Longitudinal coherence and interferometry in dispersive media

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(Received 19 August 1983)

For the nondispersive propagation of waves in one dimension, there is no essential difference between the concepts of longitudinal and temporal coherence, and these terms are often used interchangeably. For the dispersive propagation of waves, such as electrons or neutrons *in vacuo*, or light in a medium, this is no longer the case. We analyze the coherence properties of a dispersively propagating beam of radiation, and discuss the observation of its spatial and temporal coherence properties by means of a two-beam interferometer.

I. INTRODUCTION

Recent experiments in neutron and electron optics[1-3] have brought to prominence phenomena in interferometry which are not adequately described by the standard discussions of coherence in optics.[4-6] The standard analyses deal only with the dispersion-free vacuum propagation of light which, in one dimension, is characterized by a wave-form-preserving translation at uniform speed, and thus makes longitudinal (spatial) and temporal coherence properties essentially equivalent.

This equivalence does not apply to the dispersive propagation of, say, electrons or neutrons *in vacuo*, nor to light in a material medium. Further, very few experiments, if any, have ever studied temporal coherence properties directly, as this would require the artificial insertion of a variable time delay in some part of the recording process. This is seldom practicable.[7] Nevertheless, we indicate below how a two-beam, e.g., Michelson or Mach-Zehnder interferometer, with a dispersive cell in one arm and vacuum in the other, may effectively study the temporal (as well as spatial) coherence of electromagnetic waves.

In addition, we draw attention to the fact[3] that, in spite of the growth of the size of wave packets under dispersive propagation,[8] the coherence length, as measured by an interferometer, does not depend on its overall position downstream in the beam.

II. THEORY

Let $\psi(x,t)$ be the wave amplitude at position x and time t for whatever type of radiation we wish to analyze. The wave equation obeyed by $\psi(x,t)$ will be denoted by

$$W\{x,t\}\psi(x,t)=0 \quad . \tag{1}$$

For example, for light *in vacuo*,

$$W\{x,t\}\equiv\frac{\partial^2}{c^2\partial t^2}-\frac{\partial^2}{\partial x^2} \quad ; \tag{2a}$$

for nonrelativistic particles of mass m,

$$W\{x,t\}\equiv i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \quad ; \tag{2b}$$

and for light in a dispersive medium the Fourier transform of W is given by

$$W\{k,\omega\}\equiv[n(\omega)\omega/c]^2-k^2 \quad . \tag{2c}$$

The mutual coherence function Γ is defined by

$$\Gamma(xt;x't')\equiv\langle\psi(x,t)\psi^*(x',t')\rangle \quad , \tag{3}$$

where $\langle\ \rangle$ denotes an ensemble average. It follows, from the restriction that W is a linear operator, that

$$W\{x,t\}\Gamma(xt;x't')=0 \quad . \tag{4}$$

In other words, the fact that the mutual coherence function obeys the same wave equation as the amplitude function[4-6] is generally true for any linear wave equation. In particular, if ψ develops dispersively in time, then Γ will develop dispersively also.

In the light of this, and using a simple-minded picture of spreading wave packets, one could easily be misled into thinking that the coherence length will also spread upon propagation in a dispersive medium. This, however, is not the case, as will be shown below.

If the system is stationary in time, the coherence function Γ will depend on T, where

$$T=t-t' \quad . \tag{5a}$$

If the system is translationally invariant the coherence function Γ will depend on

$$X=x-x' \quad . \tag{5b}$$

We shall restrict our attention to translationally invariant, time stationary systems. We note that translational invariance precludes (significant) absorption.

