Resonant scattering in the presence of an electromagnetic field

Leonard Rosenberg
Department of Physics, New York University, New York, New York 10003
(Received 7 April 1983)

The theory of resonant reactions, in the projection-operator formulation of Feshbach, is generalized to account for the presence of an external electromagnetic field. The theory is used as the basis for the construction of low-frequency approximations for the transition amplitude. Results obtained here for scattering in a laser field confirm earlier versions of the low-frequency approximation when the resonances are isolated. However, if there are several closely spaced resonances additional terms must be included (their importance magnified by the appearance of near singularities) which account for the effect of radiative transitions between pairs of nearly degenerate resonant states. The weak-field limit of this result yields a low-frequency approximation for single-photon spontaneous bremsstrahlung which, through the inclusion of correction terms associated with closely spaced resonances, provides an improvement over the Feshbach-Yennie version derived some time ago. A separate treatment is required to deal with the limiting case of a static external field and this is worked out here in the context of a time-dependent formulation of the scattering problem. Linear and quadratic Stark splitting of the resonance positions, and resonance broadening due to the tunneling mechanism, are expected to play a significant role in the static limit and these effects are included in the approximation derived here for the transition amplitude.

I. INTRODUCTION

It was suggested some time ago by Feshbach and Yennie,[1] in the context of nuclear reaction theory, that the process of spontaneous bremsstrahlung of low-frequency photons can serve as a probe of resonant scattering dynamics. More recently, similar suggestions have been made in connection with electron-atom scattering in a low-frequency laser field (stimulated bremsstrahlung).[2,3] The effect of a strong *static* electric field on resonant states of the H^- system has been studied experimentally[4] and theoretically.[5,6] In view of this interest it seems worthwhile to have available a general formulation of the theory of resonant scattering in the presence of a field (either external or spontaneously produced), and this will be developed below. It will allow one to examine, somewhat more systematically than has been done in the past, the domain of validity of various low-frequency approximations for the transition amplitude which have been proposed. It turns out that in certain cases the approximations must be modified (if they are to preserve the level of accuracy claimed for them) through the addition of certain correction terms and this will be discussed in detail.

As is well understood, the dominant effect of a low-frequency field, over a wide range of field strengths, is to modify the wave functions which describe the motion of the projectile before and after the collision.[7] Our main concern here will be with the additional dynamical effect which the field may have on the scattering system during a resonance. Such an interaction will affect the probability for emission and absorption of radiation and may also introduce an observable change in the position and width of a resonance. The most significant modification of earlier theoretical treatments[1-3] occurs when there are a number of closely spaced resonances in the absence of the field. The effect of the field during the resonance is enhanced as a result of this near-degeneracy. In particular, we shall see that additional terms must be added to the standard Feshbach-Yennie approximation[1] for single-photon bremsstrahlung to account for photon emission by the resonant system. Analogous correction terms, corresponding to absorption and stimulated emission of photons during the resonance, must be included in the low-frequency approximation for scattering in a laser field. Actually, it is this problem which we shall study in detail since the result for spontaneous bremsstrahlung can be derived from it very simply by taking the appropriate weak-field limit.

Our analysis is based on the Feshbach projection-operator formalism,[8] generalized to include the effect of the field.[9,10] In Sec. II we consider scattering in a strong external radiation field which we describe using the occupation-number representation. This choice (as opposed to the classical description to which it is equivalent in the intense-field regime) simplifies passage to the case of single-photon spontaneous bremsstrahlung. While the formalism is quite general we have in mind a situation in which the resonance width Δ is comparable to or smaller than the photon energy $\hbar\omega$. Furthermore, when we deal with several closely spaced resonances the spacing is understood to be not substantially larger than $\hbar\omega$. In such cases only a small subset of photon states—those which bring the particle-field system into resonance—will contribute significantly to the resonant component of the scattering amplitude. A truncation of the infinite sum over intermediate photon states [such a sum appears in Eq. (2.50) below] may then be justified.

For $\hbar\omega \ll \Delta$ a very large number of photon states must be included and a somewhat different approach is required to deal effectively with this case. Here, in this extreme low-frequency limit, the field is essentially static during the lifetime of the resonant state. As a result the tunneling mechanism, in which the projectile is "drawn out" of the resonant state in which it is temporarily bound, with the lifetime of that state thereby shortened, may play a more significant dynamical role and some allowance should be made for this effect in the approximation procedure. (In the opposite limit, $\hbar\omega \gg \Delta$, the field changes sign many times during the resonance lifetime and the tunneling mechanism is presumably less important.) In Sec. III the theory is reformulated to allow for a convenient treatment of the limiting case of a static external field. The procedure is not simply one in which the frequency ω is allowed to approach zero in the formalism of Sec. II. That limit is a singular one; it corresponds to subjecting the projectile in the initial or final state to a uniform acceleration over an infinite time interval. Such singular behavior may be avoided, however, if the field is taken to be a pulse of finite width. One can then construct wave packets which spend only a finite amount of time in the field.[11] In order to treat the field in this more realistic fashion it is convenient to describe it classically within the context of time-dependent scattering theory. The present treatment generalizes one given earlier[12] in which the scattering was explicitly assumed to be nonresonant. In addition to the tunneling effect, which leads to a broadening of resonance lines, the approximation developed here takes into account transitions between nearly degenerate resonant states which results in a linear Stark shift of the resonance positions. Direct numerical studies of the dependence of resonance parameters on the strength of an applied static field have been made previously.[5,6,13] The present discussion, in addition to providing a scattering-theoretic background for such studies, leads to an approximation for the scattering amplitude in which the effect of the field on the resonant intermediate state, as well as on the asymptotic states, is accounted for explicitly. Section IV contains a summary and discussion of our results.

II. SCATTERING IN A LOW-FREQUENCY RADIATION FIELD

A. Effective-potential formulation

In order to provide some background we begin here with a brief summary of the time-independent effective-potential approach to the problem of scattering in a low-frequency laser field.[10] A number of simplifying assumptions will be introduced to allow us to focus more directly on those features of the problem of present interest. Thus, the target is assumed to be neutral and infinitely massive. The field consists of a beam of photons each having frequency ω and linear polarization vector $\vec{\lambda}$. The momentum carried by the photon is assumed to be negligible; this corresponds to the neglect of projectile-recoil effects of order v/c. In that approximation the A^2 contribution to the particle-field interaction plays no role and will be dropped here.[14–16] We confine our attention to the case of elastic scattering. That is, the target, initially in its ground state $|\chi\rangle$, returns to that state after the collision. Pauli exchange processes will not be explicitly accounted for; the particles are treated as if they were distinguishable. These restrictions do not reflect any intrinsic limitations of the theory.

As a result of the collision the projectile, of charge e and mass μ, has its momentum changed from $\vec{p}$ to $\vec{p}\,'$ while the photon number of the field changes from n to n'. Upon entering the field the projectile, as a result of multiple absorption and emission of photons, has its plane-wave state modified to the form

$$|\psi_{n,\vec{p}}\rangle = \sum_{l=-\infty}^{\infty} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{il\phi} e^{i\rho_{\vec{p}} \sin\phi} |n+l\rangle \, |\vec{p}(\phi)\rangle \; . \tag{2.1}$$

Here we define

$$\omega\rho_{\vec{p}} = -\left[\frac{e}{\mu c}\right] \vec{\mathscr{A}}\cdot\vec{p} \; , \tag{2.2a}$$

$$\vec{\mathscr{A}} = \left[\frac{8\pi c I}{\omega^2}\right]^{1/2} \vec{\lambda} \; , \tag{2.2b}$$

where I, the field intensity, is related to the photon number n and quantization volume L^3 (in units where $\hbar = 1$) by $I = n\omega c/L^3$. The strength parameter $\rho_{\vec{p}}$ represents the ratio of the particle-field interaction energy to the photon energy and can be of order unity or greater, even for moderate field strengths, if the frequency is sufficiently low. This is a reflection of the near-degeneracy of the asymptotic particle-field states in the low-frequency domain and necessitates a nonperturbative treatment of these states. Equation (2.1) may be obtained as follows.[15] One first solves the Schrödinger equation, in the Coulomb gauge, to account for the projectile-field interaction. (With recoil effects ignored the projectile momentum $\vec{p}$ remains fixed.) The solution takes the form of an expansion in photon occupation-number states with expansion coefficients—they turn out to be Bessel functions—which represent the probability amplitudes for the virtual emission or absorption of a specified number of photons. The shift in momentum from $\vec{p}$ to $\vec{p}(\phi) \equiv \vec{p} - (e/c)\vec{\mathscr{A}}\cos\phi$ in Eq. (2.1) arises from the introduction of a gauge transformation generated by the operator $\exp(i\vec{A}\cdot\sum_j e_j \vec{r}_j/c)$, where e_j and r_j are, respectively, the charge and position vector of the jth particle in the system, and $\vec{A}$ is the operator representing the vector potential at the origin of coordinates. In addition to introducing momentum shifts in the initial and final states this gauge transformation has the effect of converting the "$\vec{p}\cdot\vec{A}$" interaction of the Coulomb gauge to the "$\vec{r}\cdot\vec{E}$" form of the electric-field gauge. This is a useful step to take since for low frequencies, and in the absence of near-degeneracies, the transformed interaction is effectively weaker.[17] By neglecting this interaction completely in intermediate states as well as its effect on the target system in initial and final states one arrives at the low-frequency approximation derived previously.[15]

