e^{-2(\partial/\partial D)}-1\right]V\right.\\
&\qquad\left.+\frac{\partial}{\partial V^+}\left[2\frac{\partial^2}{\partial V\partial V^+}+2e^{-2(\partial/\partial D)}-1\right]V^+-2\left[(e^{-2(\partial/\partial D)}-1)e^{2(\partial/\partial D)}\frac{\partial^4}{\partial V^2\partial(V^+)^2}\right]\frac{D}{2}\right]\\
&\quad+\frac{\omega_{21}}{2}\left[N(e^{2(\partial/\partial D)}-1)+\frac{\partial}{\partial V}V+\frac{\partial}{\partial V^+}V^++2(e^{2(\partial/\partial D)}-1)\frac{D}{2}\right]\\
&\quad+\gamma_p\left[\frac{\partial}{\partial V}V+\frac{\partial}{\partial V^+}V^++\frac{2\partial^2}{\partial V\partial V^+}e^{2(\partial/\partial D)}\frac{D}{2}+N\frac{\partial^2}{\partial V\partial V^+}e^{2(\partial/\partial D)}\right]\,.
\end{aligned} \tag{6}$$

To justify ignoring higher derivatives, a scaling argument is used. We look first at the two-photon laser, where ω_{12} is nonzero to describe incoherent pumping and $\mathscr{E}$ is zero. The steady-state deterministic solutions (stable, unstable, and stable, respectively), which ignore fluctuations, are[31]

$$|\alpha|^2=\begin{cases}n_0[c+(c^2-1)^{1/2}], & c>1\\ n_0[c-(c^2-1)^{1/2}], & c>1\\ 0, & c>0\end{cases} \tag{7}$$

where

$$C=\frac{gD_0}{2K}\left[\frac{\gamma_{\|}}{\gamma_{\perp}}\right]^{1/2},\quad n_0=\left[\frac{\gamma_{\|}\gamma_{\perp}}{4g^2}\right]^{1/2}$$

and we have defined

$$\begin{aligned}
\gamma_{\|}&=\omega_{12}+\omega_{21}\,,\\
\gamma_{\perp}&=\frac{\gamma_{\|}}{2}+\gamma_p\,,\\
D_0&=N\frac{(\omega_{12}-\omega_{21})}{(\omega_{12}+\omega_{21})}\,.
\end{aligned} \tag{8}$$

$\gamma_{\|}$ and $\gamma_{\perp}$ are the longitudinal and transverse damping rates, respectively, and D_0 is the unsaturated inversion for all atoms. A first-order phase transition is observed at $C=1$. If a small noise limit can be justified the transition will be very sharp and the intensities will very closely follow the upper stable branch above threshold. It is the upper branch that we consider. Here the c-number variables are of the order described below:

$$
\begin{aligned}
&|\alpha|^2 = n_0 ,\\
&D = \frac{D_0}{2} = \frac{K}{g}\left[\frac{\gamma_\perp}{\gamma_\parallel}\right]^{1/2} = \frac{N}{K'} ,\\
&K' = g\frac{N}{K}\left[\frac{\gamma_\parallel}{\gamma_\perp}\right]^{1/2} ,\\
&V = \frac{K}{2g} .
\end{aligned}
\tag{9}
$$

We consider K' as fixed, but N, the number of atoms, very large. Hence, the scalings with regard to N are as follows:

$$
g \propto \frac{1}{N}, \quad D \propto N, \quad V \propto N, \quad |\alpha|^2 \propto N . \tag{10}
$$

It is thus possible to write the generalized equation (6) as an expansion in $1/N$. Truncating terms of order greater than $1/N$, a Fokker-Planck equation is obtained as follows ($E = \mathscr{E}/K$):

$$
\begin{aligned}
\frac{\partial f}{\partial t} = \Bigg[\bigg[&-\frac{\partial}{\partial \alpha}[K(E-\alpha)+2g\alpha^+ V] - \frac{\partial}{\partial V}(-\gamma_\perp V + gD\alpha^2) - \frac{\partial}{\partial D}\{-\gamma_\parallel(D-D_0) - 2g[V^+\alpha^2 + V(\alpha^+)^2]\} + \text{c.c.} \bigg] \\
&+ g\frac{\partial^2}{\partial \alpha^2}V + g\frac{\partial^2}{\partial(\alpha^+)^2}V^+ + 2Kn_{\text{th}}\frac{\partial^2}{\partial\alpha\,\partial\alpha^+} + g\frac{\partial^2}{\partial V^2}V\alpha^2 + g\frac{\partial^2}{\partial(V^+)^2}V^+(\alpha^+)^2 - 2\frac{\partial^2}{\partial V\,\partial D}\omega_{12}V \\
&+ \frac{\partial^2}{\partial V\,\partial V^+}[N\omega_{12} + \gamma_p(D+N)] + \frac{\partial^2}{\partial D^2}\{\omega_{12}(N-D) + \omega_{21}(N+D) - 2g[V^+\alpha^2 + V(\alpha^+)^2]\} \Bigg] f ,
\end{aligned}
\tag{11}
$$

where c.c. means the complex conjugate of the previous expression. The second-derivative terms, which are ignored in the deterministic approach, are of order $1/N$ while the major drift or first-derivative terms are of order zero. Thus, the effect of noise will be small in the large N limit and our approach is consistent. A small injected signal E has been included in Eq. (11). Such a small signal does not affect the two-photon laser deterministic curve appreciably but allows the lower stable branch to assume nonzero values and lowers the threshold value slightly.[31]

