Degenerate perturbative treatment of the hydrogenic Zeeman effect

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Degenerate perturbation theory is applied to study the first 14 energy levels of the hydrogen atom in a uniform magnetic field up to the second order. The twofold degeneracy of all the levels among them in terms of the oscillator or parabolic states is completely removed. The results obtained with the use of the Padé approximant are compared with those found in the literature. Level crossings are discussed.

I. INTRODUCTION

Rayleigh-Schrödinger perturbation theory has been applied to study the hydrogen atom in a homogeneous electric or magnetic field. While the hydrogenic Stark effect has been extensively investigated,[1] the Zeeman effect has not received as much attention. The reason is perhaps twofold: nonseparability and degeneracy. Larsen[2] and Pokatilov and Rusanov[3] calculated the first few energy levels by variational methods. Cabib, Fabri, and Fiorio[4] determined the $1s$ and $2s$ levels by numerical integration. Praddaude[5] reduced the Schrödinger equation to an infinite set of algebraic equations by a suitable expansion of the wave function and calculated numerically the energy of the first 14 levels. Perturbatively, Bednár,[6] using the eigenfunctions of SO(2,1) operators, calculated explicitly the first three Rayleigh-Schrödinger expansion coefficients for the ground-state energy. Avron *et al.*[7] derived asymptotic formulas for the high-order ground-state perturbation coefficients by an algebraic SO(4,2) method and computer implementation. Adams *et al.*[8] then extended the Bender-Wu analysis to the $3s$-$3d$ degenerate levels. More recently, Bender, Mlodinow, and Papanicolaou[9] studied a semiclassical perturbative expansion of the ground-state energy in each azimuthal space. They also briefly discussed the Stark effect. This paper reports a rigorous quantum-mechanical perturbative treatment of the 14 levels studied by Praddaude.[5] The present calculation is carried up to the second order and can be extended to higher orders. However, with the use of the Padé approximant, our second-order results compare well with those of Praddaude for $B=0.1$ ($\sim 10^8$ G).

The problem is formulated in Sec. II. In Sec. III we present the results followed by a discussion. Some details of our calculation are presented in the Appendix.

II. FORMULATION

The spinless hydrogenic Zeeman Hamiltonian, in atomic units, can be written in the form

$$H=-\tfrac{1}{2}\Delta-1/r+\tfrac{1}{2}BL_z+(B^2/8)(r^2-z^2)\ , \tag{1}$$

where B is the strength of the uniform magnetic field in the z direction measured in units of 2.35×10^9 G and $L_z=-i\partial/\partial\varphi$. By means of the following coordinates,

$$\mu=(2r/n)^{1/2}\cos\tfrac{1}{2}\theta\ , \tag{2a}$$

$$\nu=(2r/n)^{1/2}\sin\tfrac{1}{2}\theta\ , \tag{2b}$$

the Schrödinger equation for the physical system can be transformed into the form

$$(H_0+H_m+H')\,|\,\psi\,\rangle=2\lambda\,|\,\psi\,\rangle\ , \tag{3}$$

where the unperturbed Hamiltonian,

$$H_0=\frac{1}{2}\left[\frac{\partial^2}{\partial\mu^2}+\frac{\partial^2}{\partial\nu^2}+\frac{1}{\mu}\frac{\partial}{\partial\mu}+\frac{1}{\nu}\frac{\partial}{\partial\nu}+\left[\frac{1}{\mu^2}+\frac{1}{\nu^2}\right]\left[\frac{\partial^2}{\partial\varphi^2}-\mu^2-\nu^2\right]\right], \tag{4}$$

is that of a pair of phase-coupled two-dimensional harmonic oscillators,[10]

$$H_m=-iB\partial/\partial\varphi\ , \tag{5}$$

and

$$H'=B^2\lambda^4(\mu^2+\nu^2)[(\mu^2+\nu^2)^2-(\mu^2-\nu^2)^2]/32 \tag{6}$$

is the perturbing potential. The perturbed eigenvalue has the following Rayleigh-Schrödinger expansion:

