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Classical dilute relativistic plasmas in equilibrium. II. Thermodynamic functions

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With the use of the two-particle distribution function calculated in a preceding paper, some thermodynamic functions (energy, free energy, pressure, specific heat, compressibility, and the speed of sound) are calculated for a two-component, classical, dilute, and slightly relativistic plasma up to order kT/mc^2. The results are discussed and compared with previous work of other authors.

I. INTRODUCTION

In this paper II, we remain within the framework of classical relativistic statistical mechanics given in Ref. 1. Using the two-particle distribution function calculated in Ref. 2 (we shall refer henceforth to Ref. 2 as paper I), we find here the thermodynamic functions of a neutral, moderately hot, dilute plasma, up to order kT/mc^2.

The main matter in dealing with systems of such a plasma is that the interaction is a long-range one. This has, as a consequence, that no thermodynamic limit exists *a priori.* In fact, unless some sort of screening can be shown to exist, a magnitude such as the macroscopic energy grows with N faster than N, where N is the number of particles of the system, and so the energy "per particle" grows with N. This problem can be avoided in nonrelativistic plasmas taking a system with vanishing total charge. Nevertheless, it can be easily seen that terms like N^2 are present, in principle, when we consider the relativistic corrections to a Coulomb plasma, even in the case of vanishing total charge. So, in order that the well-known thermodynamics of a neutral Coulomb plasma—the Debye-Hückel distribution function and so on—makes sense, it must be assumed that the effective relativistic correction to the Coulomb interaction is screened enough. In this paper we shall prove that this assumption is consistent by calculating the specific energy in the thermodynamic limit for a moderately relativistic neutral plasma and showing that this limit exists, i.e., the energy per particle is finite. In doing so, we shall use the two-body distribution function given in paper I.

In the framework of predictive relativistic mechanics, let us consider the perturbative expansion for the electromagnetic energy H of N particles, in powers of the coupling constant e^2 (e standing for the typical charge of the particles). This expansion has the form[3,4]

$$H(1,\ldots,N)=\sum_i H_0(i)+\sum_{i,j} H_1(i,j)+\sum_{i,j,k} H_2(i,j,k)+\ldots, \tag{1}$$

where $H_0(i)$ is the free-particle relativistic energy and $H_1(i,j)$ stands for the e^2 terms, $H_2(i,j,k)$ for the e^4 terms, and so on. The expression for $H_1(i,j)$ can be found in Ref. 3. $H_2(i,j,k)$ and higher-order terms will not be needed here.

As has been pointed out in Ref. 5, the perturbative expansion (1) can be assumed to be an expansion in the dimensionless parameter $\epsilon_h=e^2/mh$ where m is the typical mass of the particles and h the mean impact parameter in the collisions (we take c, the speed of light, equal to 1). Then, since h is of the same order as the mean distance $\bar{r}$ between particles, i.e., $\bar{r}=(V/N)^{1/3}$ where V is the volume of the system, and we deal with the case of a dilute plasma, we will see that it is sufficient to work to first order in ϵ_h. Then in Eq. (1) only two-particle interactions will be taken into account. We can also remember that Eq. (1) is not the usual v expansion (v stands for the typical velocity of the particles): Actually each term may contain all powers in v.

Another approximation that we make is the decoupling of the relativistic Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, used in Refs. 1 and 2. In these references only two-body correlations are considered, which means that we take the first term in an expansion of the distribution functions in powers of the dilution parameter $\epsilon_d=e^2/kT\bar{r}$ (k is the Boltzmann constant and T the absolute temperature). Thus the reduced S-body distribution function can be expanded in the form

$$F^{(S)}(1,\ldots,S)=F^{(1)}(1)\cdots F^{(1)}(S)\left[1+\sum_{\substack{i,j=1\\ i<j}}^{S} G^{(2)}(i,j)+\sum_{(i<j<k)=1}^{S} G^{(3)}(i,j,k)+O(\epsilon_d^3)\right]. \tag{2}$$

In the expansion (2), the R-particle correlation function $G^{(R)}$ is of order ϵ_d^{R-1}. Then in a dilute plasma, we can work with only two-particle correlations.

The two parameters ϵ_h and ϵ_d introduced here are related by

$$\epsilon_h=\epsilon_d kT/m\ . \tag{3}$$

If the plasma is slightly relativistic $\epsilon=kT/m$ is another small parameter of the same order as the mean-square velocity $\langle v^2\rangle$.

We deal with a homogeneous plasma in equilibrium. Then for the one-particle distribution function $F^{(1)}$, appearing in Eq. (2), we take the expression

$$F^{(1)}=\frac{\beta' m}{4\pi V K_2(\beta' m)}\exp(-\beta' m\gamma)\ , \quad (4)$$

where $\gamma=(1-v^2)^{-1/2}$ and K_2 is the modified second-order Bessel function with argument $\beta' m$. As has been pointed out in paper I, Eq. (4) is the relativistic Maxwellian distribution function except for the constant β' which replaces here the Boltzmann factor $\beta=1/kT$. We shall discuss this point in Secs. II and III.