We now indicate how that earlier result, for nonresonant scattering, may be recovered in the context of the effective-potential formalism. With the target-field interaction in initial and final states ignored at the outset[18] the amplitude for the transition $\{n,\vec{p}\}\rightarrow\{n',\vec{p}\,'\}$ may be represented as

$$\mathscr{T}_{n',\vec{p}\,';n,\vec{p}}=\langle\psi_{n',\vec{p}\,'}|\mathscr{T}(E)|\psi_{n,\vec{p}}\rangle\ . \tag{2.3}$$

Here $E=(p^2/2\mu)+n\omega$ represents the total energy which, in the initial state, consists of separate contributions from the projectile and field subsystems. (The target energy does not appear here since, for convenience, we have chosen the target ground state as our reference level.) The scattering operator satisfies the equivalent one-body Lippmann-Schwinger equation

$$\mathscr{T}(E)=\mathscr{V}(E)+\mathscr{V}(E)\mathscr{G}(E)\mathscr{T}(E)\ . \tag{2.4}$$

In general,[10] the interaction of the projectile with the field is accounted for not only in the effective potential $\mathscr{V}$ but in the single-particle propagator $\mathscr{G}$ as well since that interaction persists even when the projectile and target are well separated. However, as indicated above, the (gauge-transformed) particle-field interaction is neglected in intermediate states in formulating the low-frequency approximation for nonresonant scattering. (Inclusion of this interaction in intermediate states could be accomplished perturbatively but in the absence of resonances this would introduce corrections of higher order in the photon frequency.[17] The importance of such corrections to the basic low-frequency approximation is a matter of some interest, deserving further experimental and theoretical investigation, but we shall not pursue this question here.) Accordingly, $\mathscr{G}$ is replaced by the free propagator

$$\mathscr{G}\cong(E-H_F-K)^{-1}\ , \tag{2.5}$$

where $H_F=\omega a^\dagger a$ is the field Hamiltonian and K is the projectile kinetic-energy operator. The effective potential is defined by

$$\mathscr{V}(E)=\langle\chi|[V+VG^Q(E)V]|\chi\rangle\ , \tag{2.6}$$

where V is the projectile-target interaction energy and G^Q is the resolvent operator for the full system, modified by the "removal" of the target ground state. This is accomplished by means of the projection operator $Q=1-|\chi\rangle\langle\chi|$ so that

$$G^Q(E)=[Q(E-H)Q]^{-1}\ , \tag{2.7}$$

with H representing the Hamiltonian of the system. We have the decomposition

$$H=H_0+H_F+H_I\ , \tag{2.8}$$

where H_0 is the Hamiltonian of the projectile-target system in the absence of the field. In the electric-dipole approximation the (gauge-transformed) particle-field interaction takes the form

$$H_I=-e\vec{E}\cdot\vec{R}\ , \tag{2.9}$$

where $e\vec{R}$ is the dipole operator of the system of particles and $\vec{E}$, the electric field at the origin of coordinates, is given in terms of the photon creation and annihilation operators as

$$\vec{E}=i\left[\frac{\omega}{c}\right]\left[\frac{2\pi c^2}{\omega L^3}\right]^{1/2}\vec{\lambda}(a-a^\dagger)\ . \tag{2.10}$$

In the approximation in which H_I is neglected in the definition of the effective potential, Eq. (2.6), as well as in the propagator, Eq. (2.5), the field affects the $\mathscr{T}$ operator only through the appearance of the energy term H_F in Eq. (2.4). The solution can therefore be expressed in terms of the field-free scattering operator T. Explicitly, we have, for arbitrary photon states $|l\rangle$ and $|l'\rangle$, $\langle l'|\mathscr{T}(E)|l\rangle\rightarrow T(E-l\omega)\delta_{l'l}$ for $H_I\rightarrow 0$. An exact formal solution of the Lippmann-Schwinger integral equation for $T(\Omega)$ may be constructed with the aid of the resolvent operator; this provides us with the representation

$$\begin{aligned}T(\Omega;\vec{q}\,',\vec{q})&\equiv\langle\vec{q}\,'|T(\Omega)|\vec{q}\rangle\\&=\langle\vec{q}\,'|\langle\chi|[V+V(\Omega-H_0)^{-1}V]|\chi\rangle|\vec{q}\rangle\end{aligned} \tag{2.11}$$

for the off-shell T-matrix element. With $\mathscr{T}$ expressed in terms of T in the manner shown above, and with the aid of the representation (2.1) of the asymptotic states, we obtain the low-frequency approximation

$$\begin{aligned}\mathscr{T}_{n',\vec{p}\,';n,\vec{p}}\cong\sum_l\int_0^{2\pi}\frac{d\phi'}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}&\exp\{-i[(l+n-n')\phi'+\rho_{\vec{p}\,'}\sin\phi']\}\\&\times\exp[i(l\phi+\rho_{\vec{p}}\sin\phi)]T(E-(n+l)\omega;\vec{p}\,'(\phi'),\vec{p}(\phi))\ .\end{aligned} \tag{2.12}$$

Further analysis shows that the off-energy-shell components of the T matrix cancel to good approximation over a wide range of field stengths.[19] The result may be expressed in terms of the Bessel function

$$J_l(\rho)=\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(l\phi-\rho\sin\phi)} \tag{2.13}$$

as

$$\mathscr{T}_{n',\vec{p}\,';n,\vec{p}}\cong\sum_l J_{n'-n-l}(\rho_{\vec{p}\,'})J_{-l}(\rho_{\vec{p}})T(p_l^2/2\mu;\vec{p}\,'_l,\vec{p}_l)\ , \tag{2.14}$$

where

$$\vec{p}_l=\vec{p}-\frac{\mu l\omega}{\vec{p}\cdot\vec{\lambda}}\vec{\lambda}\ , \tag{2.15a}$$

$$\vec{p}\,'_l=\vec{p}\,'+\frac{\mu(n'-n-l)\omega}{\vec{p}\,'\cdot\vec{\lambda}}\vec{\lambda}\ . \tag{2.15b}$$

If we ignore terms of second order in ω and make use of energy conservation in the form $(p^2/2\mu)+n\omega=(p'^2/2\mu)+n'\omega$ we see that the on-shell condition $p_l^2/2\mu=p_l'^2/2\mu$ is satisfied. The origin of the shifted momenta $\vec{p}_l$ and $\vec{p}\,'_l$ which appear in Eq. (2.14) can be traced to the introduction of the above-mentioned gauge transformation whose generator acts as a momentum-translation operator on the initial- and final-state wave functions. These momentum shifts introduce corrections of order ω to the lowest-order approximation obtained by Krüger and Jung.[2] Details of the derivation of Eq. (2.14) can be found in Ref. 15.

The amplitude $T(\Omega;\vec{q}\,',\vec{q})$, when on the energy shell, may be taken to be a function of two scalar variables, the energy Ω and the momentum transfer squared $\tau=(\vec{q}\,'-\vec{q})^2$. In the absence of resonances the T matrix in Eq. (2.14) may be expanded in a Taylor series about $\Omega=p^2/2\mu$ and $\tau=(p'-\vec{p})^2$, with terms of order ω^2 ignored. The sum over l may then be performed with the use of some well-known properties of the Bessel function (recursion relation and addition formula). The result may be expressed as

$$\mathscr{T}_{n',\vec{p}\,';n,\vec{p}}\cong J_{n'-n}(\rho_{\vec{p}\,'-\vec{p}})T(\Omega_\omega;\vec{p}\,'_\omega,\vec{p}_\omega) \tag{2.16}$$

with

$$\begin{aligned}\vec{p}_\omega&=\vec{p}+\frac{\mu(n'-n)\omega}{(\vec{p}\,'-\vec{p})\cdot\vec{\lambda}}\vec{\lambda}\ ,\\ \vec{p}\,'_\omega&=\vec{p}\,'+\frac{\mu(n'-n)\omega}{(\vec{p}\,'-\vec{p})\cdot\vec{\lambda}}\vec{\lambda}\ ,\end{aligned} \tag{2.17}$$

and $\Omega_\omega=p_\omega^2/2\mu=p_\omega'^2/2\mu$. This is the version of the low-frequency approximation obtained by Kroll and Watson[20] and is valid, it should be emphasized, in the absence of resonances.

It is not difficult to verify that the Bessel functions in Eq. (2.14) sum up the perturbation series for the projectile-field interaction in asymptotic states; the result of first-order perturbation theory is then obtained with the use of the small argument limit of the Bessel function, with I replaced by $\omega c/L^3$ in the expression (2.2b) for $\vec{\mathscr{A}}$. In this way we obtain the low-frequency approximation for single-photon emission in the form

$$\begin{aligned}\mathscr{T}^{\rm FY}_{1,\vec{p}\,';0,\vec{p}}=-e\left[\frac{2\pi}{\mu^2\omega L^3}\right]^{1/2}&\\ \times\Bigg[\frac{\vec{p}\,'\cdot\vec{\lambda}}{\omega}T&\left[\frac{p^2}{2\mu};\vec{p}\,'+\frac{\mu\omega}{\vec{p}\,'\cdot\vec{\lambda}}\vec{\lambda},\vec{p}\right]\\ -\frac{\vec{p}\cdot\vec{\lambda}}{\omega}T&\left[\frac{p'^2}{2\mu};\vec{p}\,',\vec{p}-\frac{\mu\omega}{\vec{p}\cdot\vec{\lambda}}\vec{\lambda}\right]\Bigg]\ .\end{aligned} \tag{2.18}$$

This is just the Feshbach-Yennie result[1] specialized to the case of linear polarization of the emitted photon.