$$2\lambda=2n+\sum\lambda_p(B^2/8)^p\ . \tag{7}$$

The eigenstates of H_0 can be denoted by $|N_1N_2m\rangle$, where N_1,N_2,m are the oscillator quantum numbers. These states correspond to the parabolic states $|n_1n_2m\rangle$ and the set of quantum numbers are related to the more familiar ones as follows[11]:

$$N_1=2n_1+|m|+1=n+n_1-n_2 , \tag{8a}$$

$$N_2=2n_2+|m|+1=n-n_1+n_2 . \tag{8b}$$

Since $N_1+N_2=N=2n$, the oscillator state $|N_1N_2m\rangle$ is doubly degenerate when $N_1\neq N_2$. In such a case, the unperturbed state is a linear combination of $|1\rangle=|N_1N_2m\rangle$ and $|2\rangle=|N_2N_1m\rangle$, i.e.,

$$|\psi_0\rangle=(|1\rangle+|2\rangle)/2^{1/2} , \tag{9}$$

and degenerate perturbation theory applies. There are cases among the 14 levels considered where $N_1=N_2$. For these cases, the nondegenerate perturbation theory suffices. It should be noted that we choose to work with the oscillator quantum numbers because of convenience. The results obtained can be easily converted into the more familiar forms by utilizing Eq. (8).

To calculate the matrix elements in the oscillator representation,[11] we express the coordinate-dependent part of the perturbing potential, i.e, $G=(\mu^2+\nu^2)[(\mu^2+\nu^2)^2-(\mu^2-\nu^2)^2]$, in terms of SO(4,2) generators. If we denote the relevant generators L_{34}, L_{35}, L_{46}, and L_{56}, respectively, by M, V, W, and N, then it can be shown that

$$G=8(N+W)[(N+W)^2-(M+V)^2] . \tag{10}$$

Denote the matrix element

$$\langle N_1+a,N_2+b\,|\,G\,|\,N_1+c,N_2+d\rangle$$

by $(abcd)$. Then, with the help of the 21 nonvanishing matrix elements of G given in the Appendix, we calculate the following quantities:

$$M_{11}^{(1)}=M_{22}^{(1)}=(0000) , \tag{11}$$

$$M_{12}^{(1)}=M_{21}^{(1)}=(00N_2-N_1,N_1-N_2) , \tag{12}$$

$$M_{11}^{(2)}=M_{22}^{(2)}=\sum_{N'\neq N}(00N_1'-N_1,N_2'-N_2)^2/(N_1+N_2-N_1'-N_2') , \tag{13}$$

$$M_{12}^{(2)}=M_{21}^{(2)}$$
$$=\sum_{N'\neq N}(00N_1'-N_1,N_2'-N_2)(N_1'-N_1,N_2'-N_2,N_2-N_1,N_1-N_2)/(N_1+N_2-N_1'-N_2') . \tag{14}$$

Details of the calculation are indicated in the Appendix. Their explicit expressions for the various cases considered are given below and their numerical results are shown in Table I.

According to the perturbation theory, the expansion coefficients λ_p for $p=1,2$ in Eq. (7) can be obtained from the eight $M_{ij}^{(p)}$ defined above. Depending on the value of $q=\frac{1}{2}(N_1-N_2)=n_1-n_2$, they are given as follows:

(a) $q=0$ (nondegenerate case)

$$\lambda_1=M_{11}^{(1)} , \tag{15a}$$

$$\lambda_2=M_{11}^{(2)} , \tag{15b}$$

TABLE I. Numerical values of $M_{ij}^{(p)}$ and expansion coefficients.

