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Resistive and viscous convection in a cylindrical plasma

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It is shown that the combined effect of resistivity, viscosity, and thermal conductivity leads to large-scale stationary convection. This is done for a cylindrical current-carrying plasma in a shearless magnetic field. Convection is shown to take place for large wave numbers and $B_\theta/B_z >> 1$.

It has been recently shown that, in a current-carrying cylindrical plasma, viscosity and thermal conductivity lead to stationary convection.[1,2] It has also been shown that, in such systems, resistivity and thermal conductivity also lead to convection.[3] In the first case, convection occurs for large wave numbers and $(B_\theta/B_z) >> 1$. In the second case, it was shown that convection takes place for small wave numbers and, in the tokamak limit, $(B_\theta/B_z) << 1$.

The stability of a current-carrying plasma column limited by conducting walls in a shearless magnetic field has been extensively studied.[4–8] The first analytical results are due to Tayler[4] and Shafranov[5] who solved the ideal magnetohydrodynamic (MHD) equations assuming incompressible motion. It turns out, however, that there is no linear solution of the ideal MHD equations close to $k_{\parallel} = \vec{k}\cdot\vec{B}^{(0)} = 0$.[2,9] However, viscosity has been shown to remove the singularity at $k_{\parallel}=0$, giving rise to three marginal modes for each m value,[1,2] as illustrated in Fig. 1.

It will now be shown that the combined effect of resistivity, viscosity, and thermal conductivity gives rise, for each m value, to four states which, under some conditions, trigger large-scale stationary convection in the plasma.

The equations describing the system are

$$\left(\rho\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}\right)\vec{v}=\vec{J}\times\vec{B}-\vec{\nabla}p-\mu_\perp\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) , \tag{1a}$$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0 , \tag{1b}$$

$$\frac{\partial p}{\partial t}+\vec{v}\cdot\vec{\nabla}p-\tfrac{2}{3}\kappa\nabla^2 p-S_0=-\gamma p\vec{\nabla}\cdot\vec{v}+\tfrac{2}{3}\eta\vec{J}^2 , \tag{1c}$$

$$\vec{E}+\vec{v}\times\vec{B}=\eta\vec{J} , \tag{1d}$$

$$\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}=\vec{0} , \tag{1e}$$

$$\vec{\nabla}\cdot\vec{B}=0 , \tag{1f}$$

$$\vec{\nabla}\times\vec{B}=4\pi\vec{J} . \tag{1g}$$

In these equations, η is the resistivity, γ the adiabaticity coefficient, S_0 a constant heat source required to maintain the equilibrium pressure profile, and $\mu_\perp$ the perpendicular part of the viscosity tensor. The other parts will be shown to be unimportant.

The system consists of a cylindrical plasma column bounded by conducting walls. The equilibrium is characterized by

$$\vec{B}^{(0)}=B_l\left(\frac{r}{a}\right)^2\hat{e}_\theta+B_0\hat{e}_z , \quad p^{(0)}=p_0-\frac{B_l}{4\pi}\left(\frac{r}{a}\right)^2 , \tag{2}$$

where B_0, B_l, and p_0 are constants and a is the radius of the cylinder. The rotational transform is constant and, therefore, the magnetic field is shearless:

$$q=\frac{2\pi rB_z^{(0)}}{LB_\theta^{(0)}}=\frac{2\pi aB_0}{LB_l} , \tag{3}$$

where L is the length of the cylinder.

Assuming a nearly constant density, $\rho\simeq\rho_0$, and linearizing Eqs. (1) for perturbations of the form

$$f^{(1)}(\vec{r},t)=f^{(1)}(r)\exp(im\theta+ikz+\Omega t) ,$$

the following equation for the perturbed velocity is obtained:

$$[(\hat{\Omega}+\mu_\perp\beta^2a^2)(\hat{\Omega}+\eta\beta^2a^2)+(m-nq)^2]\vec{\zeta}=-\nabla\hat{p}^{(1)}+2i(m-nq)(\zeta_r\hat{e}_\theta-\zeta_\theta\hat{e}_r) . \tag{4}$$

In this equation, the following definitions have been used:

$$\hat{\Omega}=\left(\frac{4\pi a^2\rho_0}{B_l^2}\right)^{1/2}\Omega,\quad \hat{\mu}_\perp=\left(\frac{4\pi}{\rho_0a^2B_l^2}\right)^{1/2}\mu_\perp ,$$
$$\hat{\zeta}=\frac{\vec{v}^{(1)}}{\Omega+\eta\beta^2/4\pi},\quad \hat{\eta}=\left(\frac{\rho_0}{4\pi a^2B_l^2}\right)^{1/2}\eta . \tag{5}$$

Also,

$$\hat{p}^{(1)}=\frac{4\pi a^2}{B_l^2}\left(p^{(1)}+\frac{1}{8\pi}(\vec{B}\cdot\vec{B})^{(1)}\right) \tag{6}$$