Note that we have *not* assumed that the field-free scattering amplitude is a slowly varying function of the energy parameter in deriving Eqs. (2.14) and (2.18). However, the field interaction has been ignored in intermediate states and the validity of this approximation must be examined more carefully for resonant scattering. We now turn to this analysis.

B. Resonant scattering

For definiteness we shall suppose that there are two closely spaced resonances of the Feshbach type[8] near the scattering energy of interest. Thus, following Feshbach, we assume the existence of two bound states satisfying

$$QH_0Q\,|B_j\rangle=e_j\,|B_j\rangle;\quad \langle B_j\,|B_k\rangle=\delta_{jk},\quad j,k=1,2\ . \tag{2.19}$$

The resonances may be traced to the appearance of singularities in $\mathscr{V}(E)$. To display these singularities explicitly we introduce the projection operators

$$p=\sum_{j=1}^{2}|B_j\rangle\langle B_j|\ ,\quad q=Q-p\ . \tag{2.20}$$

We then have the identity[21]

$$G^Q=G^q+(1+G^qH_I)pG^Qp(1+H_IG^q)\ , \tag{2.21}$$

where

$$G^q(E)=[q(E-H)q]^{-1}\ . \tag{2.22}$$

Interaction of the resonant system with the field is described by the propagator

$$pG^Qp=[p(E-H-H_IG^qH_I)p]^{-1}\ . \tag{2.23}$$

This may be expanded in terms of a conveniently chosen set of basis states which we take to be of the form

$$|B_{jn}\rangle=\sum_{m=-\infty}^{\infty}\sum_{k=1}^{2}a_m^{(k)}(j,n)\,|B_k\rangle\,|m+n\rangle\ . \tag{2.24}$$

We require that H be diagonal in this basis in the limit of exact degeneracy, $e_1=e_2$; in this way we account nonperturbatively for the dominant contribution of the effect of transitions between states $|B_1\rangle$ and $|B_2\rangle$ induced by the low-frequency field. This requirement (along with the condition that $|B_{jn}\rangle\to|B_j\rangle\,|n\rangle$ for $H_I\to 0$) fixes the expansion coefficients as

$$a_m^{(k)}(j,n)=\begin{cases}(-i)^m\frac{1}{2}[J_{-m}(\sigma)+J_m(\sigma)], & k=j\\ (-i)^m\frac{1}{2}[J_{-m}(\sigma)-J_m(\sigma)], & k\neq j\end{cases} \tag{2.25}$$

as will be verified below in Eq. (2.32). The parameter σ is defined by

$$\omega\sigma=-e\left[\frac{\omega}{c}\right]\vec{\mathscr{A}}\cdot\langle B_2\,|\vec{R}\,|B_1\rangle\ ; \tag{2.26}$$

we assume that $\langle B_2\,|\vec{R}\,|B_1\rangle=\langle B_1\,|\vec{R}\,|B_2\rangle$ is real and $\langle B_j\,|\vec{R}\,|B_j\rangle=0,\ j=1,2$.

We wish to evaluate matrix elements of H in the basis (2.24). This is done most conveniently by writing

$$|B_{1n}\rangle=\frac{1}{\sqrt{2}}(|B_{+n}\rangle+|B_{-n}\rangle)\,,$$
$$|B_{2n}\rangle=\frac{1}{\sqrt{2}}(|B_{+n}\rangle-|B_{-n}\rangle)\,, \tag{2.27a}$$

where

$$|B_{\pm n}\rangle=\frac{1}{\sqrt{2}}(|B_1\rangle\pm|B_2\rangle)|b_{\pm n}\rangle\,. \tag{2.27b}$$

It follows from the choice (2.25) that

$$|b_{\pm n}\rangle=\sum_{m=-\infty}^{\infty}(-i)^m J_{\mp m}(\sigma)|m+n\rangle\,. \tag{2.27c}$$

The recursion relation

$$J_{m-1}(\sigma)+J_{m+1}(\sigma)=\frac{2m}{\sigma}J_m(\sigma) \tag{2.28}$$

may now be used to verify the eigenvalue equation

$$[\omega a^\dagger a\pm\tfrac{1}{2}i\omega\sigma n^{-1/2}(a-a^\dagger)]|b_{\pm n}\rangle=n\omega|b_{\pm n}\rangle\,. \tag{2.29}$$

(We use the large-n approximation $a|n+m\rangle\cong n^{1/2}\times|n+m-1\rangle$, $a^\dagger|n+m\rangle\cong n^{1/2}|n+m+1\rangle$.) Other useful relations, which may be established with the aid of the integral representation (2.13) for the Bessel function, are

$$\langle b_{\pm n'}|b_{\pm n}\rangle=\delta_{n'n}\,,$$
$$\langle b_{+n'}|b_{-n}\rangle=\langle b_{-n'}|b_{+n}\rangle^*=(-i)^{n'-n}J_{n'-n}(2\sigma)\,. \tag{2.30}$$

With these relations in hand we readily verify that

$$\langle B_{j'n'}|B_{jn}\rangle=\delta_{j'j}\delta_{n'n}\,, \tag{2.31}$$

$$\langle B_{j'n'}|H|B_{jn}\rangle=\left[\frac{e_1+e_2}{2}+n\omega\right]\delta_{n'n}\delta_{j'j}+m_{j',n';j,n}i^{(n'-n)}\left[\frac{e_1-e_2}{2}\right]J_{n'-n}(2\sigma) \tag{2.32a}$$

with the matrix $\underline{m}$ defined by

$$m_{j',n';j,n}=\begin{cases}(-1)^{j+1}\frac{1}{2}[1+(-1)^{n'-n}], & j'=j\\ (-1)^{j}\frac{1}{2}[1-(-1)^{n'-n}], & j'\neq j\,.\end{cases} \tag{2.32b}$$

The propagator pG^Qp, defined by Eq. (2.23), may be expanded in terms of the basis functions $|B_{jn}\rangle$. For notational simplicity we denote the pair of indices j,n by the single index α and write

$$pG^Qp=\sum_{\alpha'}\sum_{\alpha}|B_{\alpha'}\rangle(\underline{d}^{-1})_{\alpha'\alpha}\langle B_\alpha|\,, \tag{2.33}$$

where

$$d_{\alpha'\alpha}\equiv\langle B_{\alpha'}|(E-H-H_IG^qH_I)|B_\alpha\rangle\,. \tag{2.34}$$

The decomposition (2.21) for G^Q leads to a corresponding decomposition for the effective potential (2.6) of the form

$$\mathscr{V}=\mathscr{V}^q+\sum_{\alpha'}\sum_{\alpha}|\gamma_{\alpha'}\rangle(\underline{d}^{-1})_{\alpha'\alpha}\langle\bar{\gamma}_\alpha|\,, \tag{2.35}$$

where

$$\mathscr{V}^q=\langle\chi|(V+VG^qV)|\chi\rangle \tag{2.36}$$

and

$$|\gamma_{\alpha'}\rangle=\langle\chi|(V+VG^qH_I)|B_{\alpha'}\rangle\,, \tag{2.37a}$$

$$\langle\bar{\gamma}_\alpha|=\langle B_\alpha|(V+H_IG^qV)|\chi\rangle\,. \tag{2.37b}$$

To display the resonant contribution to the scattering operator in an explicit form we combine Eqs. (2.4) and (2.35), and make use of some algebraic techniques familiar from the theory of integral equations with separable kernels.[22] Thus, we define the nonresonant contribution to the scattering operator as the solution of

$$\mathscr{T}^q=\mathscr{V}^q+\mathscr{V}^q\mathscr{G}\mathscr{T}^q\,, \tag{2.38}$$

and introduce the level-shift matrix

$$\Delta_{\alpha'\alpha}=\langle\bar{\gamma}_{\alpha'}|(\mathscr{G}+\mathscr{G}\mathscr{T}^q\mathscr{G})|\gamma_\alpha\rangle\,. \tag{2.39}$$

Then $\mathscr{T}(E)$ may be expressed as

$$\mathscr{T}=\mathscr{T}^q+\sum_{\alpha'}\sum_{\alpha}(1+\mathscr{T}^q\mathscr{G})|\gamma_{\alpha'}\rangle(\underline{D}^{-1})_{\alpha'\alpha}\langle\bar{\gamma}_\alpha|(1+\mathscr{G}\mathscr{T}^q) \tag{2.40}$$

with

$$\underline{D}=\underline{d}-\underline{\Delta}\,. \tag{2.41}$$

C. Low-frequency approximation

The strength of the coupling between the laser field and the nearly degenerate resonant states is measured by the parameter σ which, according to Eq. (2.26), is defined as the ratio of the electric-dipole interaction energy to the photon energy. The field interaction strength is effectively magnified at low frequencies; σ will be of order unity or greater for a wide range of experimentally accessible field intensities. (Recall the analogous remarks made earlier in connection with the strength parameter $\rho_{\vec{p}}$.) The low-frequency approximation for nonresonant scattering outlined in Sec. II A may now be generalized. The two strength parameters $\rho_{\vec{p}}$ and σ are treated to all orders; the interaction H_I is treated as a small perturbation wherever else it appears, i.e., wherever it couples nondegenerate states. In the following we work to lowest order in this nearly degenerate perturbation theory and neglect all matrix elements of H_I taken between nondegenerate states.