	$\|q\|$	$M_{11}^{(1)}$	$M_{12}^{(1)}$	$M_{11}^{(2)}$	$M_{12}^{(2)}$	λ_1	λ_2	E_1	E_2
$1s$	0	16		$-2\,656/3$		16	$-2\,656/3$	2	$-53/3$
$2s$	1	80	32	$-33\,248/3$	$-20\,864/3$	112	$-54\,112/3$	28	$-30\,592/3$
$2p(0)$	1	80	32	$-33\,248/3$	$-20\,864/3$	48	$-12\,384/3$	12	$-2\,688$
$2p(\pm1)$	0	96		$-13\,184$		96	$-13\,184$	24	$-7\,424$
$3s$	0	336		$-101\,664$		336	$-101\,664$	90	$-383\,940$
$3s$	2	192		$-48\,576$	$-3\,456$	192	$-52\,032$		
$3p(0)$	2	192		$-48\,576$	$-3\,456$	192	$-45\,120$	72	$-225\,990$
$3d(0)$	0	336		$-101\,664$		336	$-101\,664$	108	$-338\,499$
$3d(0)$	2	192		$-48\,576$	$-3\,456$	192	$-52\,032$		
$3p(\pm1)$	1	288	96	$-79\,104$	$-43\,028$	384	$-122\,132$	144	$-3\,687\,039/8$
$3d(\pm1)$	1	288	96	$-79\,104$	$-43\,028$	192	$-36\,076$	72	$-1\,258\,497/8$
$3d(\pm2)$	0	288		$-73\,536$		288	$-73\,536$	108	$-295\,974$

where

$$M_{11}^{(1)}=4n(3n^2-3q^2+1-m^2)\ , \tag{16}$$

$$M_{11}^{(2)}=(-1106n^5-1316n^3-234n+1692n^3q^2-586nq^4+388nq^2+132m^2nq^2+796m^2n^3 +356m^2n-122m^4n)/3\ ; \tag{17}$$

(b) $|q|=1$ (degeneracy removed in the first order)

$$\lambda_1=M_{11}^{(1)}\pm M_{12}^{(1)}\ , \tag{18a}$$

$$\lambda_2=M_{11}^{(2)}\pm M_{12}^{(2)}\ , \tag{18b}$$

where

$$M_{12}^{(1)}=4n(n^2-m^2)\ , \tag{19}$$

$$M_{12}^{(2)}=(-572n^5+320n^3+662m^2n^3 +320m^2n-94m^4n)/3\ ; \tag{20}$$

(c) $|q|=2$ (degeneracy removed in the second order)

$$\lambda_1=M_{11}^{(1)}\ , \tag{21a}$$

$$\lambda_2=M_{11}^{(2)}\pm M_{12}^{(2)}\ , \tag{21b}$$

where

$$M_{12}^{(2)}=-18n^5+36n^3-18n+36m^2n^3 +36m^2n-18m^4n\ . \tag{22}$$

It should be noted that the diagonal quantities in Eqs. (16) and (17) are also applicable to cases (b) and (c). As it can be seen, we need to calculate only four of these $M_{ij}^{(p)}$ since they are invariant with respect to interchange of the indices.

It is understood that the two values of the coefficient in Eqs. (18) and (21b) are actually the roots of the secular equation obtained from the 2×2 perturbation matrix in degenerate perturbation theory. The positive and negative signs correspond, respectively, to the symmetric and antisymmetric combinations of the unperturbed states $|1\rangle$ and $|2\rangle$. The choice depends upon whether the degenerate state in question is symmetric or antisymmetric. The choice can be made by referring to its expansion in terms of the parabolic states which, as noted, are equivalent to the oscillator states. Such expansions for all the degenerate states considered are as follows[12]:

$2s$ $(q=1)$,

$$|200\rangle=(|010\rangle+|100\rangle)/2^{1/2}\ ;$$

$2p(0)$ $(q=1)$,

$$|210\rangle=(|010\rangle-|100\rangle)/2^{1/2}\ ;$$

$3s$ $(q=2,0)$,

$$|300\rangle=(|020\rangle+|200\rangle+|110\rangle)/3^{1/2}\ ;$$

$3p(0)$ $(q=2)$,

$$|310\rangle=(|020\rangle-|200\rangle)/2^{1/2}\ ;$$

$3d(0)$ $(q=2,0)$,

$$|320\rangle=(|020\rangle+|200\rangle)/6^{1/2}-(\tfrac{2}{3})^{1/2}|110\rangle\ ;$$