On the other hand, the two-particle correlation function $G^{(2)}(i,j)$ has been calculated in paper I for a homogeneous plasma in equilibrium. For a two-component plasma whose particles have masses m_1 and m_2 and whose charges have the same absolute value e, the results are

$$G^{(2)}(1,2)=G_C(1,2)+G_R(1,2)\ , \quad (5a)$$

$$G_C(1,2)=-\frac{\beta' e_1 e_2}{r}e^{-\kappa r}\ , \quad (5b)$$

$$G_R(1,2)=\frac{\beta' e_1 e_2}{4\pi r}\int d^2\Omega_{\vec{n}}\frac{(\vec{n}\cdot\vec{v}_1)(\vec{n}\cdot\vec{v}_2)-\vec{v}_1\cdot\vec{v}_2}{(1+\vec{n}\cdot\vec{v}_1)(1+\vec{n}\cdot\vec{v}_2)} \times\left[2\delta\left[\vec{n}\cdot\frac{\vec{r}}{r}\right]-\alpha r e^{-\alpha|\vec{n}\cdot\vec{r}|}\right]\ , \quad (5c)$$

where

$$\kappa=(4\pi\rho\beta' e^2)^{1/2}\quad (\rho=N/V) \quad (6)$$

and

$$\alpha^2=\tfrac{1}{2}\kappa^2\sum_{a=1}^{2}\frac{K_0(\beta' m_a)}{(\beta' m_a)^2K_2(\beta' m_a)}\ , \quad (7)$$

where K_0 and K_2 are the modified zeroth- and second-order Bessel functions, respectively. Here G_C and G_R are the Debye-Hückel and the relativistic correlation functions given in paper I. The constant κ is the inverse Debye-Hückel screening distance (except for the change $\beta\rightarrow\beta'$) and α the inverse relativistic one.

According to Eqs. (1) and (2) we can write for the exact (i.e., to all orders in ϵ) macroscopic energy of a dilute plasma

$$E=\sum_{i,j,k}\int d^6i\, d^6j\, d^6k[H_0(i)+H_0(j)+H_0(k)+H_1(i,j)+H_1(j,k)+H_1(i,k)+H_2(i,j,k)] \times F^{(1)}(i)F^{(1)}(j)F^{(1)}(k)[1+G^{(2)}(i,j)+G^{(2)}(j,k)+G^{(2)}(i,k)+G^{(3)}(i,j,k)]\ , \quad (8)$$

where in an evident notation d^6i means $d^3r_i d^3u_i$ and $\vec{u}=\gamma\vec{v}$. In the integrand of Eq. (8) we have retained terms up to order ϵ_d^2 in the dilution parameter ϵ_d (remember that $\epsilon_h=\epsilon\epsilon_d$ and that H_1 is of order ϵ_h and H_2 is of order ϵ_h^2). Next we give some arguments to show why after integration these terms should be sufficient in order to calculate the energy up to order $\epsilon_d^{3/2}$ in the dilution parameter. Actually the H_2 and $G^{(3)}$ terms will not need to be considered.

Let us consider the normalization condition

$$\int F^{(3)}(1,2,3)d^6 2\, d^6 3=F^{(1)}(1)\ . \quad (9)$$

In particular this implies according to Eq. (2) that

$$\int F^{(1)}(2)G^{(2)}(1,2)\, d^6 2=0\ .$$

It can be seen that $G^{(2)}(1,2)$ given by (5a)—(5c) verifies this identity in the thermodynamic limit. From Eq. (9) we readily obtain

$$\sum_i\int H_0(i)F^{(3)}(i,j,k)d^6j\, d^6k=E_0, \quad (10)$$

where E_0 is the energy of the plasma as if the interaction was absent, but changing β to β'. When substituting in Eq. (10) an expression like (2) for $F^{(3)}$ we obtain the identities

$$\sum_{i,j}\int[H_0(i)+H_0(j)]F^{(1)}(i)F^{(1)}(j)G^{(2)}(i,j)d^6i\, d^6j=0\ , \quad (11)$$

$$\sum_{i,j,k}\int[H_0(i)+H_0(j)+H_0(k)]F^{(1)}(i)F^{(1)}(j) \times F^{(1)}(k)G^{(3)}(i,j,k)d^6i\, d^6j\, d^6k=0\ . \quad (12)$$

Actually, with $G^{(2)}(1,2)$ given by (5a)—(5c), we find that (11) is only satisfied for a neutral plasma. There is no inconsistence in this, since for a non-neutral plasma we have no thermodynamic equilibrium and then the stationary correlation function $G^{(2)}(1,2)$ given by (5a)—(5c) loses all its meaning.

According to this, we assume that the unknown three-particle correlation function $G^{(3)}(1,2,3)$ will satisfy Eq. (12), at least for a neutral plasma. Let us come back to Eq. (8) giving the macroscopic energy of the plasma E. Taking into account Eqs. (11) and (12), we obtain

$$E=E_0+a+b+c\ , \quad (13)$$

where

$$a=\sum_{i<j}\int F^{(1)}(i)F^{(1)}(j)H_1(i,j)d^6i\, d^6j\ , \quad (13')$$

$$b=\sum_{i<j}\int F^{(1)}(i)F^{(1)}(j)H_1(i,j)G^{(2)}(i,j)d^6i\, d^6j\ , \quad (13'')$$

$$c=\sum_{i<j<k}\int F^{(1)}(i)F^{(1)}(j)F^{(1)}(k)H_2(i,j,k)d^6i\, d^6j\, d^6k\ . \quad (13''')$$

In going from Eq. (8) to Eq. (13) we have neglected the terms whose integrands go as ϵ_d^3 or ϵ_d^4, i.e., terms of the form $H_2G^{(2)}$ and $H_1G^{(3)}$, on one hand, and $H_2G^{(3)}$, on the other hand. We will say something about the order in ϵ_d of the integrals of these kinds of terms in a moment.