The generalized equation (6) can be reduced in another important physical limit. Allowing $\mathscr{E}$ to be variable and the transition rate ω_{12} to be zero, the equation describes two-photon absorptive optical bistability.[31,36–39] For this situation, the parameters of Eq. (8) simplify as follows:

$$
\begin{aligned}
&\gamma_\parallel = \omega_{21} ,\\
&D_0 = -N ,\\
&\gamma_\perp = \frac{\gamma_\parallel}{2} + \gamma_p .
\end{aligned}
\tag{12}
$$

Since there is no pumping, the total unsaturated inversion D_0 is negative. For this reason, we redefine C for the special case of two-photon optical bistability as follows:

$$
C = \frac{gN}{2K}\left[\frac{\gamma_\parallel}{\gamma_\perp}\right]^{1/2} . \tag{13}
$$

The steady-state deterministic solution is[31]

$$
y = x\left[1 + \frac{2C|x|^2}{1+|x|^4}\right] , \tag{14}
$$

where the scaled variables are defined as

$$
y = \frac{E}{\sqrt{n_0}}, \quad x = \frac{\alpha}{\sqrt{n_0}} . \tag{15}
$$

For $C > 2.71$, absorptive bistability is observed. Considering C as fixed and N large, the scalings (10) are arrived at. The generalized Fokker-Planck equation for the two-photon optical bistability, truncated to first order in $1/N$, becomes

$$
\begin{aligned}
\frac{\partial f}{\partial t} = \Bigg[\bigg[&-\frac{\partial}{\partial \alpha}[K(E-\alpha)+2g\alpha^+ V] - \frac{\partial}{\partial V}(-\gamma_\perp V + gD\alpha^2) - \frac{\partial}{\partial D}\{-\gamma_\parallel(D+N) - 2g[V^+\alpha^2 + V(\alpha^+)^2]\} + \text{c.c.} \bigg] \\
&+ g\frac{\partial^2}{\partial \alpha^2}V + g\frac{\partial^2}{\partial(\alpha^+)^2}V^+ + 2Kn_{\text{th}}\frac{\partial^2}{\partial\alpha\,\partial\alpha^+} + g\frac{\partial^2}{\partial V^2}V\alpha^2 + g\frac{\partial^2}{\partial(V^+)^2}V^+(\alpha^+)^2 + \frac{\partial^2}{\partial V\,\partial V^+}\gamma_p(D+N) \\
&+ \frac{\partial^2}{\partial D^2}\{\gamma_\parallel(N+D) - 2g[V^+\alpha^2 + V(\alpha^+)^2]\} \Bigg] f .
\end{aligned}
\tag{16}
$$

This equation is just the special case $\omega_{12}=0$ of Eq. (11) for the two-photon laser and the two different physical systems are treated simultaneously in Sec. III.

It is important to note at this stage the origin of the various noise terms in Eq. (11) and to compare them with those derived for the one-photon laser[34] and bistability[44,45] by the same method. Present in the two-photon process but not in the one-photon process is

$$g\frac{\partial^2}{\partial\alpha^2}V \tag{17}$$

and its conjugate. This term is a direct consequence of the quadratic nature of the interaction Hamiltonian H_2 of Eq. (1). Also resulting from H_2 are

$$g\frac{\partial^2}{\partial V^2}V\alpha^2 , \tag{18}$$

$$g\frac{\partial^2}{\partial D^2}[V^+\alpha^2+V(\alpha^+)^2] , \tag{19}$$

although similar terms appear in the one-photon equations. Resulting from the atom-reservoir interaction are

$$\frac{\partial^2}{\partial V\partial V^+}\gamma_p(D+N) , \tag{20}$$

$$\frac{\partial^2}{\partial V\partial V^+}N\omega_{12} , \tag{21}$$

$$\frac{\partial^2}{\partial D^2}[\omega_{12}(N-D)+\omega_{21}(N+D)] . \tag{22}$$

Identical reservoir terms appear in the one-photon equation. For the one-photon laser, however, the transition is second order and suitable scaling shows that the term (21) is dominant over other noise terms, for large N. Since for one-photon optical bistability $\omega_{12}=0$, all second derivative terms present must be considered.

III. ADIABATIC ELIMINATION OF ATOMIC VARIABLES

Since we are interested only in the field properties, we proceed to eliminate the atomic variables under the assumption $\gamma_\perp,\gamma_{||}\gg K$. This elimination is most easily carried out in the equivalent Langevin or Stochastic differential equation form of the Fokker-Planck equation (11). These equations are

$$\begin{aligned}\dot\alpha&=K(E-\alpha)+2g\alpha^+V+\Gamma_\alpha ,\\ \dot V&=-\gamma_\perp V+gD\alpha^2+\Gamma_V ,\\ \dot D&=-\gamma_{||}(D-D_0)-2g[V^+\alpha^2+V(\alpha^+)^2]+\Gamma_D ,\end{aligned} \tag{23}$$