$3p(\pm1)$ $(q=1)$,

$$|310\rangle=(|011\rangle+|001\rangle)/2^{1/2}\ ;$$

$3d(\pm1)$ $(q=1)$,

$$|320\rangle=(|011\rangle-|001\rangle)/2^{1/2}\ .$$

The coefficients λ_1 and λ_2 can therefore be determined in accordance with the above. In the case of the states $3s$ and $3d(0)$, since they are a mixture of a doubly degenerate state and a nondegenerate state, their eigenvalues are taken to be the weighted average as follows:

$3s$,

$$\lambda_i=\tfrac{2}{3}\lambda_i(q=2)+\tfrac{1}{3}\lambda_i(q=0)\ ;$$

$3d(0)$,

$$\lambda_i=\tfrac{1}{3}\lambda_i(q=2)+\tfrac{2}{3}\lambda_i(q=0)\ .$$

The values of λ_1 and λ_2 for all levels are shown in Table I.

III. RESULTS AND DISCUSSION

The perturbed energy E can now be calculated by using

$$E=-1/(2\lambda^2)+\Delta E_m\ , \tag{23}$$

where ΔE_m is, according to Eq. (5), given by

$$\Delta E_m=\tfrac{1}{2}Bm\ . \tag{24}$$

Utilizing Eq. (7), we obtain the following Rayleigh-

Schrödinger expansion in powers of $F=B^2/8$:

$$E=-1/(2n^2)+\tfrac{1}{2}Bm+E_1F+E_2F^2+O(F^3)\ . \tag{25}$$

The expansion coefficients can be determined from

$$E_1=n\lambda_1/8\ , \tag{26}$$

$$E_2=n^4(5\lambda_1^2/4+n\lambda_2)/32\ . \tag{27}$$

Their values are given in Table I. The perturbation series is known to be divergent but Borel summable. Using E_1 and E_2 we construct the first Padé approximant to determine the energy correction

$$\Delta E^{(1)}=E_1F/(1-E_2F/E_1)\ , \tag{28}$$

so that

$$E=-1/(2n^2)+\tfrac{1}{2}Bm+\Delta E^{(1)}\ . \tag{29}$$

The values of E for the 14 levels are given in Table II. They are, in general, lower than the values obtained by Praddaude.[5] For $B=0.1$, good agreement exists for many of the levels. It is believed that the energy values can be made higher by carrying the calculation to higher orders. For example, using the higher-order coefficients in Ref. 7 for the $1s$ level, we calculate the second Padé approximant[13]

$$\Delta E^{(2)}=\frac{[E_1F+(E_2+b_1E_1)F^2]}{(1+b_1F+b_2F^2)} \tag{30}$$

to replace $\Delta E^{(1)}$ in Eq. (29), where

$$b_1=(E_2E_3-E_1E_4)/(E_1E_3-E_2^2)\ ,$$

$$b_2-(E_2E_4-E_3^2)/(E_1E_3-E_2^2)\ .$$

Consequently, we get for the $1s$ level

$$E=-0.4975\quad (B=0.1)$$

$$E=-0.3536\quad (B=1.0)\ .$$

It has been observed that, at low fields, the low-lying levels with the same value of m do not cross.[14] While, according to Ref. 5, the two levels $3s$ and $3p(0)$ cross somewhere in the range $0.1<B<1.0$, our results show that the no-crossing rule prevails up to $B=1.0$. We believe that the same rule will persist in the strong-field limit because the pth Padé approximant for the energy correction tends to a constant as $B\to\infty$. For $p=1$, the constant is $-E_1^2/E_2$ according to Eq. (28). The binding energies for the $3s$ and $3p(0)$ levels in the strong-field limit are -0.0345 and -0.0327, respectively. With $3s$ being the lower level, the two levels do not cross at any field strength.