In working out the consequences of the approximation scheme just described, results will be expressed in terms of certain field-free scattering parameters. These may be introduced by projecting out the resonant states from the field-free scattering operator $T(\Omega)$, following a procedure analogous to that which led to Eq. (2.40). We have (assuming the resonant states to be uncoupled in the absence of an external field)

$$T(\Omega)=T^q(\Omega)+\sum_{j=1}^{2}|\Gamma_j\rangle(\Omega-z_j)^{-1}\langle\bar{\Gamma}_j|\,, \tag{2.42}$$

with

$$z_j = e_j + \Delta_j \ . \tag{2.43}$$

The strong energy dependence associated with the resonance appears explicitly here. In addition the vertex functions Γ_j and $\bar{\Gamma}_j$, and the level shifts Δ_j, carry a dependence, presumably much weaker, on the energy parameter Ω. We make the simplifying and reasonable assumption that in evaluating these resonance parameters Ω is understood to be fixed at the energy e_j.

To obtain the transition amplitude we evaluate matrix elements of the low-frequency approximation for $\mathcal{T}(E)$ with respect to the field-modified asymptotic states, as shown in Eq. (2.3). The result is the sum of nonresonant and resonant terms:

$$\mathcal{T}_{n',\vec{p}\,';n,\vec{p}} \cong \mathcal{T}^{\rm NR}_{n',\vec{p}\,';n,\vec{p}} + \mathcal{T}^{R}_{n',\vec{p}\,';n,\vec{p}} \ . \tag{2.44}$$

Clearly, the argument which led to Eq. (2.16) can be repeated for the nonresonant contribution, starting with the replacement of $\langle l' | \mathcal{T}^q(E) | l \rangle$ by $T^q(E - l\omega)\delta_{l'l}$. Then $\mathcal{T}^{\rm NR}_{n',\vec{p}\,';n,\vec{p}}$ will take the form

$$\mathcal{T}^{\rm NR}_{n',\vec{p}\,';n,\vec{p}} = J_{n'-n}(\rho_{\vec{p}\,'-\vec{p}})T^q(\Omega_\omega;\vec{p}\,'_\omega,\vec{p}_\omega) \ . \tag{2.45}$$

It remains now to express the resonant contribution in terms of the field-free resonance parameters introduced above.

Consider first the construction of the matrix $\underline{D} = \underline{d} - \underline{\Delta}$. Following the prescription outlined above we replace $d_{\alpha'\alpha}$ with the approximation $\langle B_{\alpha'} | (E - H) | B_\alpha \rangle$. Furthermore, the level-shift matrix (2.39) is evaluated by ignoring the effect of the field on the operators $\mathcal{G}$ and $\mathcal{T}^q$, and writing $|\gamma_\alpha\rangle \cong \langle \chi | V | B_\alpha \rangle$ and $\langle \bar{\gamma}_{\alpha'} | \cong \langle B_{\alpha'} | V | \chi \rangle$. It is convenient to separate off the diagonal part of $\underline{D}$ by setting

$$D_{\alpha'\alpha} = D_\alpha \delta_{\alpha'\alpha} - M_{\alpha'\alpha} \tag{2.46}$$

with $M_{\alpha\alpha} = 0$. Explicitly, we find, using the relations (2.31) and (2.32), that

$$D_{jn} = E - \frac{z_1 + z_2}{2} - n\omega + (-1)^j \left[\frac{z_1 - z_2}{2} \right] J_0(2\sigma) \ , \tag{2.47}$$

$$M_{j',n';j,n} = m_{j',n';j,n} i^{(n'-n)} \left[\frac{z_1 - z_2}{2} \right] J_{n'-n}(2\sigma)(1 - \delta_{n'n}) \tag{2.48}$$

with $\underline{m}$ given by Eq. (2.32b). If the separation $z_1 - z_2$ between resonances is small compared with ω then $\underline{M}$ may be treated as a first-order quantity and the inversion of $\underline{D}$ required in Eq. (2.40) may be accomplished approximately as

$$(\underline{D}^{-1})_{\alpha'\alpha} \cong D_\alpha^{-1}\delta_{\alpha'\alpha} + D_{\alpha'}^{-1} M_{\alpha'\alpha} D_\alpha^{-1} \ . \tag{2.49}$$

Equation (2.40) still involves an infinite sum over photon states but only a finite (presumably small) number of states will contribute significantly (i.e., lead to a small energy denominator) if the width of the resonance is not appreciably greater than ω.

The resonance contribution to the transition amplitude may be expressed in the form

$$\mathcal{T}^{R}_{n',\vec{p}\,';n,\vec{p}} = \sum_{\alpha'} \sum_{\alpha} \Gamma_{n',\vec{p}\,';\alpha'} (\underline{D}^{-1})_{\alpha'\alpha} \bar{\Gamma}_{\alpha;n,\vec{p}} \ . \tag{2.50}$$

The field-modified vertex function describing the (temporary) capture of the projectile into the resonant state is defined as

$$\bar{\Gamma}_{j,m;n,\vec{p}} = \langle \bar{\gamma}_{jm}(1 + \mathcal{G}\mathcal{T}^q) | \psi_{n,\vec{p}} \rangle \tag{2.51}$$

and is evaluated, in the low-frequency approximation, and with the use of the adjoint of Eq. (2.24), as

$$\bar{\Gamma}_{j,m;n,\vec{p}} \cong \sum_{l=-\infty}^{\infty} \sum_{k=1}^{2} a_l^{(k)*}(j,m) \langle m + l | \langle \bar{\Gamma}_k | \psi_{n,\vec{p}} \rangle \ . \tag{2.52}$$

This expression may be simplified by making use of the integral representation (2.13) for the Bessel functions which appear in the definition (2.25) of the expansion coefficients. Let us recall the representation (2.1) of the continuum state $\psi_{n,\vec{p}}$, and write $\bar{\Gamma}_j(\vec{p}) \equiv \langle \bar{\Gamma}_j | \vec{p} \rangle$ for the field-free vertex function. We then find

$$\bar{\Gamma}_{j,m;n,\vec{p}} \cong \tfrac{1}{2} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(m-n)\phi} \{ e^{i(\rho_{\vec{p}}\sin\phi - \sigma\cos\phi)} [\bar{\Gamma}_j(\vec{p}(\phi)) + \bar{\Gamma}_{\hat{j}}(\vec{p}(\phi))] + e^{i(\rho_{\vec{p}}\sin\phi + \sigma\cos\phi)} [\bar{\Gamma}_j(\vec{p}(\phi)) - \bar{\Gamma}_{\hat{j}}(\vec{p}(\phi))] \} \ , \tag{2.53}$$

where $\hat{j} = 1$ for $j = 2$ and $\hat{j} = 2$ for $j = 1$. Further simplification is obtained if we replace $\vec{p}(\phi) \equiv \vec{p} - (e/c)\vec{\mathcal{A}}\cos\phi$ by $\vec{p}$ (an excellent approximation for moderate field intensities[23]). Then, writing

$$\rho_{\vec{p}} \sin\phi \pm \sigma\cos\phi = (\rho_{\vec{p}}^2 + \sigma^2)^{1/2} \sin(\phi \pm \eta) \tag{2.54}$$

with $\tan\eta = \sigma/\rho_{\vec{p}}$ we shift the integration variable in Eq. (2.53) from ϕ to $\phi + \eta$ in the first term and to $\phi - \eta$ in the second term. The integration may then be performed with the result

$$\bar{\Gamma}_{j,m;n,\vec{p}} \cong J_{n-m}[(\rho_{\vec{p}}^2 + \sigma^2)^{1/2}] \times \{ \bar{\Gamma}_j(\vec{p})\cos[(m-n)\eta] + i\bar{\Gamma}_{\hat{j}}(\vec{p})\sin[(m-n)\eta] \} \ . \tag{2.55}$$

To obtain the function $\Gamma_{n,\vec{p};j,m}$ one takes the complex conjugate of $\bar{\Gamma}_{j,m;n,\vec{p}}$, in either the form (2.53) or (2.55), and then replaces $\bar{\Gamma}_j^*(\vec{p})$ by $\Gamma_j(\vec{p})$. Specifically, in the approximation corresponding to that shown in Eq. (2.55), we have

$$\Gamma_{n,\vec{p};j,m} \cong J_{n-m}[(\rho_{\vec{p}}^2+\sigma^2)^{1/2}] \times\{\Gamma_j(\vec{p})\cos[(m-n)\eta] - i\Gamma_j(\vec{p})\sin[(m-n)\eta]\} . \quad (2.56)$$