where for optical bistability $D_0=-N$. The nonzero correlations of the stochastic functions $\Gamma_\alpha,\Gamma_V,\Gamma_D$ are obtained from the nonzero components of the diffusion matrix of Eqs. (11) or (16). Since we are using a generalized representation,[43] standard methods of Itô calculus are applied to obtain

$$\begin{aligned}&\langle\Gamma_\alpha(t)\Gamma_{\alpha^+}(t')\rangle=2Kn_{\rm th}\delta(t-t') ,\\ &\langle\Gamma_\alpha(t)\Gamma_\alpha(t')\rangle=2gV\delta(t-t') ,\\ &\langle\Gamma_{\alpha^+}(t)\Gamma_{\alpha^+}(t')=2gV^+\delta(t-t') ,\\ &\langle\Gamma_{V^+}(t)\Gamma_V(t')\rangle=[N\omega_{12}+\gamma_p(D+N)]\delta(t-t') ,\\ &\langle\Gamma_V(t)\Gamma_V(t')\rangle=2gV\alpha^2\delta(t-t') ,\\ &\langle\Gamma_{V^+}(t)\Gamma_{V^+}(t')\rangle=2gV^+(\alpha^+)^2\delta(t-t') ,\\ &\langle\Gamma_V(t)\Gamma_D(t')\rangle=-2\omega_{12}V\delta(t-t') ,\\ &\langle\Gamma_{V^+}(t)\Gamma_D(t')\rangle=-2\omega_{12}V^+\delta(t-t') ,\\ &\langle\Gamma_D(t)\Gamma_D(t')\rangle=\{\omega_{12}(N-D)+\omega_{21}(N+D)\\ &\qquad\qquad -2g[V^+\alpha^2+V(\alpha^+)^2]\}\delta(t-t') .\end{aligned} \tag{24}$$

Without these fluctuating terms, the equations are the Maxwell-Bloch equations for the two-photon system. Such equations have been analyzed previously[31] and the steady-state solutions deduced.

To proceed with the adiabatic elimination, the assumption that the atoms decay at a much faster rate than the field, allows us to set $\dot V=\dot D=0$. Solving, we find

$$\begin{aligned}D&=\frac{D_0}{\Pi}+\frac{\Gamma_0}{\Pi\gamma_{||}}-\frac{2g(\alpha^+)^2}{\Pi\gamma_{||}\gamma_\perp}\Gamma_V-\frac{2g\alpha^2}{\Pi\gamma_{||}\gamma_\perp}\Gamma_{V^+} ,\\ V&=\frac{gD_0\alpha^2}{\gamma_\perp\Pi}+\frac{g\alpha^2\Gamma_0}{\gamma_{||}\gamma_\perp\Pi}\\ &\quad+\frac{\Gamma_V}{\gamma_\perp}\left[1-\frac{2g^2|\alpha|^4}{\gamma_{||}\gamma_\perp\Pi}\right]-\frac{2g^2\alpha^4}{\Pi\gamma_{||}\gamma_\perp^2}\Gamma_{V^+} ,\\ \dot\alpha&=\mathscr{E}-K\alpha+\frac{2g^2D_0|\alpha|^2\alpha}{\gamma_\perp\Pi}+F ,\\ \Pi&=1+\frac{4g^2|\alpha|^4}{\gamma_{||}\gamma_\perp} ,\end{aligned} \tag{25}$$

where

$$F=\Gamma_\alpha+\frac{2g\alpha^+}{\gamma_\perp}\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]\Gamma_V+\frac{2g^2|\alpha|^2\alpha}{\Pi\gamma_{\|}\gamma_\perp}\Gamma_D-\frac{4g^3|\alpha|^2\alpha^3}{\Pi\gamma_{\|}\gamma_\perp^2}\Gamma_{V^+}\,.$$

The correlations of the stochastic force F,F^+ are

$$\begin{aligned}
\langle F(t)F(t')\rangle=&\ \langle\Gamma_\alpha(t)\Gamma_\alpha(t')\rangle+\left[\frac{2g\alpha^+}{\gamma_\perp}\right]^2\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]^2\langle\Gamma_V(t)\Gamma_V(t')\rangle+\left[\frac{2g^2|\alpha|^2\alpha}{\Pi\gamma_{\|}\gamma_\perp}\right]^2\langle\Gamma_D(t)\Gamma_D(t')\rangle\\
&+\left[\frac{4g^3|\alpha|^2\alpha^3}{\Pi\gamma_{\|}\gamma_\perp^2}\right]^2\langle\Gamma_{V^+}(t)\Gamma_{V^+}(t')\rangle-\frac{4g\alpha^+}{\gamma_\perp}\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]\frac{4g^3|\alpha|^2\alpha^3}{\Pi\gamma_{\|}\gamma_\perp^2}\langle\Gamma_V(t)\Gamma_{V^+}(t')\rangle\\
&-\frac{4g\alpha^+}{\gamma_\perp}\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]\frac{2g^2|\alpha|^2\alpha}{\Pi\gamma_{\|}\gamma_\perp}\langle\Gamma_V(t)\Gamma_D(t')\rangle\\
&-\frac{8g^3|\alpha|^2\alpha^3}{\gamma_{\|}\gamma_\perp^2\Pi}\,\frac{2g^2|\alpha|^2\alpha}{\Pi\gamma_{\|}\gamma_\perp}\langle\Gamma_{V^+}(t)\Gamma_D(t')\rangle\,,
\end{aligned}\tag{26}$$