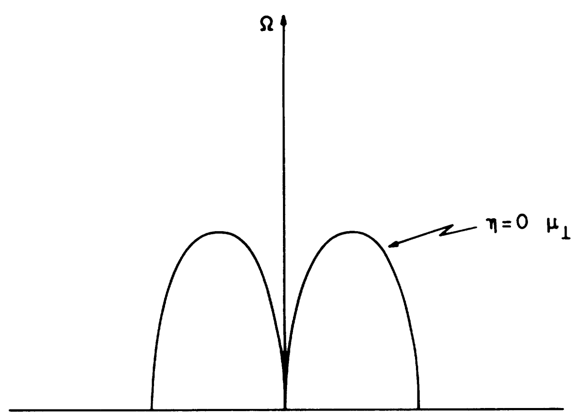

FIG. 1. Growth rate Ω vs $nq-m$ for $\mu_\perp\neq0$ and $\eta=0$.

and

$$k = -\frac{2\pi n}{L}, \quad n = 0, 1, 2, \dots \quad (7)$$

Assuming $\vec{\nabla}\cdot\vec{v}=0$, Eq. (4) yields

$$\nabla^2 \hat{p}^{(1)} + k^2\sigma^2\hat{p}^{(1)} = 0 \ , \quad (8)$$

whose solution, regular at $r=0$, is

$$\hat{p}^{(1)} = \alpha J_m(k(\sigma^2-1)^{1/2} r) \quad (9)$$

with α a constant and

$$\sigma^2 = \frac{2|m-nq|}{(\hat{\Omega}+\hat{\mu}\beta^2a^2)(\hat{\Omega}+\hat{\eta}\beta^2a^2)+(m-nq)^2} \ . \quad (10)$$

Throughout, it has been assumed that $\vec{\nabla}\times\vec{B}^{(1)}=\beta\vec{B}^{(1)}$. Equation (4) shows that the assumption is valid provided that $\beta = k\sigma$.

The boundary conditions are

$$\xi_r(r=a)=0 \quad (11)$$

and

$$\xi_\theta(r=a)=0 \ . \quad (12)$$

From the components of Eq. (4) it follows that the boundary conditions can only be satisfied for $\sigma^2=1$.

Thus, setting $\sigma^2=1$ in Eq. (10) yields the following dispersion relation:

$$\hat{\Omega} = -\tfrac{1}{2}(\hat{\mu}+\hat{\eta})k^2a^2 + \left\{ (\hat{\mu}-\hat{\eta})^2\frac{k^4a^4}{4} + 2|m-nq| - (m-nq)^2 \right\}^{1/2} \ . \quad (13)$$

On the other hand, the boundary conditions, Eqs. (11) and (12), and the recurrence relation among Bessel's functions,

$$xJ_m'(x) = xJ_{m-1}(x) - mJ_m(x) \ , \quad (14)$$

yield

$$\sigma^2 = 1 + \frac{Z_{m-1}^2}{k^2a^2} \ , \quad (15)$$

where Z_{m-1} is the value of the argument of J_{m-1} at the point where the function takes its first zero.

Therefore, the analysis is valid provided that $k^2a^2 \gg Z_{m-1}^2$ and the boundary conditions are satisfied within the order of Z_{m-1}^2/k^2a^2.

It is interesting to notice that, as expected, the dispersion relation given by Eq. (13) reduces to the one obtained in Refs. 1 and 2 for $\eta=0$. Moreover, resistivity reduces both the range and the growth rate of the unstable spectrum (see Fig. 2).

From Eq. (13) it follows that there are, for each m value, four states which are both marginally stable, $\mathrm{Re}\hat{\Omega}=0$, and stationary, $\mathrm{Im}\hat{\Omega}=0$. These states are denoted by α_1, α_2, $-\alpha_1$, $-\alpha_2$ in Fig. (2) and are given by

$$\alpha_1 = 1-(1-\lambda^2)^{1/2} \quad (16)$$

and

$$\alpha_2 = 1+(1-\lambda^2)^{1/2} \ , \quad (17)$$

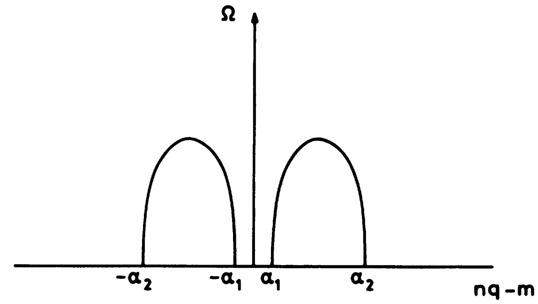

FIG. 2. Growth rate Ω vs $nq-m$ for $\mu_\perp \neq 0$ and $\eta \neq 0$.

where $\lambda^2 = k^4a^4\hat{\mu}_\perp\hat{\eta}$.