For such systems we have

$$\begin{aligned}\Gamma(X,T)&\equiv\Gamma(x'+X,t'+T;x',t')\\&=\langle\psi(x'+X,t'+T)\psi^*(x',t')\rangle \quad .\end{aligned} \tag{6}$$

Let us denote $\psi(x,t)$ by a Fourier integral

$$\psi(x,t)=\int A(\omega)\exp\{i[k(\omega)x-\omega t]\}d\omega \quad . \tag{7}$$

If we use the stationarity requirement and perform a finite time average and take the limit, in the usual way (Wiener-Kintchine theorem), we may write the mutual coherence function in terms of the spectral distribution function:

$$\Gamma(X,T)=\int_{-\infty}^{\infty} S(\omega)\exp\{i[k(\omega)X-\omega T]\}d\omega \ , \tag{8}$$

where

$$S(\omega)=2\pi\langle|A(\omega)|^2\rangle \ . \tag{9}$$

We see, from Eq. (8), that the spatial and temporal coherence properties of the field are completely determined by the spectral distribution function $S(\omega)$ and have no direct relation to the size and duration of the possible wave packets that comprise the field. This will be illustrated below in connection with a two-beam interferometer.

In the case of nondispersive propagation

$$k(\omega)=\omega/c \ , \tag{10}$$

where c is the constant phase velocity, we have

$$\Gamma(X,T)=\int_{-\infty}^{\infty} S(\omega)\exp[i\omega(X/c-T)]d\omega$$

$$\equiv\Gamma(X-cT,0)\equiv\Gamma(0,T-X/c) \ . \tag{11}$$

Thus the coherence function is a function of the single variable $(X-cT)$, and is completely determined by its spatial or temporal dependence alone, i.e., space and time displacement intervals play an equivalent role.

In the case of dispersive propagation, $k(\omega)$ is not linear in ω, and the equivalence of space and time intervals, manifest in Eq. (11), does not hold. In the dispersive case, the roles of X and T are distinct.

III. APPLICATIONS TO INTERFEROMETERS

If the wave field before a two-beam interferometer is given by $\psi(x,t)$, the wave field after passing through the interferometer is given by

$$\psi_M(x,t)=\tfrac{1}{2}[\psi(x,t)+\psi(x+X,t)] \ , \tag{12}$$

where X is the path difference between the interferometer arms. The average output intensity thereafter is given by

$$I=\tfrac{1}{2}[\Gamma(0,0)+\mathrm{Re}\Gamma(X,0)] \ . \tag{13}$$

If we write the complex degree of coherence as

$$\gamma(X,T)\equiv\Gamma(X,T)/\Gamma(0,0) \tag{14}$$

and

$$I_0=\tfrac{1}{2}\Gamma(0,0) \ , \tag{15}$$

then

$$I=I_0[1+\mathrm{Re}\gamma(X,0)] \ . \tag{16}$$

The magnitude of $\gamma(X,0)$ corresponds to Michelson's fringe visibility function

$$V(X)\equiv\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}=|\gamma(X,0)| \ . \tag{17}$$

It will be noted that the coordinate x, which appears in Eq. (12), does not appear in Eqs. (16) or (17). The coordinate x, which measures the distance downstream from the source to the interferometer, and hence influences the overall size of the dispersing wave packet, plays no role in determining the time-averaged interferometer output. The coherence length, on the other hand, is determined by the range of X for which $|\gamma(X,0)|$ is significantly different from zero. Thus coherence length is independent of wave-packet size. It may be worth emphasizing that, in general, $|\gamma(X,T)|$ for $T\neq 0$ is not measurable in a simple interferometer, employing one detector. However, in the case of radiation for which nondispersive propagation is possible, such as electromagnetic waves *in vacuo*, a time delay T may be simulated by an additional nondispersive path difference. Such is not the case for beams of material particles for which even the vacuum acts as a dispersive medium, by virtue of the Schrödinger equation (2b).

For the case of light and other electromagnetic waves the behavior of $|\gamma(X,T)|$ for $T\neq 0$ may be studied as follows:

Consider the inclusion of a dispersive cell in the nonvariable arm of a Michelson interferometer, or in one arm of a Mach-Zehnder interferometer. Let D be the length of path that light traverses in this dispersive cell.