$$\begin{aligned}
\langle F^+(t)F(t')\rangle=&\ \langle\Gamma_{\alpha^+}(t)\Gamma_\alpha(t')\rangle+\left[\frac{4g^2}{\gamma_\perp^2}|\alpha|^2\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]^2+\frac{16g^6}{\Pi^2\gamma_{\|}^2\gamma_\perp^4}|\alpha|^{10}\right]\langle\Gamma_V(t)\Gamma_{V^+}(t')\rangle\\
&+\frac{4g^4|\alpha|^6}{\Pi^2\gamma_{\|}^2\gamma_\perp^2}\langle\Gamma_D(t)\Gamma_D(t')\rangle-\frac{8g^4}{\gamma_\perp^3\Pi\gamma_{\|}}|\alpha|^2\alpha^4\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]\langle\Gamma_{V^+}(t)\Gamma_{V^+}(t')\rangle\\
&-\frac{8g^4}{\gamma_{\|}\gamma_\perp^3\Pi}|\alpha|^2(\alpha^+)^4\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]\langle\Gamma_V(t)\Gamma_V(t')\rangle\\
&+\left[\frac{4g^3}{\gamma_{\|}\gamma_\perp^2\Pi}|\alpha|^2\alpha^2\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]-\frac{8g^5|\alpha|^6}{\Pi^2\gamma_{\|}^2\gamma_\perp^3}|\alpha|^6\alpha^2\right]\langle\Gamma_D(t)\Gamma_{V^+}(t')\rangle\\
&+\left[\frac{4g^3}{\gamma_{\|}\gamma_\perp^2\Pi}|\alpha|^2(\alpha^+)^2\left[1-\frac{2g^2|\alpha|^4}{\gamma_{\|}\gamma_\perp\Pi}\right]-\frac{8g^5}{\Pi^2\gamma_{\|}^2\gamma_\perp^3}|\alpha|^6(\alpha^+)^2\right]\langle\Gamma_D(t)\Gamma_V(t')\rangle\,.
\end{aligned}$$

Since the correlations (24) depend upon the atomic variables themselves, we approximate V and D by their steady-state semiclassical values. Such an approximation is valid in the limit of large N where fluctuations are small. The correlations become

$$\begin{aligned}
\langle\Gamma_\alpha(t)\Gamma_\alpha(t')\rangle&=\frac{2g^2D_0}{\gamma_\perp\Pi}\alpha^2\delta(t-t')=D_{\alpha\alpha}\delta(t-t')\,,\\
\langle\Gamma_{\alpha^+}(t)\Gamma_{\alpha^+}(t')\rangle&=\frac{2g^2D_0(\alpha^+)^2}{\gamma_\perp\Pi}\delta(t-t')=D_{\alpha^+\alpha^+}\delta(t-t')\,,\\
\langle\Gamma_{V^+}(t)\Gamma_V(t')\rangle&=\left[N\omega_{12}+\gamma_p\left[\frac{D_0}{\Pi}+N\right]\right]\delta(t-t')=D_{V^+V}\delta(t-t')\,,\\
\langle\Gamma_V(t)\Gamma_V(t')\rangle&=\frac{2g^2D_0}{\gamma_\perp\Pi}\alpha^4\delta(t-t')=D_{VV}\delta(t-t')\,,\\
\langle\Gamma_{V^+}(t)\Gamma_{V^+}(t')\rangle&=\frac{2g^2D_0}{\gamma_\perp\Pi}(\alpha^+)^4\delta(t-t')=D_{V^+V^+}\delta(t-t')\,,\\
\langle\Gamma_V(t)\Gamma_D(t')\rangle&=-\frac{2\omega_{12}gD_0}{\gamma_\perp\Pi}\alpha^2\delta(t-t')=D_{VD}\delta(t-t')\,,\\
\langle\Gamma_{V^+}(t)\Gamma_D(t')\rangle&=-\frac{2\omega_{12}gD_0}{\gamma_\perp\Pi}(\alpha^+)^2\delta(t-t')=D_{V^+D}\delta(t-t')\,,\\
\langle\Gamma_D(t)\Gamma_D(t')\rangle&=\left[2\gamma_{\|}N-2(\omega_{12}-\omega_{21})\frac{D_0}{\Pi}-\frac{4g^2D_0}{\Pi\gamma_\perp^2}2\gamma_\perp|\alpha|^4\right]\delta(t-t')=D_{DD}\delta(t-t')
\end{aligned}\tag{27}$$

for the two-photon laser and

$$\langle \Gamma_\alpha(t)\Gamma_\alpha(t')\rangle = -\frac{2g^2N}{\gamma_\perp \Pi}\alpha^2\delta(t-t') = D_{\alpha\alpha}\delta(t-t') ,$$

$$\langle \Gamma_{\alpha^+}(t)\Gamma_{\alpha^+}(t')\rangle = -\frac{2g^2N}{\gamma_\perp \Pi}(\alpha^+)^2\delta(t-t') = D_{\alpha^+\alpha^+}\delta(t-t') ,$$

$$\langle \Gamma_{V^+}(t)\Gamma_V(t')\rangle = \frac{4\gamma_p N g^2}{\Pi\gamma_{\|}\gamma_\perp}|\alpha|^4\delta(t-t') = D_{V^+V}\delta(t-t') , \tag{28}$$