Note that if there were only one resonance the terms $\mathcal{T}^{\rm NR}$ and $\mathcal{T}^{R}$ could be combined using Eq. (2.42), and the result (2.14) would be regained. It also follows that the Feshbach-Yennie approximation for single-photon spontaneous bremsstrahlung (which, as pointed out earlier, can be derived from the external-field version by taking the weak-field limit) is verified in the case of an isolated resonance. The advantage of the present derivation lies in the fact that the resonance contribution has been extracted explicitly and the low-frequency approximation has been applied only to those components (the nonresonant "background" scattering amplitude, the vertex functions, and the level shift) which are free of near singularities. Feshbach and Yennie, recognizing that photon emission during the resonant collision might introduce corrections, wrote down their basic result [Eq. (21) of Ref. 1] with an additional, unspecified "error term" appended to it. We now see that this error term vanishes, to the level of accuracy of the underlying low-frequency approximation, for isolated resonances. If, however, there are closely spaced resonances the system can make a transition from one resonant state to another (assuming the selection rules are satisfied) and emit a low-frequency photon. The radiative coupling strength is small but its effect is magnified by the presence of an additional small resonance denominator. This process, which is easily computed perturbatively, contributes to the Feshbach-Yennie error term. The improved approximation obtained in this way is

$$\mathcal{T}_{1,\vec{p}\,';0,\vec{p}} \cong \mathcal{T}^{\rm FY}_{1,\vec{p}\,';0,\vec{p}} + \left[\Gamma_1(\vec{p}\,')\frac{1}{E-z_1-\omega}(H_I)_{12}\frac{1}{E-z_2}\bar{\Gamma}_2(\vec{p})+\Gamma_2(\vec{p}\,')\frac{1}{E-z_2-\omega}(H_I)_{21}\frac{1}{E-z_1}\bar{\Gamma}_1(\vec{p})\right] \quad (2.57)$$

with $\mathcal{T}^{\rm FY}$ given by Eq. (2.18) and with

$$(H_I)_{jk} \equiv \langle B_j\,|\,\langle 1\,|\,H_I\,|\,0\rangle\,|\,B_k\rangle = ie\left[\frac{2\pi\omega}{L^3}\right]^{1/2}\vec{\lambda}\cdot\langle B_j\,|\,\vec{R}\,|\,B_k\rangle . \quad (2.58)$$

To check that Eq. (2.57) is obtained in the weak-field limit of the more general result derived above it is convenient to begin by expressing the resonant term in Eq. (2.44), somewhat schematically, as

$$\mathcal{T}^R(\rho,\sigma) \cong \mathcal{T}^R(\rho,0)+[\mathcal{T}^R(0,\sigma)-\mathcal{T}^R(0,0)] ,$$

where ρ stands for the field dependence contained in $\rho_{\vec{p}}$ and $\rho_{\vec{p}\,'}$, and we work to first order in ρ and σ. The first term, $\mathcal{T}^R(\rho,0)$, combines with $\mathcal{T}^{\rm NR}$ to give the Feshbach-Yennie amplitude, Eq. (2.18). (The detailed verification is straightforward and will not be reproduced here.) The remaining terms give rise to the two resonant emission amplitudes shown in Eq. (2.57). The derivation of Eq. (2.57), which represents one of the main results of this paper, illustrates, in the relatively simple context of single-photon bremsstrahlung, how low-frequency approximation techniques can be generalized to account for resonances. The result may be useful in the analysis of certain experimental studies of resonant reactions; we return briefly to this point in Sec. IV, below. More generally, the strong-field version of the low-frequency approximation derived above can be expressed as the sum of two parts, the first being the form (2.14) derived previously and the second being a sum of correction terms corresponding to transitions of the system from one resonant state to the other.

III. SCATTERING IN A STATIC EXTERNAL FIELD

A. Effective-potential formulation

It may be expected that approximation procedures applicable to scattering in a low-frequency plane-wave radiation field will have useful analogs when the field is taken to its zero-frequency limit, i.e., a constant crossed field. That this is, in fact, the case has been demonstrated earlier for nonresonant scattering by a structureless target.[12] That treatment will now be extended to the case of resonant scattering by a compound system in the framework of the theory developed in Sec. II of the present paper. We begin, in this subsection, by transcribing the effective-potential formalism into the language of time-dependent scattering theory. The approximation procedure of Ref. 12 will then be introduced into this formalism. This will provide the necessary background for the extension of the theory to the resonant case, taken up in Sec. III B.

The amplitude for scattering in a *classical* external field may be represented, in close analogy with the form (2.3), as

$$\mathcal{T}_{\vec{p}\,',\vec{p}} = -i\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt\langle\psi_{\vec{p}\,'}(t')\,|\,\mathcal{T}(t',t)\,|\,\psi_{\vec{p}}(t)\rangle . \quad (3.1)$$

[The correspondence between the amplitudes shown in Eqs. (2.3) and (3.1) is essentially one of Fourier transformation.[7]] The modified plane-wave states are given by

$$|\,\psi_{\vec{p}}(t)\rangle = \exp[-i(p^2/2\mu)t+i\Phi_{\vec{p}}(t)]\,|\,\vec{p}(t)\rangle \quad (3.2)$$

with

$$\vec{p}(t)=\vec{p}-e\vec{A}(t)/c \quad (3.3)$$

and

$$\Phi_{\vec{p}}(t) = -\int^t\left[-\frac{e}{\mu c}\vec{p}\cdot\vec{A}(t')+\frac{e^2A^2(t')}{2\mu c^2}\right]dt' . \quad (3.4a)$$

The vector potential is chosen as $\vec{A}(t)=-\vec{E}_0ct$ with $\vec{E}_0$ representing the homogeneous static electric field.[24] With the integration constant in Eq. (3.4a) discarded we then have

$$\Phi_{\vec{p}}(t)=-\left[\frac{e}{2\mu}\right]\vec{E}_0\cdot\vec{p}t^2-\frac{e^2}{6\mu}E_0^2t^3\ . \qquad (3.4b)$$

The scattering operator satisfies the integral equation

$$\mathscr{T}=\mathscr{V}+\mathscr{V}\mathscr{G}\mathscr{T}\ . \qquad (3.5)$$

Here, in order to emphasize the analogy with Eq. (2.4), we write operator relations of the form $C(t',t)=\int_{-\infty}^{\infty}A(t',t'')B(t'',t)dt''$ in the condensed notation $C=AB$. The projectile-field propagator is

$$\mathscr{G}(t',t)=-i\Theta(t'-t)\int d^3q\,|\psi_{\vec{q}}(t')\rangle\langle\psi_{\vec{q}}(t)|\ , \qquad (3.6)$$

where Θ is the step function. Note that (in analogy with the time-independent treatment of Sec. II) we work not in the Coulomb gauge but rather in the electric-field gauge generated from it by the transformation $\exp[ie\vec{A}(t)\cdot\vec{R}/c]$, with $e\vec{R}\equiv\sum_j e_j\vec{r}_j$ representing the dipole operator of the system. This tranformation has the effect of modifying the form of the particle-field interaction (see below) and also leads to the momentum shift indicated in Eq. (3.3).

Turning now to the analog of Eq. (2.6) for the effective potential we write the gauge-transformed interaction operator $H_I(t',t)$ as $h_I\delta(t'-t)$ with $h_I=-e\vec{E}_0\cdot\vec{R}$. The Hamiltonian is then $H(t',t)=(H_0+h_I)\delta(t'-t)$. In terms of the projection operator $Q=1-|\chi\rangle\langle\chi|$ we introduce the modified resolvent G^Q as the solution of

$$Q\left[i\frac{\partial}{\partial t'}-(H_0+h_I)\right]QG^Q(t',t)=\delta(t'-t)Q \qquad (3.7)$$

and express the effective potential as

$$\mathscr{V}(t',t)=\langle\chi|[V\delta(t'-t)+VG^Q(t',t)V]|\chi\rangle\ . \qquad (3.8)$$

It is convenient at this point to introduce the Fourier transformation

$$\mathscr{T}(t',t)=\int_{-\infty}^{\infty}\frac{d\Omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega't'}\tilde{\mathscr{T}}(\Omega',\Omega)e^{i\Omega t}\ . \qquad (3.9)$$

The dominant effect of the field is on the asymptotic states—the field has a long time to act and its cumulative effect must be treated nonperturbatively, as in Eq. (3.2). We assume, however, that the field is weak enough (compared with the strength of the interactions which bind the target and scatter the projectile) so that, as a first approximation, it may be ignored in intermediate states. In this approximation we have

$$\tilde{\mathscr{T}}(\Omega',\Omega)\cong 2\pi\delta(\Omega'-\Omega)T(\Omega)\ , \qquad (3.10)$$

where $T(\Omega)$ is the field-free scattering operator introduced in Sec. II. Equation (3.1) then becomes

$$\mathscr{T}_{\vec{p}',\vec{p}}\cong -i\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt\exp\{i[S_{\vec{p}}(\Omega,t)-S_{\vec{p}'}(\Omega,t')]\}T(\Omega;\vec{p}'(t'),\vec{p}(t)) \qquad (3.11)$$

with

$$S_{\vec{p}}(\Omega,t)=(\Omega-p^2/2\mu)t+\Phi_{\vec{p}}(t)\ , \qquad (3.12)$$

and with $T(\Omega;\vec{q}',\vec{q})$ given by Eq. (2.11). The T-matrix in Eq. (3.11) is off the energy shell. However, we may recognize that due to rapid oscillations of the exponential factor the dominant contributions to the time integrations come from the neighborhood of the points of stationary phase[25]

$$\Omega-(p^2/2\mu)+\dot{\Phi}_{\vec{p}}(t)=0\ , \qquad (3.13a)$$

$$\Omega-(p'^2/2\mu)+\dot{\Phi}_{\vec{p}'}(t')=0\ . \qquad (3.13b)$$

Now these are just the conditions which place the T matrix in Eq. (3.11) on the energy shell. That is, in terms of the scalar variables