The no-crossing rule has been successfully used by Simola and Virtamo[15] in linking the Landau orbitals to hydrogenic levels. These authors studied the hydrogenic levels in the strong-field limit by taking the Coulomb potential as the perturbation after separating the unperturbed wave function into a transverse part ($\perp B$) described by the Landau orbital and a longitudinal part ($\|B$). Since the unperturbed states are no longer hydrogenic, their identification was made on the basis of the no-crossing rule in conjunction with the evenness or oddness of $l+m$ as well as the number of nodes in the wave function. Consider, for example, the four $m=0$ levels $3s$, $3p(0)$, $3d(0)$, and $4p(0)$. The number of nodes in the wave function is 4, 3, 6, and 5, respectively. Therefore, in the strong-field limit, they are arranged in the order of increasing energies as $3p(0)$, $3s$, $4p(0)$, and $3d(0)$. The result is that in this scheme the $3s$ level is higher than the $3p(0)$ level for strong fields as in Ref. 5 which shows that the levels cross as discussed

TABLE II. Energies in a.u. Those in columns 3 and 5 are obtained from Ref. 5.

	$B=0.1$		$B=1.0$	
ls	−0.4951	−0.4975	−0.2624	−0.3312
$2s$	−0.1009	−0.0981	−0.0498	0.3395
$2p(0)$	−0.1133	−0.1124	−0.0733	0.2399
$2p(-1)$	−0.1534	−0.1508	−0.5493	0.0434
$2p(+1)$	−0.0534	−0.0508	0.4507	1.0434
$3s$	−0.0378	−0.0249	−0.0345	0.4336
$3p(0)$	−0.0373	−0.0199	−0.0327	0.4098
$3p(-1)$	−0.0696	−0.0312	−0.5107	0.3745
$3p(+1)$	0.0304	0.0688	0.4893	1.3745
$3d(0)$	−0.0281	0.0122	−0.0212	1.1469
$3d(-1)$	−0.0815	−0.0578	−0.5228	0.2934
$3d(+1)$	0.0185	0.0422	0.4772	1.2934
$3d(-2)$	−0.1251	−0.0878	−1.0163	0.1469
$3d(+2)$	0.0749	0.1122	0.9837	2.1469

earlier. Since no numerical results are given for $B<1.0$ in Ref. 15, it is not known whether the $3s$ level remains higher in the weak-field limit.

On the other hand, our results and the results of Refs. 5 and 15 show that the levels $2p(0)$ and $3d(-2)$ cross. With the use of Eq. (29), it can be shown that the crossing takes place at $B=0.093$. This consensus is in contrast to the conclusion reached in Ref. 8, in which the two levels are taken to be combinations of two tunneling states $|\mathrm{I}>$ and $|\mathrm{II}>$ as follows:

$$|3s\rangle=-0.402\,|\mathrm{I}\rangle-0.915\,|\mathrm{II}\rangle\ ,$$

$$|3d\rangle=-0.915\,|\mathrm{I}\rangle+0.402\,|\mathrm{II}\rangle\ ,$$

instead of the parabolic states mentioned in Sec. II.

We believe that our second-order results are useful for relatively weak fields. For stronger fields and/or higher levels, higher-order calculations are needed. Clearly, the controversy over the Zeeman level structure also needs to be studied and definitively resolved. Our degenerate perturbative approach with the use of oscillator (or the underlying parabolic) quantum numbers provides a way for resolving the issue on level crossing.[16]