$$\langle \Gamma_{V^+}(t)\Gamma_{V^+}(t')\rangle = -\frac{2g^2N}{\gamma_\perp \Pi}(\alpha^+)^4\delta(t-t') = D_{V^+V^+}\delta(t-t') ,$$

$$\langle \Gamma_V(t)\Gamma_V(t')\rangle = -\frac{2g^2N}{\gamma_\perp \Pi}\alpha^4\delta(t-t') = D_{VV}\delta(t-t') ,$$

$$\langle \Gamma_D(t)\Gamma_D(t')\rangle = \frac{16g^2N}{\Pi\gamma_\perp}|\alpha|^4\delta(t-t') = D_{DD}\delta(t-t')$$

for two-photon bistability. We have chosen to ignore thermal noise n_{th}.

Defining the functions S and $\mathcal{X}$ as

$$\begin{aligned}\langle F(t)F(t')\rangle &= 2\mathcal{X}\alpha^2\delta(t-t') ,\\ \langle F^+(t)F(t')\rangle &= S|\alpha|^2\delta(t-t') ,\end{aligned} \tag{29}$$

and the Fokker-Planck equation equivalent to the Langevin equation, (25) is written as

$$\frac{\partial P}{\partial t} = \left\{ \left[\frac{\partial}{\partial\alpha}\left[K(E-\alpha) + \frac{2KC\alpha|\alpha|^2}{n_0\left[1+\frac{|\alpha|^4}{n_0}\right]} \right] + \text{c.c.} \right] + \frac{\partial^2}{\partial\alpha\,\partial\alpha^+}S|\alpha|^2 + \frac{\partial^2}{\partial\alpha^2}\mathcal{X}\alpha^2 + \frac{\partial^2}{\partial(\alpha^+)^2}\mathcal{X}(\alpha^+)^2 \right\} P . \tag{30}$$

Note that S and $\mathcal{X}$ are real functions of $|\alpha|^2$, and that only $\mathcal{X}$ contains the unique two-photon term.

The thermal noise in the field n_{th} has been set equal to zero in Eq. (30). Thus, Eq. (30) represents the best possible situation for squeezing and in any practical device thermal fluctuations would act as a further counter to squeezing.

IV. STEADY-STATE SOLUTIONS

We now look for steady-state solutions to Eq. (30) using the method of potentials. In the steady state, time derivatives are zero, and we wish to solve an equation of the form

$$\frac{\partial}{\partial x_j}\left[-A_i(\vec{x})P(\vec{x}) + \frac{1}{2}\frac{\partial}{\partial x_j}D_{ij}(\vec{x})P(\vec{x}) \right] = 0 . \tag{31}$$

This is satisfied if

$$D_{ij}(\vec{x})\frac{\partial}{\partial x_j}\ln P = 2A_i(\vec{x}) - \frac{\partial}{\partial x_j}D_{ij}(\vec{x}) . \tag{32}$$

Denoting

$$P(\vec{x}) = \exp[-\phi(\vec{x})] ,$$

we wish to solve

$$-\frac{\partial\phi}{\partial x_j}(\vec{x}) = 2D_{ij}^{-1}(\vec{x})\left[A_j(\vec{x}) - \frac{1}{2}\frac{\partial D_{jk}}{\partial x_k} \right] = F_j(\vec{x}) . \tag{33}$$

The system of equations can be solved by integration if the following "potential conditions" are satisfied:

$$\frac{\partial F_j(\vec{x})}{\partial x_i} = \frac{\partial F_i(\vec{x})}{\partial x_j} . \tag{34}$$

A. $E=0$. Two-photon laser

The first special case to consider is no injected signal, $E=0$. The two functions F_α, F_{α^+} are given as follows:

$$F_\alpha = \frac{f_1(I)}{\alpha} + f_2(I)\alpha^+ , \quad I = \alpha^+\alpha \tag{35}$$

and F_{α^+} is obtained by interchanging α and α^+. f_1 and f_2 are functions of intensity I only and, for brevity, are not written out explicitly. Potential conditions are satisfied, since

$$\frac{\partial F_\alpha}{\partial \alpha^+}=f_1'(I)+f_2(I)+If_2'(I)=\frac{\partial F_{\alpha^+}}{\partial \alpha} . \quad (36)$$

Thus, the potential solution found from integrating the following exists:

$$\frac{\partial \phi}{\partial \alpha}=F_\alpha, \quad \frac{\partial \phi}{\partial \alpha^+}=F_{\alpha^+} . \quad (37)$$

To determine ϕ, we need the following integrals:

$$\int f_1 \frac{(\alpha^+\alpha)}{\alpha} d\alpha+\text{c.c.}=\int \frac{f_1(I)}{I} dI ,$$
$$\int f_2(\alpha^+\alpha)\alpha^+ d\alpha+\text{c.c.}=\int f_2(I)dI . \quad (38)$$

A close inspection reveals that ϕ is a function of intensity I only. Hence, one can say immediately that no squeezing is possible.