The integrals a and c are obviously of order ϵ_d and ϵ_d^2, respectively. However, the integral b is not of order ϵ_d^2 which is the order of the term in the integrand $H_2G^{(2)}$. In order to see this, notice that H_1 behaves like $1/r$ times a function which does not depend on r (see Ref. 3 where the expression for H_1 can be found). Then, let us substitute (5a)—(5c) in the integral b and change the integration variable $\vec{r}$ to the new variable $\kappa\vec{r}$ when evaluating the integral which contains G_C and similarly from $\vec{r}$ to $\alpha\vec{r}$ in the integral containing G_R. This produces an overall factor κ^{-1} or α^{-1}, respectively, out of the integrals. Now, according to (6), κ can be written $\kappa\simeq(2\sqrt{\pi}/\bar{r})\epsilon_d^{1/2}$, i.e., it goes as $\epsilon_d^{1/2}$ in the dilution parameter, and the same is true for α [see Eq. (7)]. This means that b is, in fact, of order $\epsilon_d^{3/2}$. So, we can neglect c in front of b, i.e., we calculate E up to order $\epsilon_d^{3/2}$ and so we only retain a and b.

What about the order of the remaining integrals, i.e., in short, terms $H_2G^{(2)}$, $H_1G^{(3)}$, and $H_2G^{(3)}$? From the above argument we see that the integral of $H_2G^{(2)}$ will be of order $\epsilon_d^{5/2}$ and so we can neglect it. As long as terms $H_1G^{(3)}$ and $H_2G^{(3)}$ are concerned we do not know the expression of $G^{(3)}$. Nevertheless it seems natural to assume that its contribution to the energy E will be of higher order than $\epsilon_d^{3/2}$ (see also the general comments at the end of Sec. V).

Hitherto, we have not said how hot, i.e., how relativistic our plasma is, and so we have been working with an arbitrary parameter $\epsilon=kT/m$. Henceforth we deal with slightly relativistic plasmas.[6] Then, in Sec. II we find the macroscopic energy of a neutral dilute plasma to first order in ϵ. In Sec. III and in the same approximation, we find the equation of state and we obtain the new Boltzmann factor β' using the virial theorem.

Section IV is devoted to the study of two response functions (compressibility and specific heat), from which the stability of the plasma follows. The speed of sound is also calculated.

Comparison with other works on relativistic plasmas and the discussion of the results are done in Sec. V.

II. ENERGY UP TO ORDER ϵ

For the sake of simplicity, let it be a two-component plasma whose particles all have the same square charge e^2. Also let us assume that the plasma is a neutral, dilute, slightly relativistic one, which is homogeneous and in equilibrium.

In this section we evaluate the energy of such a plasma to first order in ϵ. To this goal, we expand $H_1(1,2)$ in powers of v, and so we have for the energy $H(1,2)$ of any two particles labeled 1 and 2, the following expression:

$$H(1,2)=H_0(1)+H_0(2)+H_C(1,2)+H_D(1,2)+O(\epsilon_h^2,v^4)\ , \tag{14}$$

where

$$H_0(i)=m_i\gamma_i\ , \tag{15a}$$

$$H_C(1,2)=\frac{e_1e_2}{r}\ , \tag{15b}$$

$$H_D(1,2)=\frac{e_1e_2}{2r}\left[\vec{v}_1\cdot\vec{v}_2+\frac{(\vec{r}\cdot\vec{v}_1)(\vec{r}\cdot\vec{v}_2)}{r^2}\right] \tag{15c}$$

are the relativistic kinetic energy, the Coulomb interaction, and the Darwin interaction. Here m_i, e_i, and $\vec{v}_i$ are the mass, the charge, and the velocity of particle i, while $\vec{r}\equiv\vec{r}_1-\vec{r}_2$. For later convenience in (15a) we have considered the exact relativistic kinetic energy.

The one-particle distribution functions, corresponding to the noncanonical variables $\vec{x}_i,\vec{u}_i$, have been given in Eq. (4). As explained before, the modified Boltzmann factor comes from the presence of relativistic terms in the interaction. In particular, the change from canonical coordinates $\vec{p}_i,\vec{q}_i$ (see Ref. 7) to the usual coordinates $\vec{x}_i,m_i\vec{u}_i$ (which, as is well known, are not canonical for the case of relativistic interaction) can change β to another parameter β' as is discussed in Ref. 1. Also, the interaction by itself can give rise to a similar effect.[8] The new parameter β' will be some function of β, the density of the plasma $\rho=N/V$, and mechanical parameters. In addition, β' must become β when $\rho\to0$ (free-gas case). In fact, we shall see in Sec. III that $\beta'/\beta=1+O(\epsilon_d^{3/2}\epsilon^2)$ by using the virial theorem [see Eq. (42)]. We must also remark that the distribution function (4) is normalized with respect to the variables $\vec{r}_i,\vec{u}_i$.