APPENDIX

Let us define

$$|ab)=|N_1+a,N_2+b,m\rangle\ ,$$

$$(abcd)=\langle N_1+a,N_2+b,m\,|\,G\,|\,N_1+c,N_2+d,m\rangle\ ,$$

$$a_\pm=[(N_1\pm1)^2-m^2]^{1/2}\ ,$$

$$b_\pm=[(N_1\pm3)^2-m^2]^{1/2}\ ,$$

$$c_\pm=[(N_1\pm5)^2-m^2]^{1/2}\ ,$$

$$A_\pm=[(N_2\pm1)^2-m^2]^{1/2}\ ,$$

$$B_\pm=[(N_2\pm3)^2-m^2]^{1/2}\ .$$

Then we can write the action of N, W, M, and V as follows:

$$N\,|00)=\tfrac{1}{2}(N_1+N_2)\,|00)\ ,$$

$$W\,|00)=\tfrac{1}{4}[\,a_+\,|20)+a_-\,|-20)+A_+\,|02)+A_-\,|0-2)]\ ,$$

$$M\,|00)=\tfrac{1}{2}(N_1-N_2)\,|00)\ ,$$

$$V\,|00)=\tfrac{1}{4}[\,a_+\,|20)+a_-\,|-20)-A_+\,|02)-A_-\,|0-2)]\ .$$

It then follows that there are, all together, 21 non-vanishing matrix elements of the perturbing potential, given below:

$$(0000)=4N_1N_2(N_1+N_2)+N_2(a_+^2+a_-^2)+N_1(A_+^2+A_-^2)\ , \tag{A1}$$

$$(0020)=a_+[\,2N_2(N_1+N_2+2)+2N_1N_2+\tfrac{1}{2}A_+^2+\tfrac{1}{2}A_-^2]\ , \tag{A2}$$

$$(00-20)=a_-[\,2N_2(N_1+N_2-2)+2N_1N_2+\tfrac{1}{2}A_+^2+\tfrac{1}{2}A_-^2]\ , \tag{A3}$$

$$(0002)=A_+[\,2N_1(N_1+N_2+2)+2N_1N_2+\tfrac{1}{2}a_+^2+\tfrac{1}{2}a_-^2]\ , \tag{A4}$$

$$(000-2)=A_-[\,2N_1(N_1+N_2-2)+2N_1N_2+\tfrac{1}{2}a_+^2+\tfrac{1}{2}a_-^2]\ , \tag{A5}$$

$$(0040)=N_2a_+b_+\ , \tag{A6}$$

$$(00-40)=N_2a_-b_-\ , \tag{A7}$$

$$(0004)=N_1A_+B_+\ , \tag{A8}$$

$$(000-4)=N_1A_-B_-\ , \tag{A9}$$

$$(0022)=2a_+A_+(N_1+N_2+2)\ , \tag{A10}$$

$$(00-2-2)=2a_-A_-(N_1+N_2-2)\ , \tag{A11}$$

$$(002-2)=2a_+A_-(N_1+N_2)\ , \tag{A12}$$

$$(00-22)=2a_-A_+(N_1+N_2)\ , \tag{A13}$$

$$(0042)=\tfrac{1}{2}a_+b_+A_+\ , \tag{A14}$$

$$(00-42)=\tfrac{1}{2}a_-b_-A_+\ , \tag{A15}$$

$$(004-2)=\tfrac{1}{2}a_+b_+A_-\ , \tag{A16}$$

$$(00-4-2)=\tfrac{1}{2}a_-b_-A_-\ , \tag{A17}$$

$$(0024)=\tfrac{1}{2}a_+A_+B_+\ , \tag{A18}$$

$$(00-24)=\tfrac{1}{2}a_-A_+B_+\ , \tag{A19}$$

$$(002-4)=\tfrac{1}{2}a_+A_-B_-\ , \tag{A20}$$

$$(00-2-4)=\tfrac{1}{2}a_-A_-B_-\ . \tag{A21}$$

The matrix element $M_{11}^{(1)}$ is given by (A1), which, after a little algebra, becomes

$$M_{11}^{(1)}=2(N_1+N_2)(3N_1N_2+1-m^2)=M_{22}^{(1)}\ .$$

Using Eq. (8), we obtain the result shown in Eq. (16). To calculate $M_{11}^{(2)}$ from Eq. (13), we square all the above matrix elements except (A1), (A12), and (A13) and divide each by its appropriate denominator. It can be seen that the terms fall into four groups. It is convenient to sum them separately and combine the subsums. The result is given in Eq. (17).