B. $E\neq 0$. Two-photon laser with injected signal or two-photon optical bistability

Still in search of squeezing, an external phase is introduced into the system in the form of an injected signal E. For this case,

$$F_\alpha=\frac{f_1(I)}{\alpha}+f_2(I)\alpha^+ -\frac{4X\mathcal{E}}{(4X^2-S^2)\alpha^2}-\frac{2S\mathcal{E}}{(4X^2-S^2)\alpha^+\alpha} \quad (39)$$

and F_{α^+} is obtained by interchanging α and α^+. The potential condition is not satisfied, except in the limit for which X is independent of I and dominates over S. However, this is a nonphysical limit since terms (S_0 and X_0), for example, of S and X which are independent of I are related as follows:

$$S_0=4X_0+\frac{4g^2N}{\gamma_\perp} ,$$
$$S_0=\frac{4g^2N\omega_{12}}{\gamma_\perp^2} , \quad (40)$$
$$X_0=\frac{g^2D_0}{\gamma_\perp} .$$

It is still possible to obtain limited information regarding the Fokker-Planck equation (30) by linearizing about a stable semiclassical steady state.

V. LINEARIZED ANALYSIS

Expressions for the statistics of the field in the limit of small fluctuations may be obtained by linearizing the Langevin equation (25) or the Fokker-Planck equation (30). This procedure is justified since the original scaling (Sec. II) of variables showed second-order derivative terms of Eq. (11) to be of order $1/N$ higher than the zeroth-order drift terms. Thus, in the limit of a large number N of atoms, the effect of noise will be small, and we may expand about a stable steady-state deterministic solution. The linear theory will apply only to regions other than threshold.

A. Deterministic steady-state results

The deterministic steady-state solutions[31] are summarized: The deterministic equation, in which all fluctuations are ignored, is written

$$\dot\alpha=\mathcal{E}-\alpha f(I), \quad I=|\alpha|^2$$
$$f(I)=K\left[1-\frac{2CI}{n_0[1+(I^2/n_0^2)]}\right] , \quad (41)$$

where the sign of C changes [in accordance with the changed definition (13)] for the optical bistability case. The steady-state solution, or state equation, is

$$\mathcal{E}=\alpha f(I) . \quad (42)$$

The stability of the state equation is determined by standard linearization procedure. We substitute $\alpha=\alpha_0+\delta\alpha$ in Eq. (41), where α_0 satisfies (42). The result is

$$\frac{d}{dt}\begin{bmatrix}\delta\ \alpha\\ \delta\ \alpha^+\end{bmatrix}=-\underline{A}(\alpha_0)\begin{bmatrix}\delta\ \alpha\\ \delta\ \alpha^+\end{bmatrix} ,$$
$$\underline{A}(\alpha)=\begin{bmatrix}If'(I)+f(I) & \alpha^2 f'(I)\\ (\alpha^+)^2 f'(I) & If'(I)+f(I)\end{bmatrix} . \quad (43)$$

The conditions for stability are

$$\text{Tr}\underline{A}>0 , \quad (44a)$$

$$\det\underline{A}=f(I)[2If'(I)+f(I)]>0 . \quad (44b)$$

These criteria imply $f(I)>0$, $f(I)+2If'(I)>0$, and $f(I)+If'(I)>0$. Note that for the laser ($E=0$), $\det\underline{A}=0$ and the linear theory presented in Sec. V B diverges.

B. Linearized fluctuation analysis

The effect of fluctuations can be estimated by linearizing about a stable solution α_0 of the state equation (42). To first order we write for the fluctuations $\delta\alpha = \alpha - \alpha_0$,

$$\frac{d}{dt}\begin{bmatrix}\delta\ \alpha \\ \delta\ \alpha^+\end{bmatrix} = -\underline{A}(\alpha_0)\begin{bmatrix}\delta\ \alpha \\ \delta\ \alpha^+\end{bmatrix} + \underline{D}^{1/2}(\alpha_0)\mathscr{E}_i(t)\ , \tag{45}$$

where $\mathscr{E}_i(t)$ are delta-correlated random Gaussian functions, $\underline{A}$ is the linearized drift as in Eq. (43), which we abbreviate as

$$\underline{A}(\alpha_0) = \begin{bmatrix} -A & B\alpha_0^2 \\ B(\alpha_0^+)^2 & -A \end{bmatrix},$$

$$B = f'(I), \quad -A = If'(I) + f(I)\ , \tag{46}$$

$$I = |\alpha_0|^2\ .$$

$\underline{D}(\alpha_0)$ is the diffusion array evaluated at α_0:

$$\underline{D}(\alpha_0) = \begin{bmatrix} 2\chi\alpha_0^2 & S|\alpha_0|^2 \\ S|\alpha_0|^2 & 2\chi(\alpha_0^+)^2 \end{bmatrix},$$

where S and χ, defined by Eq. (29), are functions of I. The correlation matrix is deduced[46–48]:

$$\underline{C} = \begin{bmatrix} \langle a^2\rangle - \langle a\rangle^2 & \langle a^\dagger a\rangle - |\langle a\rangle|^2 \\ \langle a^\dagger a\rangle - |\langle a\rangle|^2 & \langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2 \end{bmatrix} = \begin{bmatrix} \langle(\delta\alpha)^2\rangle & \langle\delta\alpha^+\delta\alpha\rangle \\ \langle\delta\alpha^+\delta\alpha\rangle & \langle(\delta\alpha^+)^2\rangle \end{bmatrix}$$

$$= \frac{\underline{D}\det\underline{A} + [\underline{A} - \underline{I}\,\mathrm{Tr}\underline{A}]\underline{D}[\underline{A}^T - \underline{I}\,\mathrm{Tr}\underline{A}]}{2\,\mathrm{Tr}\underline{A}\det\underline{A}} = \frac{1}{2\det\underline{A}}\begin{bmatrix} (-2\chi A - BSI)\alpha_0^2 & -SIA - 2\chi BI^2 \\ -SIA - 2\chi BI^2 & (-2\chi A - BSI)(\alpha_0^+)^2 \end{bmatrix}, \tag{47}$$

where stability demands $\det\underline{A} > 0$, $-A > 0$.