Up to order $\epsilon_d v^4$, the correlation function reads as

$$G^{(2)}(1,2)=G_C(1,2)+G_2(1,2)+G_4(1,2)\ , \tag{16}$$

where G_C was defined in (5b) and

$$G_2(1,2)=\frac{\beta'e_1e_2}{r}e^{-\alpha r}\left[\frac{(\vec{r}\cdot\vec{v}_1)(\vec{r}\cdot\vec{v}_2)}{r^2}-\vec{v}_1\cdot\vec{v}_2+\frac{1}{\alpha r}\left[3\frac{(\vec{r}\cdot\vec{v}_1)(\vec{r}\cdot\vec{v}_2)}{r^2}-\vec{v}_1\cdot\vec{v}_2\right]+\frac{1}{\alpha^2r^2}\left[3\frac{(\vec{r}\cdot\vec{v}_1)(\vec{r}\cdot\vec{v}_2)}{r^2}-\vec{v}_1\cdot\vec{v}_2\right]\right]+\frac{\beta'e_1e_2}{r}\frac{1}{\alpha^2r^2}\left[\vec{v}_1\cdot\vec{v}_2-3\frac{(\vec{r}\cdot\vec{v}_1)(\vec{r}\cdot\vec{v}_2)}{r^2}\right]\ , \tag{17a}$$

$$G_4(1,2)=\frac{\beta'e_1e_2}{4\pi r}\int d^2\Omega_{\vec{n}}[(\vec{n}\cdot\vec{v}_1)(\vec{n}\cdot\vec{v}_2)-(\vec{v}_1\cdot\vec{v}_2)][(\vec{n}\cdot\vec{v}_1)^2+(\vec{n}\cdot\vec{v}_1)(\vec{n}\cdot\vec{v}_2)+(\vec{n}\cdot\vec{v}_2)^2]\times\left[2\delta\left[\vec{n}\cdot\frac{\vec{r}}{r}\right]-\alpha re^{-\alpha r|\vec{n}\cdot(\vec{r}/r)|}\right] \tag{17b}$$

are their v^2 and v^4 relativistic corrections, respectively.

Let us substitute $H(1,2)$ and $G^{(2)}(1,2)$ given by (14) and (16), respectively, in Eq. (13). Then, it can be easily seen that these terms in (13) which do not contain the terms $G_2(1,2)$ or $G_4(1,2)$ in the integrand, once the integration is performed, change the expansion in powers of v to an expansion for E powers of ϵ, in such a manner that v^2 becomes proportional to ϵ. When terms $G_2(1,2)$ or $G_4(1,2)$ are present, after integration, an overall α^{-1} factor comes out of the integral. Now, according to (7), α goes as ϵ. Then, those terms containing G_2 or G_4 are one order less in ϵ than the corresponding ones where they are absent.

In all, up to order $\epsilon_d^{3/2}\epsilon$, there are six terms that can contribute to the energy,

$$\int d^6 1\, d^6 2[H_0(1)+H_0(2)]F^{(1)}(1)F^{(1)}(2)\ , \tag{18a}$$

$$\int d^6 1\, d^6 2\, H_C(1,2)F^{(1)}(1)F^{(1)}(2)\ , \tag{18b}$$

$$\int d^6 1\, d^6 2\, H_C(1,2)F^{(1)}(1)F^{(1)}(2)G_C(1,2)\ , \tag{18c}$$

$$\int d^6 1\, d^6 2\, H_C(1,2)F^{(1)}(1)F^{(1)}(2)[G_2(1,2)+G_4(1,2)]\ , \tag{18d}$$

$$\int d^6 1\, d^6 2\, H_D(1,2)F^{(1)}(1)F^{(1)}(2)[1+G_C(1,2)]\ , \tag{18e}$$

$$\int d^6 1\, d^6 2\, H_D(1,2)F^{(1)}(1)F^{(1)}(2)G_2(1,2)\ , \tag{18f}$$

where the integrations must be extended to all the volume of the system and to all possible velocities. At the end, one must sum over all the particles in the plasma.

Now, we discuss each possible contribution. The term (18a) is the relativistic energy of an ideal gas except for the change $\beta \rightarrow \beta'$ and gives to the energy per particle the following contribution:

$$\frac{E^{\text{id}}}{N} \equiv E_a/N = \frac{1}{\beta'} - \tfrac{1}{2}m_1\frac{K_2'(\beta' m_1)}{K_2(\beta' m_1)} - \tfrac{1}{2}m_2\frac{K_2'(\beta' m_2)}{K_2(\beta' m_2)}$$
$$= M + \frac{3}{2\beta'} + \frac{15}{16}\frac{1}{(\beta')^2\mu} + O(\epsilon^2)\ , \tag{19}$$

where we have used the expansion of the modified Bessel functions for large argument.[9] Here $K'(x)$ means $dK(x)/dx$ and we have defined $M=(m_1+m_2)/2$ and $\mu=(m_1 m_2)/(m_1+m_2)$.