$$\xi=\Omega-p^2(t)/2\mu,\quad \xi'=\Omega-p'^2(t')/2\mu\ ,$$
$$\tau=[\vec{p}'(t')-\vec{p}(t)]^2 \qquad (3.14)$$

we have $T[\Omega;\vec{p}'(t),\vec{p}(t)]\equiv T(\Omega,\tau,\xi',\xi)$ with $T(\Omega,\tau,0,0)$ representing the physical, on-shell scattering amplitude. Since Eqs. (3.13) correspond to $\xi=\xi'=0$ the on-shell contributions are expected to dominate. As a first approximation the scattering amplitude in Eq. (3.11) may be replaced by its on-shell value $T(\Omega,\tau,0,0)$. This approximation is better than one might expect since, upon introducing a Taylor-series expansion of the T matrix about $\xi=\xi'=0$, the first-order correction terms may be seen to vanish. Thus, in the term proportional to $[\Omega-p^2(t)/2\mu]\partial T/\partial\xi$, for example, we may write

$$(\Omega-p^2/2\mu)\exp\{i[(\Omega-p^2/2\mu)t+\Phi_{\vec{p}}(t)]\}=-i\frac{d}{dt}(e^{i(\Omega-p^2/2\mu)t})e^{i\Phi_{\vec{p}}(t)}$$

and integrate by parts. Since the surface terms vanish due to rapid oscillations of the exponential as $t\to\pm\infty$ we have effectively replaced $\Omega-p^2/2\mu$ by $-\dot{\Phi}_{\vec{p}}$, corresponding to $\xi=0$. Similarly, the first-order correction term proportional ξ' drops out. The integration-by-parts procedure brings in a higher-order correction term involving the derivative of the T matrix with respect to the momentum-transfer variable. This has been dropped under the assumption that the scattering amplitude is slowly varying in the τ variable as well as in ξ and ξ'. In the absence of resonances we may also assume that $T(\Omega,\tau,0,0)$ is slowly varying in its energy argument. Expanding about $\Omega=p^2(t)/2\mu$ and keeping only the leading term (again, the first-order correction term vanishes) we find that the integration over Ω in Eq. (3.11) may be carried out very simply with the result

$$\mathscr{T}_{\vec{p}',\vec{p}}\cong -i\int_{-\infty}^{\infty}dt\exp\{i[\Phi_{\vec{p}}(t)-\Phi_{\vec{p}'}(t)+(p'^2/2\mu-p^2/2\mu)t]\}\times T(p^2(t)/2\mu;\vec{p}'(t),\vec{p}(t)) \qquad (3.15)$$

with the T matrix understood to be evaluated on the energy shell. It is not difficult to derive an approximation of the same form as (3.15) in the more realistic case where $\vec{A}(t)$ vanishes for $|t|$ larger than some finite cutoff value.[12]

It should be emphasized that the validity of the approximation (3.15) must be carefully examined when resonances are present; the derivation given in Ref. 12 was explicitly limited to the nonresonant case. We now turn to a consideration of resonant scattering in a static field within the framework of the effective-potential formalism.

B. Resonant scattering

To account for resonances we follow the method of Sec. II B and project out the resonant states; the approximation procedure leading to Eq. (3.15) is then applied only to the slowly varying, nonresonant components of the resultant expression. The starting point is again the identity (2.21) where

$$p=\sum_{j=1}^{2} |B_j\rangle\langle B_j| , \quad q=Q-p \tag{3.16}$$

and G^q satisfies

$$q\left[i\frac{\partial}{\partial t'}-(H_0+h_I)\right]qG^q(t',t)=\delta(t'-t)q . \tag{3.17}$$

The propagator in p space is a 2×2 matrix with elements $u_{jk}(t',t)\equiv\langle B_j|G^Q(t',t)|B_k\rangle$ satisfying

$$\left[i\frac{\partial}{\partial t'}-e_j\right]u_{jk}(t',t)-\int_{-\infty}^{\infty}dt''\sum_{l=1}^{2}s_{jl}(t',t'')u_{lk}(t'',t)=\delta(t'-t)\delta_{jk} , \tag{3.18}$$

with

$$s_{jk}(t',t)=\langle B_j|[h_I\delta(t'-t)+h_IG^q(t',t)h_I]|B_k\rangle . \tag{3.19}$$

Combining Eqs. (2.21) and (3.8) we are led to a formal solution of Eq. (3.5):

$$\mathscr{T}=\mathscr{T}^q+\sum_j\sum_k(1+\mathscr{T}^q\mathscr{G})|\gamma_j\rangle U_{jk}\langle\bar{\gamma}_k|(1+\mathscr{G}\mathscr{T}^q) , \tag{3.20}$$

where $U_{jk}(t',t)$ satisfies

$$\left[i\frac{\partial}{\partial t'}-e_j\right]U_{jk}(t',t)-\int_{-\infty}^{\infty}dt''\sum_{l=1}^{2}[s_{jl}(t',t'')+\Delta_{jl}(t',t'')]U_{lk}(t'',t)=\delta(t'-t)\delta_{jk} . \tag{3.21}$$

The level-shift matrix is

$$\Delta_{jk}=\langle\bar{\gamma}_j|(\mathscr{G}+\mathscr{G}\mathscr{T}^q\mathscr{G})|\gamma_k\rangle \tag{3.22}$$

and the vertex functions are given by

$$|\gamma_j(t',t)\rangle=\langle\chi|[V\delta(t'-t)+VG^q(t',t)h_I]|B_j\rangle , \tag{3.23a}$$

$$\langle\bar{\gamma}_k(t',t)|=\langle B_k|[V\delta(t'-t)+h_IG^q(t',t)V]|\chi\rangle . \tag{3.23b}$$

The decomposition (3.20) leads to a corresponding separation of nonresonant and resonant components of the transition amplitude:

$$\mathscr{T}_{\vec{p}',\vec{p}}=\mathscr{T}^{\rm NR}_{\vec{p}',\vec{p}}+\mathscr{T}^{R}_{\vec{p}',\vec{p}} . \tag{3.24}$$

Since $\mathscr{T}^q(E)$ is nonresonant an approximation for $\mathscr{T}^{\rm NR}_{p'p}$ can be derived of the form shown in Eq. (3.15), but with the field-free T matrix replaced with its nonresonant component T^q, as defined in Eq. (2.42). Turning now to the remaining terms in Eq. (3.20) we note first that the vertex functions are nonresonant. Following the basic approximation scheme we shall ignore the effect which the field has on these functions and express them, by Fourier transformation, in terms of the field-free vertex functions which appear in Eq. (2.42). Explicitly, we have

$$(1+\mathscr{T}^q\mathscr{G})|\gamma_j\rangle\cong\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega(t'-t)}|\Gamma_j(\Omega)\rangle , \tag{3.25a}$$

$$\langle\bar{\gamma}_k|(1+\mathscr{G}\mathscr{T}^q)\cong\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega(t'-t)}\langle\bar{\Gamma}_k(\Omega)| . \tag{3.25b}$$

The differential equation (3.21) defining the resonance propagator can be recast as the integral equation

$$\underline{U}=\underline{U}^{(0)}+\underline{U}^{(0)}(\underline{s}+\underline{\Delta})\underline{U} . \tag{3.26}$$

$\underline{U}^{(0)}$ is diagonal and satisfies

$$\left[i\frac{\partial}{\partial t'}-e_j\right]U^{(0)}_{jj}(t',t)=\delta(t'-t) ; \tag{3.27}$$

the Fourier representation is

$$U^{(0)}_{jk}(t',t)=\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega(t'-t)}\frac{\delta_{jk}}{\Omega-e_j} \tag{3.28}$$

with Ω in the denominator understood to carry an infinitesimal positive imaginary part. Since the level-shift matrices $\underline{s}$ and $\underline{\Delta}$ have similar representations the Fourier representation of Eq. (3.26) may be solved algebraically, leading to

$$U_{jk}(t',t)=\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega(t'-t)}U_{jk}(\Omega) \tag{3.29}$$

with

$$[\underline{U}^{-1}(\Omega)]_{jk}=(\Omega-e_j-\Delta_j)\delta_{jk}-s_{jk}(\Omega) \tag{3.30}$$

and

$$s_{jk}(\Omega)=\langle B_j\,|\,\{h_I+h_I[q(\Omega-H_0-h_I)q]^{-1}h_I\}\,|\,B_k\rangle\ . \tag{3.31}$$