To first order, the intensity fluctuations $g^2(0)$ are calculated

$$g^2(0) - 1 = \frac{2}{I}\left[\langle\delta\alpha^+\delta\alpha\rangle + \mathrm{Re}\left[\alpha_0^*\frac{\langle(\delta\alpha)^2\rangle}{\alpha_0}\right]\right]$$

$$= \frac{2\chi + S}{f(I) + 2If'(I)}\ . \tag{48}$$

Of more immediate interest is the squeezing. Defining

$$X_1 = \frac{a + a^\dagger}{2}, \quad X_2 = \frac{a - a^\dagger}{2i}\ ,$$

the fluctuations in the quadratures are

$$(\Delta X_1)^2 - \tfrac{1}{4} = 2(\langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle) + (\langle a^2\rangle - \langle a\rangle)^2 + (\langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2)\ ,$$

$$(\Delta X_2)^2 - \tfrac{1}{4} = 2(\langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle) - (\langle a^2\rangle - \langle a\rangle^2) - (\langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2)\ . \tag{49}$$

Squeezing exists if one of the variances becomes less than $\frac{1}{4}$. We choose the coherent field E to be real. This implies that α_0 is also real (only in-phase solutions are stable[31]). The variances become

$$(\Delta X_1)^2 - \tfrac{1}{4} = \frac{I(S + 2\chi)}{4[f(I) + 2If'(I)]}\ ,$$

$$(\Delta X_2)^2 - \tfrac{1}{4} = \frac{I(S - 2\chi)}{4f(I)}\ . \tag{50}$$

Since both $f(I)$ and $[f(I) + 2If'(I)]$ are positive, squeezing will exist if either $S - 2\chi$ or $S + 2\chi$ are negative. If $S + 2\chi$ is negative, antibunching is also present.

C. Laser with injected signal

For the laser, an inversion exists and D_0 is positive. From Eqs. (26), (27), and (29), expressions for $S + 2\chi$ and $S - 2\chi$ are derived. It may be shown that $S + 2\chi$, and hence ΔX_1^2 and $[g^2(0) - 1]$, is always positive. $S - 2\chi$ simplified as follows:

$$S - 2\chi = \frac{-D_{\alpha\alpha}}{\alpha^2} + D_{vv} + \frac{4g^2}{\gamma_\perp^2} - \frac{4g^2(\alpha^+)^2}{\gamma_\perp^2\alpha^2}D_{VV}$$

$$= \frac{-2g^2D_0}{\gamma_\perp\Pi} + \left[N\omega_{12} + N\gamma_p + \gamma_p\frac{D_0}{\Pi}\right]\frac{4g^2}{\gamma_\perp^2} - \frac{2g^2D_0}{\gamma_\perp\Pi}\frac{4g^2}{\gamma_\perp^2}|\alpha|^4\ . \tag{51}$$

The fluctuation $D_{\alpha\alpha}$, which originated purely as a result of the quadratic nature of the field-atom Hamiltonian, has appeared as the first term of Eq. (51). It is negative and by itself would produce a squeezing in the imaginary direction X_2. Also negative, and thus tending to cause squeezing, is the term proportional to D_{VV}. It has originated also from the field-atom Hamiltonian but is not unique to the multiphoton laser. In addition, there is the term, which is positve and opposes squeezing, arising from the atom-reservoir interaction Hamiltonian. There are two contributing parts. One is proportional to ω_{12}, the incoherent pumping rate, and describes spontaneous emission. It is this noise which is dominant, and thus destroys the possibility of squeezing, in the one-photon laser.[34] The second term is proportional to γ_p, the rate of collisional damping. Defining $f=\gamma_{\parallel}/2\gamma_{\perp}$ [and $\gamma_p=\gamma_{\perp}(1-f)$], a value of $f=1$ indicates pure radiative damping and $f=0$ is the collisional limit. Manipulation of Eq. (51) reveals that the final result is independent of f

$$S-2\chi=\frac{2g^2D_0}{\gamma_{\perp}\Pi}+\frac{4g^2N}{\gamma_{\perp}} \tag{52}$$

and for the two-photon laser ($D_0=N$) is clearly positive and hence excludes the possibility of squeezing. The spontaneous-emission noise proportional to ω_{12} has washed out any squeezing that might otherwise have been present as a result of the quadratic Hamiltonian. Expressing the result for the variance in terms of the scaled variables of Eq. (15) we find

$$(\Delta X_2)^2-\tfrac{1}{4}=\frac{|x|^2(2|x|^4+3)}{4\lambda\frac{y}{x}(1+|x|^4)}\ , \tag{53}$$

where

$$\lambda=\frac{K}{2gN}\left[\frac{\gamma_{\perp}}{\gamma_{\parallel}}\right]^{1/2} .$$

This result is in exact agreement with the recent work of Lugiato and Strini.[40] The exact agreement of the expressions is due to the linearized fluctuation analysis used here which is equivalent to the Gaussian factorization method used by Lugiato and Strini. Where a potential solution can be found, as for the two-photon laser without injected signal, the Fokker-Planck method enables fluctuations to be included to all orders.