The term (18b) gives for any pair of particles 1 and 2

$$\frac{4\pi}{2}R^2 V^{-1} e_1 e_2\ , \tag{20}$$

where R is the radius of the spherical vessel where for the moment we imagine that the plasma is contained. Actually, the integral (18b) depends on the form of the volume. Nevertheless, this raises no difficulty, since because of the neutrality of the plasma, the sum over all pairs of particles of terms like (18b) gives no contribution to the macroscopic energy.

Contribution (18c) gives[10] an energy per particle E_C/N,

$$E_C/N = -\sqrt{\pi\beta'}\rho^{1/2}e^3\ , \tag{21}$$

which is the interaction energy of a nonrelativistic Coulomb plasma.

The terms (18d) and (18e) give no contribution, because by symmetry reasons the angular integrations over the velocities vanish.

The most difficult term to evaluate is (18f), since there are divergent integrals in d^3r. This term can be calculated, introducing a cutoff in the lower limit of integration, say, r_0. However, we shall see that the result does not depend on the cutoff. The four terms appearing in the right side of (17a) give us the following contribution when we integrate them with (15c):

$$-V^{-1}\frac{1}{\beta'}\frac{(e_1e_2)^4}{m_1m_2}4\pi\left[\frac{1}{\alpha}-\frac{1}{\alpha}E_1(\alpha r_0) - \frac{1}{r_0\alpha^2}E_2(\alpha r_0)+\frac{1}{\alpha^2 r_0}\right]\ , \tag{22}$$

where $E_1(r_0\alpha)$ and $E_2(r_0\alpha)$ are the "exponential integrals" defined in Ref. 9. Now, both E_1 and E_2 diverge when we take the limit $r_0 \rightarrow 0$. However, we can use the expansions of these functions given in Ref. 9 for small argument. Doing this one can see that (22) becomes

$$-V^{-1}\frac{e^4 4\pi}{\beta' m_1 m_2}\frac{2}{\alpha}\ . \tag{23}$$

Summing over all the pairs of particles and expanding α to the lowest order in ϵ, we find a contribution to the energy per particle E_R/N,

$$E_{R/N} = -\left[\frac{\pi}{2}\right]^{1/2} e^3\rho^{1/2}\frac{1}{\mu(\beta')^{1/2}}\ . \tag{24}$$

Thus the energy per particle up to order $\epsilon\epsilon_d^{3/2}$ reads

$$E/N = M + \frac{3}{2\beta'} + \frac{15}{16(\beta')^2\mu} - (\pi\beta'\rho)^{1/2}e^3 - \left[\frac{\pi}{2}\rho\right]^{1/2}\frac{e^3}{\mu(\beta')^{1/2}} \tag{25}$$

corresponding formally (i.e., except for the change $\beta \rightarrow \beta'$) to the energy of an ideal relativistic gas (the three first terms), the Coulomb correction, and the relativistic correction, respectively. Therefore we have found a finite value for the specific energy of the plasma in the thermodynamic limit, i.e., the thermodynamic limit exists, according to our initial assumption (see the Introduction). Actually this was to be expected since, as is pointed out in paper I, the relativistic two-particle distribution function (17a), which we use here, goes like $1/r^3$ for large values of r. On the other hand, as we have discussed before, the corresponding two-particle interaction relativistic energy $H_1(1,2)$ behaves like $1/r$. Then, the integral giving the energy of a dilute plasma must converge for large values of r.

However, our result does not agree with that of Refs. 11 and 12. This point will be discussed in Sec. V.

III. PRESSURE AND VIRIAL THEOREM

In this section we find the equation of state of our dilute, slightly relativistic plasma, and we give the corrected Boltzmann factor β' as a function of β, ρ, and mechanical parameters.

To this goal, let us consider the partition function[1]

$$Z = \text{const}\int \prod_{i=1}^{N} d^3x_i\, d^3u_i\, e^{-\beta H}\ . \tag{26}$$

If we define, as usual, the free-energy function by $F=(-1/\beta)\ln Z$, we see from (26) that

$$\frac{\partial(\beta F)}{\partial\beta'}=E(\beta') . \tag{27}$$

Integrating Eq. (27) we obtain the free-energy function

$$\beta F=\int^{\beta'}E(\beta'')d\beta''+F^0 , \tag{28}$$

where F^0 does not depend on β. Then, we shall compute the pressure by

$$P=-\left[\frac{\partial F}{\partial V}\right]_\beta \tag{29}$$

taking into account that β' may depend on V.