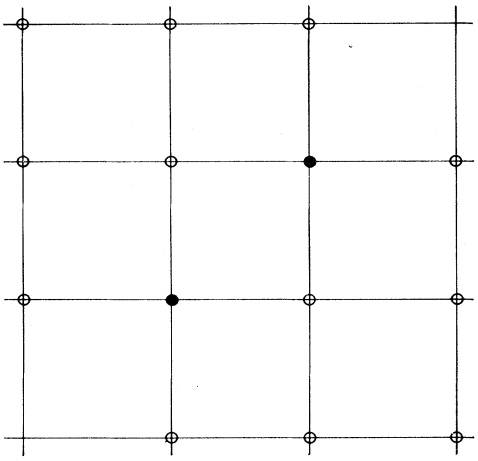

FIG. 1. Intermediate states in $M_{12}^{(2)}$, $|q|=1$. ● denotes initial/final states 02 and 20. ○ denotes intermediate states 04, 24, 44 (in the top row), etc.

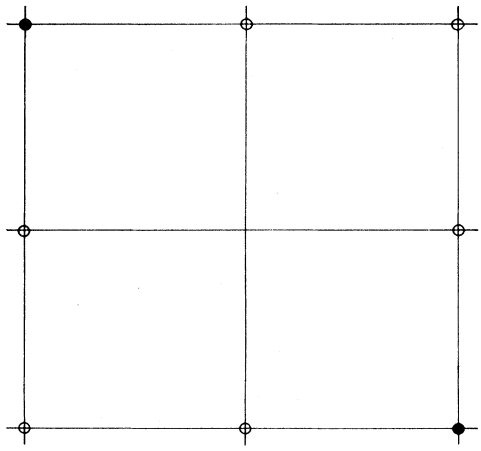

FIG. 2. Intermediate states in $M_{12}^{(2)}$, $|q|=2$. ● denotes initial/final states 04 and 40. ○ denotes intermediate states 24, 44 (in the top row), etc.

The first-order off-diagonal matrix element is nonvanishing only for the case $N_2=N_1\pm2$ or $|q|=1$. It can be obtained from (A12) or (A13). For this case, we have

$$M_{12}^{(1)}=(0220)=4a_+^2(N_1+1)=4n(n^2-m^2)=M_{21}^{(1)} ,$$

which is Eq. (19).

To calculate the quantity $M_{12}^{(2)}$ for the two cases (b) $|q|=1$ and (c) $|q|=2$, we refer to Figs. 1 and 2 to ascertain the intermediate states. It is seen that there are 12 terms and 6 terms to sum for the two cases, respectively. They are as follows:

(b) $|q|=1$,

$$-\tfrac{1}{2}(0200)(2220)$$
$$=-\tfrac{1}{2}a_+^2[\,4(N_1+2)^2+2N_1(N_1+2)+\tfrac{1}{2}a_+^2+\tfrac{1}{2}b_+^2\,]^2 ,$$

$$\tfrac{1}{2}(0200)(0020)=\tfrac{1}{2}a_+^2[4N_1^2+2N_1(N_1+2)+\tfrac{1}{2}a_+^2+\tfrac{1}{2}a_-^2]^2 ,$$

$$-(0244)(4420)/6=-a_+^2b_+^4/24 ,$$

$$(02-2-2)(-2-220)/6=a_+^2a_-^4/24 ,$$

$$-\tfrac{1}{2}(0204)(0420)=-\tfrac{1}{4}a_+^2b_+^2[6N_1(N_1+2)+\tfrac{1}{2}a_+^2+\tfrac{1}{2}a_-^2] ,$$

$$\tfrac{1}{2}(02-22)(-2220)=\tfrac{1}{4}a_+^2a_-^2[6N_1(N_1+2)+\tfrac{1}{2}a_+^2+\tfrac{1}{2}b_+^2] ,$$