In terms of the Fourier decomposition of the incident wave $\psi(x,t)$ given in Eq. (7), the interferometer output wave will now be

$$\Psi_I(x,t)=\int A(\omega)\exp[i\omega(x/c-t)]\tfrac{1}{2}(\exp(i\omega X/c)+\exp\{i[k(\omega)-\omega/c]D\})d\omega \ , \tag{18}$$

where $k(\omega)$ is the dispersion relation for the included medium. We now find that the time-averaged intensity function is given by

$$I=\int S(\omega)\tfrac{1}{2}(1+\mathrm{Re}\exp\{i[k(\omega)D-\omega(X+D)/c]\})d\omega \tag{19}$$

$$=\tfrac{1}{2}[\Gamma(0,0)+\mathrm{Re}\Gamma(D,(X+D)/c)]$$

$$=I_0[1+\mathrm{Re}\gamma(D,(X+D)/c)] \ . \tag{20}$$

Thus the temporal coherence of propagation in the dispersive medium may be analyzed by varying the nondispersive drift path $(X+D)$.

We shall evaluate Eqs. (19) and (20) for the case of a narrow spectral distribution over which the wave number $k(\omega)$ changes only slowly. Let ω_0 be the center frequency of the distribution function $S(\omega)$. Then we may expand

$$k(\omega)=k(\omega_0)+(\omega-\omega_0)\left(\frac{dk}{d\omega}\right)_0+\tfrac{1}{2}(\omega-\omega_0)^2\left(\frac{d^2k}{d\omega^2}\right)_0+\cdots$$

$$=\omega_0/u_0+(\omega-\omega_0)/v_0+\tfrac{1}{2}(\omega-\omega_0)^2\left(\frac{d^2k}{d\omega^2}\right)_0+\cdots \ , \tag{21}$$

where $u_0\equiv\omega/k(\omega_0)$ is the phase velocity and $v_0\equiv(dk/d\omega)_0^{-1}$ is the group velocity.

Let us evaluate $\Gamma(X,T)$ using only the first and second terms of Eq. (21). In this case

$$\Gamma(X,T)=\exp[i\omega_0(X/u_0-T)]\int S(\omega)\exp[i(\omega-\omega_0)(X/v_0-T)]d\omega=\exp[i\omega_0(X/u_0-T)]F(X/v_0-T) \ . \tag{22}$$

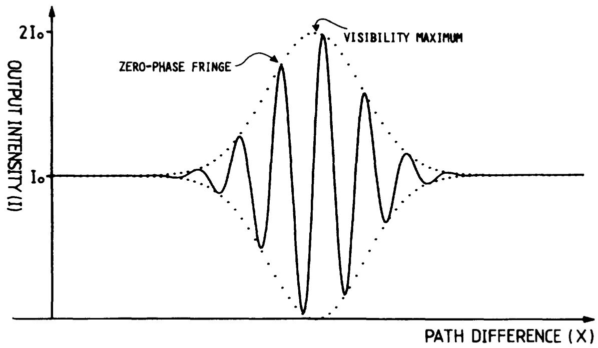

FIG. 1. Output of a Michelson interferometer with fixed dispersive path difference D vs the variable vacuum path difference X, Eq. (20), for the Gaussian spectral distribution, Eq. (24).

If $S(\omega)$ is symmetric about ω_0 then F is real. In that case we see from Eq. (20) that the interferometer fringe spacing is determined by the phase factor in Eq. (22) and hence by the *phase* velocity u_0. We see further from Eqs. (22) and (17) that F alone determines the visibility function which, in turn, depends only on the *group* velocity v_0.