First, the "ideal"-gas contribution to the pressure [term (18a) for the energy] is

$$P^{\rm id}=-\frac{1}{\beta}\left[\frac{\partial\beta'}{\partial V}\right]_\beta E^{\rm id}(\beta')-\frac{1}{\beta}\frac{\partial F^0(V)}{\partial V} \tag{30}$$

because $E^{\rm id}(\beta)$ does not depend on V. We choose the integration constant $F^0(V)$ so that for $\rho\to 0$, we recover the ideal-gas pressure. Therefore, $\partial F^0(V)/\partial V=-\rho$. Then, for the Coulomb contribution, we have

$$\beta F_C=-N\tfrac{2}{3}\sqrt{\pi}e^3\rho^{1/2}(\beta')^{3/2} \tag{31}$$

and so

$$P_C=-\frac{\sqrt{\pi}}{3}\frac{e^3\rho^{3/2}(\beta')^{3/2}}{\beta}+\frac{N\sqrt{\pi}}{\beta}\rho^{1/2}e^3(\beta')^{1/2}\left[\frac{\partial\beta'}{\partial V}\right]_\beta . \tag{32}$$

For the relativistic contribution we find

$$\beta F_R=-N\sqrt{2\pi}e^3\rho^{1/2}\frac{(\beta')^{1/2}}{\mu} \tag{33}$$

and

$$P_R=-\left[\frac{\pi}{2}\right]^{1/2}\rho^{3/2}\frac{e^3}{\mu\beta}(\beta')^{1/2}+N\left[\frac{\pi}{2}\right]^{1/2}e^3\frac{\rho^{1/2}}{\beta\mu}(\beta')^{-1/2}\left[\frac{\partial\beta'}{\partial V}\right]_\beta . \tag{34}$$

In order to evaluate β', consider the virial theorem[13]

$$E-3PV=\sum_{i=1}^{N}m_i\langle(1-v_i^2)^{1/2}\rangle . \tag{35}$$

In order that this theorem be fullfilled, we must have the plasma contained in a vessel (large enough since we take the thermodynamic limit), in such a manner that the electromagnetic field vanishes at infinity. Furthermore, since we work to first order in the product of the charges, we are neglecting the electromagnetic radiation and so the field will go to zero fast enough at infinity.

Let us consider Eq. (35), both in the case of our interacting relativistic plasma and in the case of an ideal relativistic one, and then let us subtract both equations. We have

$$E-E^0-3(P-P^0)V=\sum_{i=1}^{N}m_i[\langle(1-v^2)^{1/2}\rangle-\langle(1-v^2)^{1/2}\rangle^0] , \tag{36}$$

where the superscript zeros mean magnitudes or averages in the ideal-gas case. Carrying out the one-body averages appearing in the right-hand side and substituting the above results, we arrive, after some tedious calculations, at the following equation for β':

$$\left[\frac{\partial\beta'}{\partial V}\right]_\beta\frac{3VM}{\beta}=3\left[\frac{1}{\beta}-\frac{1}{\beta'}\right]+\sqrt{\pi}\rho^{1/2}e^3(\beta')^{1/2}(1-\beta'/\beta)-2\left[\frac{\pi}{2}\right]^{1/2}\rho^{1/2}\frac{e^3}{\mu(\beta')^{1/2}}+O(\epsilon^2) . \tag{37}$$

Let us define now $C\equiv\beta'-\beta$. Then, Eq. (37) reads

$$VkT\left[\frac{\partial C}{\partial V}\right]_\beta=\left[\frac{kT}{M}\right]C-\frac{\sqrt{\pi}}{3}\epsilon_d^{3/2}\left[\frac{kT}{M}\right]C-\frac{2}{3}\left[\frac{\pi}{2}\right]^{1/2}\epsilon_d^{3/2}\left[\frac{kT}{\mu}\right]\left[\frac{kT}{M}\right]+O(\epsilon^3) . \tag{38}$$

First, we see that in the nonrelativistic case [i.e., Eq. (38) without the last term] the only solution of Eq. (38) fulfilling the boundary condition $C(\rho=0)=0$ is $C=0$ as it must be.

On the other hand, we see from (30) that terms of order ϵ^2 in $(\partial C/\partial V)_\beta$ can contribute to the pressure and so we must retain them in Eq. (38). Then we write

$$C=C_1(\beta,\rho)\left[\frac{kT}{\mu}\right]+C_2(\beta,\rho)\left[\frac{kT}{\mu}\right]^2+O(\epsilon^3) . \tag{39}$$

Substituting it in (38), we have the two following equations for the unknown functions C_1 and C_2:

$$VkT\left[\frac{\partial C_1}{\partial V}\right]_\beta=0 , \tag{40a}$$

$$VkT\left[\frac{\partial C_2}{\partial V}\right]_\beta=\frac{\mu}{M}C_1-\frac{\sqrt{\pi}}{3}\epsilon_d^{3/2}\frac{\mu C_1}{M}-\frac{2}{3}\left[\frac{\pi}{2}\right]^{1/2}\epsilon_d^{3/2}\frac{\mu}{M} . \tag{40b}$$

From (40a) we see that the only solution for C_1 fulfilling $C_1(\rho=0)=0$ is $C_1=0$. Actually, this was to be expected, since if $C_1\neq 0$, there could be a zeroth-order contribution in ϵ to the pressure (30) coming from β'.

Then (40b) can be solved with the condition $C_2(\rho=0)=0$ given for C,

$$C=+\frac{4}{3}\left[\frac{\pi}{2}\right]^{1/2}\epsilon_d^{3/2}\frac{(kT)}{M\mu} \tag{41}$$

and therefore the "effective" temperature in the plasma reads

$$T'=(k\beta')^{-1} = T\left[1-\frac{4}{3}\left[\frac{\pi}{2}\right]^{1/2}\epsilon_d^{3/2}\frac{(kT)^2}{M\mu}+O(\epsilon^3)\right]. \quad (42)$$

In the same approximation we can write $T'=T[1-2e^2\alpha/3M+O(\epsilon^3)]$. Written in this way this result coincides formally with that obtained by Trubnikov and Kosachev,[8] (except for the sign of the correction), but they have a very different relativistic screening constant α, which makes their correction to T independent of the temperature. We shall discuss their results later.