D. Two-photon optical bistability

We consider $\omega_{12}=0$ and $D_0=-N$ the two-photon absorptive bistability limit. Using the scaled variables of Eq. (15), we find

$$\begin{aligned}S-2\chi&=\frac{2KC}{\pi n_0}(1+2|x|^4)\ ,\\ S+2\chi&=\frac{-2KC}{n_0\Pi}+\frac{4KC|x|^4(1-2f)}{n_0\Pi}+\frac{8KC|x|^8(2f-1)}{\pi^2n_0}+\frac{8KC|x|^8}{\pi^3n_0}+\frac{4KC|x|^{12}(1-2f)}{\pi^3n_0}\ .\end{aligned} \tag{54}$$

The linearized analysis predicts the following results for a real coherent driving field:

$$\begin{aligned}g^2(0)-1&=\frac{2C(-1-4f|x|^4+3|x|^8)}{n_0(1+6C|x|^2+3|x|^4+4C|x|^6+3|x|^8-2C|x|^{10}+|x|^{12})}\ ,\\ (\Delta X_1)^2-\tfrac{1}{4}&=\frac{C|x|^2(-1-4f|x|^4+3|x|^8)}{2(1+6C|x|^2+3|x|^4+4C|x|^6+3|x|^8-2C|x|^{10}+|x|^{12})}\ ,\\ (\Delta X_2)^2-\tfrac{1}{4}&=\frac{C|x|^2(1+2|x|^4)}{2(1+|x|^4+2C|x|^2)}\ .\end{aligned} \tag{55}$$

Once again these results are identical to those obtained by Lugiato and Strini. The expression for the intensity fluctuations $[g^2(0)-1]$ is inversely proportional to n_0, which determines the photon number within the cavity at the threshold of nonlinearity. For large enough n_0, fluctuations are small and this is consistent with our approach.

It is evident that three can be no squeezing in the imaginary direction X_2. However, squeezing is possible in the real direction X_1, for the same conditions that antibunching exists. This is because the spontaneous-emission noise proportional to ω_{12}, which drowned squeezing in the two-photon laser, is no longer present. The term originating due to the quadratic nature of the Hamiltonian dominates at low intensities. As intensity is increased, other quantum fluctuations become important and squeezing (and antibunching) is destroyed. The crossover point depends on f and is given by

$$|x|^2=\frac{1}{3}[2f+(4f^2+3)^{1/2}]^{1/2}\,. \qquad (56)$$

As the linear theory described here cannot accurately predict the threshold region, Eq. (56) means, in practice, that squeezing and antibunching exist at low saturation (the lower branch) but not at high saturations (the upper branch).

It is interesting to examine the behavior of Eq. (30), (48), or (50) in the mathematical limit of zero spontaneous emission, $S=0$ and

$$\chi=\frac{g^2D_0}{\gamma_\perp\Pi}\,. \qquad (57)$$

Since, for a laser, D_0 and hence χ are positive, the linearized analysis predicts squeezing in the imaginary quadrature X_2. Having chosen the coherent excitation to be real, the direction of squeezing is consistent with the prediction of photon bunching,[20,49] χ acting as a squeeze parameter. Allowing χ to change sign ($D_0=-N$) and ignoring saturation ($|\alpha|^2$ small) the equations describe a two-photon absorber. The linearized analysis now predicts photon antibunching and squeezing in the real quadrature X_1, for E real. This is in agreement with exact steady-state statistics previously calculated,[20] for an unsaturable two-photon absorber.

VI. CONCLUSION

The two-photon laser has been modeled as N two-level atoms in an optical cavity interacting, via a simple two-photon transition, with a single resonant-cavity mode. Following Haken's procedure for describing quantum-mechanical effects in a one-photon laser, a Fokker-Planck equation for the field alone is arrived at. The equation has several leading noise terms. One originates from the quadratic nature of the atom-field Hamiltonian, while others result from the atom-reservoir interaction and describe spontaneous emission. The steady-state solution of the Fokker-Planck equation in the two-photon laser limit is a function of intensity only and can predict no squeezing. A phase dependence is introduced into the system by injecting a resonant coherent signal into the cavity. A linearized analysis about a stable steady state reveals that squeezing is possible only in a mathematical limit where one can suitably neglect spontaneous emission. Since the spontaneous emission is dominant where there is a nonzero incoherent pumping causing an inversion of atoms, our results predict squeezing is not present in the present model of the two-photon laser with injected signal. It should be noted, however, that the effective two-level model is an approximation to the three-level situation in an actual two-photon laser. While the effective two-level model is a good approximation for the intermediate level well detuned from the laser transition, the effect of the quantum fluctuations have not yet been checked.

Our results indicate that for a two-photon absorber, where spontaneous emission is not present, squeezing is possible at relatively low saturations, that is, below the threshold of absorptive bistability.

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