In arriving at this representation of the resonance propagator $U_{jk}(t',t)$ we have used the relation[26]

$$G^q(t',t)=\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega(t'-t)}[q(\Omega-H_0-h_I)q]^{-1} \tag{3.32}$$

in the definition (3.19) of $\underline{s}$ and have expressed the time-dependent level shift $\underline{\Delta}$, defined in Eq. (3.22), as the Fourier transform of the field-free level shift, in accordance with the approximation scheme adopted here. Note that the particle-field interaction has been retained in higher orders in Eq. (3.31) to allow for the effect of tunneling on the energy and lifetime of the resonant state. Performing the inversion indicated in Eq. (3.30) we find

$$\underline{U}(\Omega)=\frac{1}{D(\Omega)}\begin{bmatrix}\Omega-z_2 & s_{12}\\ s_{21} & \Omega-z_1\end{bmatrix}, \tag{3.33}$$

where

$$z_j=e_j+\Delta_j+s_{jj} \tag{3.34}$$

and

$$D(\Omega)=(\Omega-z_1)(\Omega-z_2)-s_{12}s_{21}\ . \tag{3.35}$$

It follows from Eqs. (3.25) and (3.29) that the resonant part of the scattering operator $\mathcal{T}$ in Eq. (3.20) can, in the present approximation, be put in the form

$$\mathcal{T}^R(t',t)=\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega(t'-t)}\times\sum_j\sum_k|\Gamma_j(\Omega)\rangle U_{jk}(\Omega)\langle\bar{\Gamma}_k(\Omega)|\ . \tag{3.36a}$$

With the time dependence thus specified we construct the resonant contribution to the transition amplitude $\mathcal{T}_{\vec{p}',\vec{p}}$ according to the prescription (3.1) and obtain

$$\mathcal{T}^R_{\vec{p}',\vec{p}}\cong-i\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt\exp\{i[S_{\vec{p}}(\Omega,t)-S_{\vec{p}'}(\Omega,t')]\}\times\sum_j\sum_k\langle\vec{p}'(t')\,|\,\Gamma_j(\Omega)\rangle U_{jk}(\Omega)\langle\bar{\Gamma}_k(\Omega)\,|\,\vec{p}(t)\rangle\ . \tag{3.36b}$$

This form may be simplified by recognizing that the main contribution to the integral over Ω comes from the neighborhood of the resonance energy. The Ω-integration domain may then be replaced by a short segment $\Omega_1\le\Omega\le\Omega_2$ covering the resonance region, with the contribution from the remainder of the integration domain absorbed into the nonresonant term $\mathcal{T}^{\rm NR}_{\vec{p}',\vec{p}}$. The resonant term, thus redefined, is

$$\mathcal{T}^R_{\vec{p}',\vec{p}}\cong-i\int_{\Omega_1}^{\Omega_2}\frac{d\Omega}{2\pi}\sum_j\sum_k\Gamma_{\vec{p}';j}(\Omega)U_{jk}(\Omega)\bar{\Gamma}_{k;\vec{p}}(\Omega) \tag{3.37}$$

with

$$\bar{\Gamma}_{\vec{k};\vec{p}}(\Omega)\equiv\int_{-\infty}^{\infty}dt\,e^{iS_{\vec{p}}(\Omega,t)}\langle\bar{\Gamma}_k(\Omega)\,|\,\vec{p}(t)\rangle\ , \tag{3.38a}$$

$$\Gamma_{\vec{p}';j}(\Omega)\equiv\int_{-\infty}^{\infty}dt'e^{-iS_{\vec{p}'}(\Omega,t')}\langle\vec{p}'(t')\,|\,\Gamma_j(\Omega)\rangle\ . \tag{3.38b}$$

Assuming that the vertex functions and level shifts are slowly varying in energy we may replace Ω, as it appears in these functions, by a fixed value in the resonance region. Further simplification is achieved by evaluating the time integrations in Eq. (3.38) by the saddle-point method, an approximation which becomes increasingly accurate as the strength of the external field is decreased. The calculation is outlined in the Appendix. Let us remark here that the saddle points are determined by the conditions (3.13). The first of these may be written as

$$\Omega-p/2\mu=-(e/\mu c)\vec{p}\cdot\vec{A}(t)+(e^2/2\mu c^2)A^2(t) \tag{3.39}$$

with Ω now understood to lie within the resonance region. It would appear that this condition could be satisfied (by suitable choice of the time variable) for a wide range of scattering energies $p^2/2\mu$ since the interaction energy on the right-hand side is unbounded as it stands. Recall, however, the implicit assumption that the field vanishes for $|t|$ exceeding some finite cutoff value. In the weak-field limit, when the interaction energy is small compared with the scattering energy (for all values of the time), Eq. (3.39) does, in fact, restrict the initial energy $p^2/2\mu$ to the resonance region and a similar restriction holds for the final energy $p'^2/2\mu$. There is a broadening of the resonance shape with increasing field strength.

In the foregoing discussion we have assumed initial conditions appropriate to the scattering problem. Alternatively, one may consider a different situation in which a composite bound state (H^- in the experimental study of Ref. 4) is introduced into the field. The level structure of the perturbed system is then probed by means of a laser field of variable frequency. The dependence of the resonance parameters on the strength of the static field can be studied theoretically by examining the position of the poles of the resolvent operator in the complex-energy plane.[5] The resonance positions are determined, in the present formulation, by the zeros of the denominator function $D(\Omega)$ defined in Eq. (3.35). (In the context of the photodetachment experiment mentioned above, Ω

would represent the energy to which the atomic system has been excited by absorption of a laser photon.) With the Ω dependence of the level-shift matrix $\underline{s}$ ignored, the roots of the equation $D(\Omega)=0$ are immediately determined as

$$z_{\pm}=\tfrac{1}{2}(z_1+z_2)\pm[\tfrac{1}{2}(z_1-z_2)+s_{12}s_{21}]^{1/2} . \qquad (3.40)$$

Consider now the limiting case where $|z_1-z_2| \ll |s_{12}|$ and suppose that s_{jk} can be well approximated by the first term on the right-hand side of Eq. (3.31). The resonance positions are then estimated as

$$z_{\pm}\cong\tfrac{1}{2}(z_1+z_2)\pm|\langle B_1|h_I|B_2\rangle| . \qquad (3.41)$$

This formula shows how the linear Stark splitting of the resonance positions can be accounted for in the present formalism.

In the opposite limit, where $|z_1-z_2| \gg |s_{12}|$, we have $z_+ \cong z_1$ and $z_- \cong z_2$, corresponding to the case of isolated resonances. Equation (3.33) then reduces to $U_{jk}\cong(\Omega-z_j)^{-1}\delta_{jk}$. This is to be inserted into Eq. (3.36b) and the result combined with the nonresonant contribution $\mathscr{T}^{\rm NR}_{\vec{p}\,',\vec{p}}$. Recalling Eq. (2.42) we see that the approximation thus obtained is of the form (3.11) with the T matrix now understood to include the effects of the isolated resonances. It must be noted, however, that due to the rapid variation of the T matrix with energy, the further simplification introduced previously (in the nonresonant case) through the passage from Eq. (3.11) to Eq. (3.15) is not justified here. Of course, the general result (3.40) allows for a smooth interpolation between the two limiting cases discussed above.

A knowledge of the field dependence of the level-shift matrix $\underline{s}$ would enable one, through Eq. (3.40), to estimate the resonance level splitting as a function of field strength. If this procedure were carried out for the nearly degenerate Feshbach resonances of H^- (lying just below the $n=2$ level of H) the result could then be compared with the observations reported in Ref. 4. We have found that the simplest approximation, in which $\underline{s}$ is replaced by the first term on the right-hand side of Eq. (3.31), gives the general trend of the linear Stark splitting effect but fails to reproduce the experimental data in detail. This may be taken as an indication of the relevance, in this problem, of the second term in Eq. (3.31) which accounts not only for the quadratic Stark splitting but also for tunneling effects depending nonlinearly on the field strength.

IV. SUMMARY AND CONCLUSIONS

The theory of resonant reactions, in the form set up by Feshbach[8] and others,[26] has been generalized here to apply to the situation where the scattering takes place in the presence of an external field. Particular attention has been paid to the low-frequency domain since in that limit approximations for the transition amplitude can be derived, even for fields of appreciable intensity, which are simple enough so that they stand the chance of being useful in future resonance studies. The great virtue of the Feshbach procedure lies in the fact that it allows, formally at least, for a clean separation of resonant and nonresonant components of the scattering amplitude. The nonresonant parts (vertex functions and level shifts) are characterized by a weak dependence on energy since all "small energy denominators" have been removed. It follows that earlier work, devoted to the construction of low-frequency approximations for scattering in the absence of resonances, can be taken over in the resonant case and applied, not to the full amplitude, but to its nonresonant components.

The strategy outlined above has been worked out in some detail for three physically distinct situations with results which may be summarized as follows.

(i) An approximation for the amplitude for scattering in a low-frequency laser field (described in the occupation-number representation) has been obtained in the form (2.44). The nonresonant part is given in Eq. (2.45) in terms of T^q, the nonresonant component of the on-shell field-free scattering amplitude. To the amplitude shown in Eq. (2.45) we must add the resonant part $\mathscr{T}^{R}_{n',\vec{p}\,';n,\vec{p}}$ given by Eq. (2.50). Approximate expressions for the vertex functions appear in Eqs. (2.55) and (2.56). The resonance propagator is obtained by inverting the matrix $\underline{D}$ defined in Eqs. (2.46)–(2.48). This inversion is accomplished approximately in Eq. (2.49), a form which is appropriate when the separation between resonances is small compared with the photon energy. The presence of closely spaced resonances introduces corrections to the earlier version of the low-frequency approximation reproduced in Eq. (2.14); these corrections correspond physically to the effect of radiative transitions from one resonance state to another.