From (42) we see that the only ϵ-order correction to the pressure, due to the correction of β, comes from the first term of (30), because corrections to other pressure terms are of order ϵ_d^2 or ϵ^2 at least.

Then the total pressure, including this correction, reads as

$$P=kT\rho\left[1-\frac{\sqrt{\pi}}{3}\epsilon_d^{3/2}\left[1+\frac{1}{\sqrt{2}}\frac{kT}{\mu}\right]\right] \quad (43)$$

which behaves correctly for ϵ_d or ϵ going to zero.

Equation (43), again, disagrees with the result given by Krizan and Havas[11] and with that of Trubnikov and Kosachev. Concretely, instead of the term $[1+(1/\sqrt{2})kT/m]$ in (43), Krizan and Havas[11] get $[1+O(\epsilon^2)]$ while Trubnikov and Kosachev obtain $[1+O(\epsilon^{3/2})]$. In the first case the reason is that, as pointed out by the authors themselves, in Ref. 11 only short-range correlations are considered. In the second one,[12] the authors start with a noncoherent Darwin Hamiltonian, and therefore their results are meaningless. We shall discuss these points in more detail in Sec. V.

IV. OTHER THERMODYNAMIC FUNCTIONS

In order to discuss the stability of the plasma, we shall calculate the second derivatives of the free energy and we shall check that they are always non-negative, as it must be.

The specific heat is defined by $C_V=N^{-1}(\partial E/\partial T)_V$. Then

$$C_V(T)=C_V^{\rm id}(T)+C_V^C(T)+C_V^R(T), \quad (44)$$

where

$$C_V^{\rm id}(T)=(\tfrac{3}{2}+\tfrac{15}{8}kT/\mu)k, \quad (45a)$$

$$C_V^C(T)=\frac{\sqrt{\pi}e^3\rho^{1/2}}{2(kT)^{3/2}}k, \quad (45b)$$

$$C_V^R(T)=-\frac{1}{2}\left[\frac{\pi}{2}\right]^{1/2}\frac{e^3\rho^{1/2}}{\mu(kT)^{1/2}}k \quad (45c)$$

are the free, Coulomb, and relativistic contributions, respectively. Then (44) can be written as

$$C_V(T)=k\left[\frac{3}{2}\left[1+\frac{5}{4}\frac{kT}{\mu}\right]+\frac{\sqrt{\pi}}{2}\epsilon_d^{3/2}\left[1-\frac{1}{\sqrt{2}}\frac{kT}{\mu}\right]\right], \quad (46)$$

where we see that $C_V\geq 0$.

The isothermal compressibility is defined by $K_T\equiv -V^{-1}(\partial V/\partial P)_T$. Then, we obtain

$$K_T^{-1}=\rho kT\left[1-\frac{\sqrt{\pi}}{2}\epsilon_d^{3/2}\left[1+\frac{1}{\sqrt{2}}\frac{kT}{\mu}\right]\right]. \quad (47)$$

Therefore, as long as ϵ_d is a small parameter (as we have assumed) the stability of the plasma is ensured.

Now we evaluate the speed of sound in the plasma, W, i.e., $W=(\partial P/\partial D)_S^{1/2}$ where D is the mass density ($D=M\rho$) and S is the entropy. After some manipulations, using only thermodynamic properties, we find

$$W^2=\frac{1}{M}\left[\frac{1}{\rho}K_T^{-1}+\frac{T}{\rho^2C_V}\left[\frac{\partial P}{\partial T}\right]_\rho^2\right]. \quad (48)$$

Evaluating $(\partial P/\partial T)$ from Eq. (43), and using (46) and (47), we arrive at

$$W^2=\frac{kT}{M}\left[\frac{5}{3}\left[1-\frac{1}{2}\frac{T}{\mu}\right]-\frac{\sqrt{\pi}}{2}\epsilon_d^{3/2}\left[1+\frac{1}{\sqrt{2}}\frac{kT}{\mu}\right]\right], \quad (49)$$

where we have retained only relevant terms, i.e., terms up to order $\epsilon_d^{3/2}\epsilon$ into the square brackets.

V. CONCLUDING REMARKS

For all the thermodynamic functions that we have found, we see that in the limit $\epsilon=kT/m\to 0$ we recover the well-known Coulomb behavior, and in the limit where the dilution parameter $\epsilon_d\to 0$, the ideal-gas thermodynamics. In Ref. 12 Trubnikov and Kosachev, using the Darwin Lagrangian, obtain a relativistic correction to the pressure of a classical Coulomb plasma, of order $\epsilon^{3/2}$. Let us look at this point in detail. In Ref. 12 (as well as in Refs. 14—16), starting from the Darwin Lagrangian, the authors calculate the canonical momenta in order to write the Hamiltonian as a function of the canonical coordinates. In doing this calculation they keep all powers in ϵ_h. Now it is clear from Refs. 3 and 4, that the expansion of the canonical momenta in powers of ϵ_h will involve two-particle terms of order ϵ_h^2 and higher which do not come from the Darwin Lagrangian. Such terms are of the same order, even in the thermodynamic limit, as those considered in Refs. 12 and 14—16 when these authors extend the expansion of the canonical momenta beyond order ϵ_h. Therefore, we think that their results are not correct (for other criticisms on this point see Sec. IV in paper I).