$$\tfrac{1}{2}(022-2)(2-220)=\tfrac{1}{4}a_+^2a_-^2[6N_1(N_1+2)+\tfrac{1}{2}a_+^2+\tfrac{1}{2}b_+^2] ,$$

$$-\tfrac{1}{2}(0240)(4020)=-\tfrac{1}{4}a_+^2b_+^2[6N_1(N_1+2)+\tfrac{1}{2}a_+^2+\tfrac{1}{2}a_-^2] ,$$

$$-\tfrac{1}{4}(0242)(4220)=-a_+^2b_+^2(N_1+2)^2 ,$$

$$-\tfrac{1}{4}(0224)(2420)=-a_+^2b_+^2(N_1+2)^2 ,$$

$$\tfrac{1}{4}(02-20)(-2020)=a_+^2a_-^2N_1^2 ,$$

$$\tfrac{1}{4}(020-2)(0-220)=a_+^2a_-^2N_1^2 ;$$

(c) $|q|=2$,

$$-\tfrac{1}{2}(0424)(2440)=-\tfrac{1}{4}a_+^2b_+^2[\,6(N_1+4)(N_1+2)+\tfrac{1}{2}b_+^2+\tfrac{1}{2}c_+^2\,] ,$$

$$-\tfrac{1}{2}(0442)(4240)=-\tfrac{1}{2}(0424)(2440) ,$$

$$\tfrac{1}{2}(0402)(0240)=\tfrac{1}{4}a_+^2b_+^2[\,6N_1(N_1+2)+\tfrac{1}{2}a_+^2+\tfrac{1}{2}a_-^2\,] ,$$

$$\tfrac{1}{2}(0420)(2040)=\tfrac{1}{2}(0402)(0240) ,$$

$$-\tfrac{1}{4}(0444)(4440)=-\tfrac{1}{4}N_2^2a_+^2b_+^2 ,$$

$$\tfrac{1}{4}(0400)(0040)=\tfrac{1}{4}N_1^2a_+^2b_+^2 .$$

The results are given in Eqs. (20) and (22).

[1]See, for example, H. J. Silverstone, Phys. Rev. A 18, 1853 (1978) and references contained therein.
[2]D. M. Larsen, J. Phys. Chem. Solids 29, 271 (1968).
[3]E. P. Pokatilov and M. M. Rusanov, Fiz. Tverd. Tela (Leningrad) 10, 3117 (1968) [Sov. Phys.—Solid State 10, 2458 (1969)].
[4]D. Cabib, E. Fabri, and G. Fiorio, Solid State Commun. 9, 1517 (1971).
[5]H. C. Praddaude, Phys. Rev. A 6, 1321 (1972).
[6]M. Bednář, Phys. Rev. A 15, 27 (1977).
[7]J. E. Avron, B. G. Adams, J. Čížek, M. Clay, M. L. Glasser, P. Otto, J. Paldus, and E. Vrscay, Phys. Rev. Lett. 43, 691 (1979).
[8]B. G. Adams, J. E. Avron, J. Čížek, P. Otto, J. Paldus, R. K. Moats, and H. J. Silverstone, Phys. Rev. A 21, 1914 (1980).
[9]C. M. Bender, L. D. Mlodinow, and N. Papanicolaou, Phys. Rev. A 25, 1305 (1982).
[10]$\varphi=\beta+\gamma$, where β and γ are the phase angles of the two two-dimensional oscillators. See Ref. 11.
[11]A. C. Chen, J. Math. Phys. 23, 412 (1982).
[12]See, for example, C. B. Tarter, J. Math. Phys. 11, 3192 (1970).
[13]See, for example, A. Chen, Nuovo Cimento A 52, 474 (1967).
[14]M. L Zimmerman, M. M. Kash, and D. Kleppner, Phys. Rev. Lett. 45, 1092 (1980).
[15]J. Simola and J. Virtamo, J. Phys. B 11, 3309 (1978).
[16]The approach has been applied to determine the hydrogenic Stark levels by the author (unpublished).