(ii) The correction terms mentioned above are shown explicitly in Eq. (2.57) for the particular case of single-photon spontaneous bremsstrahlung. The first term on the right-hand side is the Feshbach-Yennie version of the low-frequency approximation given in Eq. (2.18). It would be of interest to see how inclusion of the correction terms affects comparison between the Feshbach-Yennie theory and observations of the bremsstrahlung cross section; a case in point would be p-^{12}C scattering in the neighborhood of the two closely spaced resonances near 1.7 MeV.[27]

(iii) A modified procedure, based on time-dependent scattering theory, has been used to treat the case of a static external field. The nonresonant contribution to the transition amplitude, $\mathscr{T}^{\rm NR}_{\vec{p}\,',\vec{p}}$, is given approximately by a form similar to that shown in Eq. (3.15), but with the T matrix replaced by its nonresonant part. The approximation derived here for the resonant part $\mathscr{T}^{R}_{\vec{p}\,',\vec{p}}$ is shown in Eq. (3.37). The vertex functions which appear in that equation are studied further in the Appendix; there a weak-field approximation is derived which requires as input the field-free vertex functions entering into the representation (2.42). The applicability of the present formalism to photodetachment as well as scattering processes has been pointed out.

ACKNOWLEDGMENTS

This work was supported in part by the U. S. Navy Office of Naval Research under Contract No. N00014-76-

C-0317 and by the National Science Foundation under Grant No. PHY-79-10413.

APPENDIX

Here we derive an approximation for the vertex functions (3.38). We begin by making use of Eqs. (3.4b) and (3.12) to write, in Eq. (3.38a),

$$S_{\vec{p}}(\Omega,t)=\alpha t+\beta t^2+(\gamma/3)t^3 \tag{A1}$$

with

$$\alpha=\Omega-p^2/2\mu,\quad \beta=-(e/2\mu)\vec{p}\cdot\vec{E}_0\,,\qquad \gamma=-(e^2/2\mu)E_0^2\,. \tag{A2}$$

The field-free vertex function is written as $\langle\overline{\Gamma}_k(\Omega)\,|\,\vec{p}(t)\rangle=\overline{\Gamma}_k(\vec{p}(t))$ where, in conformity with the notation introduced in Sec. II, the Ω dependence is suppressed, it being understood that Ω is evaluated at the resonance energy. As we have mentioned, and will confirm below, the dominant contribution to the integral in Eq. (3.38a) comes from values of the integration variable for which $\Omega=p^2(t)/2\mu$. Writing $\vec{p}=\vec{p}_\perp+\vec{p}_{||}$ with $\vec{p}_{||}=(\vec{p}\cdot\hat{E}_0)\hat{E}_0$ the condition $\Omega=p^2(t)/2\mu$ becomes $p_\perp^2+p_{||}^2=2\mu\Omega$ and the values of $\vec{p}(t)$ which satisfy this condition take the form

$$\vec{p}_\pm\equiv\vec{p}_\perp\pm(2\mu\Omega-p_\perp^2)^{1/2}\hat{E}_0\,. \tag{A3}$$

With the change of variable

$$t=\gamma^{-1/3}u-\beta/\gamma \tag{A4}$$

Eq. (3.38a) becomes [recall that $\vec{p}(t)=\vec{p}+e\vec{E}_0t$]

$$\overline{\Gamma}_{k;\vec{p}}(\Omega)=\gamma^{-1/3}\exp[-i\alpha(\beta/\gamma)+i(2\gamma/3)(\beta/\gamma)^3]I_{k;\vec{p}}\,, \tag{A5a}$$

where

$$I_{k;\vec{p}}=\int_C du\,\exp[i(\lambda u+u^3/3)]\times\Gamma_k(\vec{p}-e\vec{E}_0(\beta/\gamma)+\gamma^{-1/3}e\vec{E}_0u)\,. \tag{A5b}$$

The contour C is the straight line obtained by rotating the real axis through an angle of $\pi/3$ about the origin. The parameter λ is defined as

$$\lambda=\gamma^{2/3}[(\alpha/\gamma)-(\beta/\gamma)^2]=-e^{i2\pi/3}(e^2E_0^2/2\mu)^{-1/3}(\Omega-p_\perp^2/2\mu) \tag{A6}$$

and is assumed to be sufficiently large (weak-field condition) so that the integral in Eq. (A5b) can be estimated by the saddle-point method. The saddle points are at $u=\pm(-\lambda)^{1/2}$, which implies the condition $\Omega=p^2(t)/2\mu$. For $\Omega-p_\perp^2/2\mu>0$ (this corresponds to real t) the saddle points lie on the contour C at $u=\pm e^{i\pi/3}|\lambda|^{1/2}$. We ignore the saddle points corresponding to $\Omega-p_\perp^2/2\mu<0$. These lie at $u=\pm e^{i5\pi/6}|\lambda|^{1/2}$ and the path of steepest descent does not pass through these points.

The vertex functions are evaluated at the saddle points, with the remaining integration performed in the standard way. Note that at $u=\pm e^{i\pi/3}|\lambda|^{1/2}$, $\vec{p}(t)$ becomes $\vec{p}_\pm$ defined in Eq. (A3). The condition $p_\pm^2/2\mu=\Omega$ places these vertex functions on the energy shell; they are therefore measurable in field-free resonant scattering processes. Keeping only the leading term in the asymptotic expansion of the integral in Eq. (A5b) we find

$$I_{k;\vec{p}}\cong\frac{(\pi)^{1/2}}{|\lambda|^{1/4}}\left\{\overline{\Gamma}_k(\vec{p}_+)\exp\left[i\left[\tfrac{2}{3}|\lambda|^{3/2}+\frac{\pi}{12}\right]\right]-\overline{\Gamma}_k(\vec{p}_-)\exp\left[-i\left[\tfrac{2}{3}|\lambda|^{3/2}+\frac{5\pi}{12}\right]\right]\right\}. \tag{A7}$$

An approximate evaluation of the integral in Eq. (3.38b) can be obtained with the use of very similar methods.

[1]H. Feshbach and D. R. Yennie, Nucl. Phys. 37, 150 (1962).

[2]H. Krüger and C. Jung, Phys. Rev. A 17, 1706 (1978).

[3]M. H. Mittleman, Phys. Rev. A 20, 1965 (1979).

[4]P. A. M. Gram, J. Ć. Pratt, M. A. Yates-Williams, H. C. Bryant, J. Donahue, H. Sharifian, and H. Tootoonchi, Phys. Rev. Lett. 40, 107 (1978).

[5]J. J. Wendoloski and W. P. Reinhardt, Phys. Rev. A 17, 195 (1978).

[6]J. Callaway and A. R. P. Rau, J. Phys. B 11, L289 (1978).

[7]See, e.g., L. Rosenberg, Adv. At. Mol. Phys. 18, 1 (1982), and references therein.

[8]H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958).

[9]J. Gersten and M. H. Mittleman, Phys. Rev. A 13, 123 (1976).

[10]L. Rosenberg, Phys. Rev. A 16, 1941 (1977).

[11]R. A. Neville and F. Rohrlich, Phys. Rev. D 3, 1692 (1971).

[12]L. Rosenberg, Phys. Rev. A 24, 2995 (1981).

[13]C. Cerjan, R. Hedges, C. Holt, W. P. Reinhardt, K. Scheiber, and J. J. Wendoloski, Int. J. Quantum Chem. 14, 393 (1978).

[14]A formulation of the scattering problem which takes into account effects of projectile recoil and arbitrary polarization of the field, and includes the A^2 interaction term, appears in Ref. 15.

[15]L. Rosenberg, Phys. Rev. A 23, 2283 (1981).

[16]A low-frequency approximation for bremsstrahlung, in which the target is assumed to be charged and of finite mass, is derived in L. Rosenberg, Phys. Rev. A 27, 1879 (1983).

[17]Specifically, if one compares matrix elements of the interaction taken with respect to two states separated in energy by an amount ΔE then the $\vec{r}\cdot\vec{E}$ and $\vec{p}\cdot\vec{A}$ interactions will be in the ratio $\omega/\Delta E$. See Ref. 15 for further discussion of this point.

[18]A more general treatment is given in Ref. 10.

[19]A rather similar argument appears in Sec. III, below, in the context of the problem of scattering in a static field.

[20]N. M. Kroll and K. M. Watson, Phys. Rev. A 8, 804 (1973).

[21]L. Mower, Phys. Rev. 142, 799 (1966).

[22]See, e.g., F. Smithies, *Integral Equations, Cambridge Tracts in Mathematics and Mathematical Physics, No. 49* (Cambridge University Press, London, 1958).

[23]For example, take the field intensity to be 10^8 W/cm^2. If the photon energy is 0.1 eV and if the projectile, assumed to be an electron, has an energy of 10 eV then $(e\mathscr{A}/pc)\cong 0.01$.

[24]As discussed in detail in Ref. 12 a proper physical formulation requires that the field vanish for large $|t|$. To simplify the presentation we shall not explicitly introduce the cutoff here. Furthermore, by taking the vector potential to be proportional to t (rather than $t-z/c$) we ignore small magnetic corrections which had been included in the earlier treatment.

[25]The stationary-phase condition is examined in greater detail in the Appendix. See also, A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. 50, 255 (1966) [Sov. Phys.—JETP 23, 168 (1966)].

[26]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), Chap. 8.

[27]C. C. Trail, P. M. S. Lesser, A. H. Bond, Jr., M. K. Liou, and C. K. Liu, Phys. Rev. C 21, 2131 (1980).