In our opinion, when doing calculations in a system governed by long-range interactions, as is the case of a plasma, one must begin considering the (electromagnetic in the case of a plasma) interaction to a given approximation in the velocities, in the ϵ_h parameter used here or another convenient parameter and then proceed systematically in the expansion in order not to neglect terms of the same order as those considered. Now, in such long-range systems, when $N\to\infty$, it could happen that the small correction terms in the basic interaction give terms in the macroscopic quantities growing faster than N, which would mean that our expansions are in fact macroscopically meaningless and the thermodynamic limit does not exist. If one does not know *a priori* which is the real situation in the system considered, one can assume for the moment that the expansion used will be also valid when $N \rightarrow \infty$ (in which case some sort of screening must take place in the system) and then verify that the results obtained in this way are consistent with the previous assumption, i.e., it must be found that the thermodynamic limit exists and that the correction terms obtained for the thermodynamic functions are actual corrections in the sense that they are smaller than the leading terms.

Therefore, we think that the Jones and Pytte[14–16] criticisms to Krizan and Havas,[11,17,18] as well as those from Trubnikov and Kosachev[12,19] are groundless in the sense that in Refs. 11, 17, and 18, Krizan and Havas start correctly with the usual Darwin Hamiltonian.

Now, the two-body distribution function given in paper I, as well as the thermodynamic functions calculated here, do not agree with the corresponding results in Refs. 17 and 11, which are grounded in the ring approximation.

However, when they sum all the ring graphs, for example, in order to calculate the correlation function, they have series expansions which are only convergent for wave numbers $k > \kappa$ in the nonrelativistic part, and for k greater than their inverse relativistic screening distance in the relativistic part. In both cases, the long-range contributions (corresponding to small k) to this series have no meaning in principle. Nevertheless, in the nonrelativistic case, by interchanging integrations and infinite summations, one can obtain a finite result for the correlation function—the standard Debye-Hückel correlation function—which, on the other hand, coincides with the one calculated using the BBGKY hierarchy. This shows that the ring calculation, in the nonrelativistic case, is actually correct. However, in the relativistic case, both calculations—the ring diagram and the BBGKY ones—do not give the same result.

Then, since in the BBGKY approach there is no problem with short values of k, we think that in the relativistic case, our results (which are obtained in this way) are more reliable than the Krizan ones.

In all, we think that, since our results are coherent, the method developed here is a good one to deal with relativistic plasmas.

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[1]R. Lapiedra and E. Santos, Phys. Rev. D 23, 2181 (1981).

[2]R. Lapiedra and E. Santos, Phys. Rev. A 27, 422 (1983), referred to as paper I.

[3]See, for example, L. Bel and J. Martín, Ann. Inst. Henri Poincaré 22, 173 (1975), and references therein.

[4]L. Bel and X. Fustero, Ann. Inst. Henri Poincaré 25, 411 (1976).

[5]R. Lapiedra and A. Molina, J. Math. Phys. 20, 1308 (1979); R. Lapiedra, F. Marquès, and A. Molina, *ibid.* 20, 1316 (1979).

[6]We make, in this work, two formal and independent expansions in ϵ_d and in ϵ. In actual tokamak fusion plasmas one value for the dilution parameter ϵ_d can be about 10^{-6}, and for ϵ we have $\epsilon \lesssim 2\times 10^{-2}$.

[7]X. Fustero and E. Verdaguer, Phys. Rev. D 24, 3094 (1981); 24, 3103 (1981).

[8]V. V. Kosachev and B. A. Trubnikov, Nucl. Fusion 9, 53 (1969).

[9]M. A. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

[10]See, for example, L. Landau and E. Lifshitz, *Physique Statistique* (Editions Mir, Moscow, 1967).

[11]J. E. Krizan and P. Havas, Phys. Rev 128, 2916 (1962).

[12]B. A. Trubnikov and V. V. Kosachev Zh. Eksp. Teor. Fiz. 54, 939 (1968) [Sov. Phys.—JETP 27, 501 (1968).]

[13]L. Landau and E. Lifshitz, *Théorie des Champs* (Editions Mir, Moscow, 1970).

[14]R. D. Jones, Ph. D. thesis, Dartmouth College, 1979 (unpublished).

[15]R. D. Jones and A. Pytte, Phys. Fluids 23, 269 (1980).

[16]R. D. Jones, Phys. Rev. D 25, 591 (1982).

[17]J. E. Krizan, Phys. Rev. D 10, 298 (1974).

[18]J. E. Krizan, Phys. Rev. D 22, 3017 (1980).

[19]B. A. Trubnikov, Nucl. Fusion 8, 51 (1968); 8, 59 (1968).