Applying Eq. (22) to the dispersive cell interferometer situation given in Eq. (20), we find the explicit X and D dependence

$$I = I_0\left\{1 + \cos\omega_0\left[D\left(\frac{1}{u_0}-\frac{1}{c}\right)-\frac{X}{c}\right]F\left[D\left(\frac{1}{v_0}-\frac{1}{c}\right)-\frac{X}{c}\right]\right\} . \tag{23}$$

It is instructive to consider the case of a Gaussian spectral intensity distribution

$$S(\omega) = I_0\frac{\exp[-(\omega-\omega_0)^2/2(\Delta\omega)^2]}{[2\pi(\Delta\omega)^2]^{1/2}} . \tag{24}$$

In Fig. 1 we illustrate Eq. (20) for the case of the spectrum of Eq. (24). Note the differential displacement between the zero-phase fringe and the center of the visibility envelope.

We now evaluate the complex degree of coherence γ using the Gaussian spectral distribution of Eq. (24) to find

$$\gamma(X,T) = \frac{\exp[i\omega_0(X/u_0-T)]}{[1-i(\Delta\omega)^2(d^2k/d\omega^2)_0X]^{1/2}} \times\exp\left[-\frac{\frac{1}{2}(\Delta\omega)^2(X/v_0-T)^2}{1-i(\Delta\omega)^2(d^2k/d\omega^2)_0X}\right] . \tag{25}$$

[In Eq. (25) the quadratic terms of Eq. (21) have been retained.]

In Fig. 2 we display Eq. (20) as a function of frequency ω_0 using the above form of $\gamma(X,T)$ for the case of a dispersion relation corresponding to a single optical resonance in a

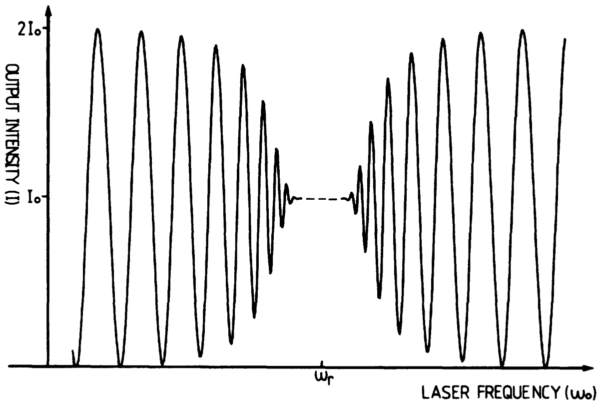

FIG. 2. Output intensity of a dispersive cell Michelson interferometer as a function of ω_0, center frequency of the Gaussian spectral distribution, in the vicinity of a single optical resonance at ω_r.

dilute gas, namely,

$$k = \omega/c + \pi r_e cNf/(\omega_r - \tfrac{1}{2}i\epsilon_r - \omega) , \tag{26}$$

where N is the particle density, f is the oscillator strength of the resonance, ω_r is the resonance position, ϵ_r is the resonance width, and r_e is the classical radius of the electron. This corresponds closely to the experimental observations of Duval and McIntosh,[9,10] who used a Cs vapor cell in a Mach-Zehnder interferometer, illuminated by a turnable dye laser.

The disappearance of fringe visibility as ω_0 approaches ω_r is due to the factor $[1-i(\Delta\omega)^2(d^2k/d\omega^2)_0X]^{1/2}$ of Eq. (25) which becomes significant when $d^2k/d\omega^2$ becomes large, as it does in the neighborhood of a resonance. (Note that the quadratic approximation to the dispersion relation is invalid for $|\omega_0-\omega_r| \leq \epsilon_r, \Delta\omega$.)

IV. CONCLUSION

We conclude that for propagation in dispersive media, the distinction between longitudinal and temporal coherence is significant. It is experimentally accessible in the case of electromagnetic radiation, and is useful in the interpretation of interferometric data.

ACKNOWLEDGMENTS

We are indebted to Dr. McIntosh and Dr. Duval for bringing their experimental data to our attention prior to publication and to Professor C. G. Shull and Professor A. Zeilinger for their careful reading of the manuscript. This work was supported by the Australian Research Grants Scheme.

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