As pointed out above, the motion has been assumed incompressible. In general, Eqs. (1) for $\vec{\nabla}\cdot\vec{v}=0$ can only be satisfied if $\gamma=\infty$. Nevertheless, it is possible to show that, for the particular states under consideration, i.e., α_1 and α_2, Eq. (1c) is also satisfied for $\vec{\nabla}\cdot\vec{v}=0$ and γ finite, provided that

$$\mu\kappa = -\frac{3}{k^4a}\frac{dp^{(0)}}{dr}\bigg|_{r=a} \quad (18)$$

and

$$\eta/\kappa = 8\pi/3 \ , \quad (19)$$

respectively. Similar conditions hold for the states $-\alpha_1$ and $-\alpha_2$.

The flow pattern corresponding to these states can be determined in the usual way and is illustrated in Fig. 3 for $m=1$.

The perturbed velocity on the center line of each tube can be shown to be given by

$$v_r = 0 \ , \quad (20)$$

$$v_\theta = (B_\theta/B_z)v_z \ , \quad (21)$$

$$v_z = (-1)^{n+1}CkaJ_m(Z_{m-1}r/a) \ , \quad (22)$$

where C is a constant.

From the dispersion relation given by Eq. (13) it follows that $nq \lesssim m$ and, since $\sigma^2 \simeq 1$, Eqs. (3) and (7) imply $(B_\theta/B_z) \gg 1$. Therefore, from Eq. (21) one concludes that $v_\theta \gg v_z$ which justifies the neglect of parallel viscosity.

In complete analogy with the demonstration given in Ref. 10 (Chaps. II and XI and Appendix I) and in Ref. 11, it is possible to show that the conditions given by Eqs. (18) and (19) can be written in terms of a critical Rayleigh number.

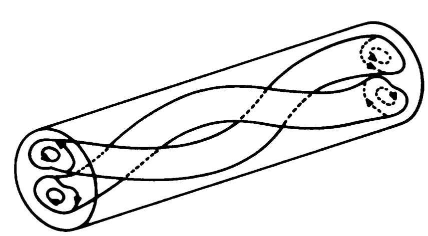

FIG. 3. Flow pattern of convective states for $m=1$.

For the state α_1 the critical Rayleigh number becomes

$$\mathscr{R}_{\mathrm{crit}}=\frac{k^6a^6}{6m^2}\ , \tag{23}$$

and for the state α_2

$$\mathscr{R}_{\mathrm{crit}}=\frac{16\pi}{3}\frac{k^2a^2}{m^2}\ . \tag{24}$$

Equations (23) and (24) have been derived assuming $k^4a^4\hat{\eta}\hat{\mu}_\perp < 1$.

In general, the treatment for $\mathscr{R}=\mathscr{R}_{\mathrm{crit}}$ is a condition for marginal stability. The linear theory does not say anything about the behavior for $\mathscr{R} > \mathscr{R}_{\mathrm{crit}}$. However, since the states under consideration are not only marginally stable ($\mathrm{Re}\Omega=0$), but also stationary ($\mathrm{Im}\Omega=0$), the complete nonlinear Eqs. (1) possess stationary convective solutions which bifurcate from the equilibrium solution for $\mathscr{R} > \mathscr{R}_{\mathrm{crit}}$.[10,11]

Finally, it is interesting to notice that the convective character of the modes α_1 and $-\alpha_1$ is determined only by viscosity and thermal conductivity while that of α_2 and $-\alpha_2$ only by resistivity and thermal conductivity [see Eqs. (18) and (19)].

A detailed account of this paper will be given elsewhere.

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[1] L. Gomberoff and M. Hernández, Phys. Rev. A 27, 1244 (1983).
[2] L. Gomberoff and M. Hernández, Phys. Fluids (in press).
[3] L. Gomberoff, J. Plasma Phys. (in press).
[4] R. J. Tayler, Proc. Phys. Soc. London Sect. B 70, 1049 (1957).
[5] V. D. Shafranov, *Plasma Physics and the Problem of Controlled Thermonuclear Research* (Pergamon, New York, 1958), Vol. II, p. 197.
[6] J. P. Friedberg, Phys. Fluids 13, 1812 (1970).
[7] J. P. Goedbloed and H. J. L. Hagebeuk, Phys. Fluids 15, 1090 (1972).
[8] M. Ohta, Y. Shimomura, and T. Takeda, Nucl. Fusion 12, 271 (1972).
[9] L. Gomberoff and E. K. Maschke, in *Field Theory Quantization and Statistical Physics*, edited by E. Tirapegui (Reidel, Dordrecht, 1981), p. 123.
[10] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon, Oxford, 1961).
[11] E. K. Maschke and R. B. Paris, in *Proceedings of the Fifth Conference on Plasma Physics and Controlled Nuclear Fusion Research, Tokyo, 1974* (IAEA, Vienna, 1975).