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Estimation of the Kolmogorov entropy from a chaotic signal

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A new method for estimating the Kolmogorov entropy directly from a time signal is proposed and tested on examples. The method should prove valuable for characterizing experimental chaotic signals.

While there has been recently a dramatic growth in new mathematical concepts related to chaotic systems,[1] the detailed comparison between models and experimental data has lagged somewhat. After observing a seemingly chaotic signal in the laboratory, the researcher is faced with the question of how to characterize the signal, how to be sure that it is chaotic (rather than multiperiodic or random), and how to quantify "how" chaotic the signal is. In this Rapid Communication we propose a method to estimate the Kolmogorov entropy K directly from a time signal. If one obtains a finite, positive K entropy one can answer some of the above questions with a degree of confidence. Although we have dissipative systems in mind, the idea presented below should prove useful for conservative systems as well.

The Kolmogorov entropy is defined as follows: Consider a dynamical system with F degrees of freedom. Suppose that the F-dimensional phase space is partitioned to boxes of size ϵ^F. Suppose that there is an attractor in phase space and that the trajectory $\vec{x}(t)$ is in the basin of attraction. The state of the system is now measured at intervals of time τ. Let $p(i_1,i_2,\ldots,i_d)$ be the joint probability that $\vec{x}(t=\tau)$ is in box i_1, $\vec{x}(t=2\tau)$ is in box $i_2,\ldots,$ and $\vec{x}(t=d\tau)$ is in box i_d. The Kolmogorov entropy is then[2]

$$K=-\lim_{\tau\to 0}\lim_{\epsilon\to 0}\lim_{d\to\infty}\frac{1}{d\tau}\sum_{i_1,\ldots,i_d}p(i_1,\ldots,i_d)\times\ln p(i_1,\ldots,i_d)\quad. \tag{1}$$

As is well known, $K=0$ in an ordered system, K is infinite in a random system, but K is a constant $\neq 0$ in a chaotic (deterministic) system.

For analytically defined models, it is very easy to estimate K from the tangent (or "variational") equations describing the evolution of the distance between two (infinitely close) points. But it is very difficult to determine K directly from a measured time signal.

In this paper we shall thus define a new quantity K_2 which has the following properties: (i) $K_2\geqslant 0$; (ii) $K_2\leqslant K$; (iii) K_2 is infinite for random systems; and (iv) $K_2\neq 0$ for chaotic systems. It will turn out that for typical cases K_2 is numerically close to K. Thus K_2 has an advantage over the topological entropy h. Since $h>K$, $h>0$ is a necessary but not sufficient condition for observable chaos. $K_2>0$ is a *sufficient* condition for chaos. The most important property of K_2, however, is that it can be extracted fairly easily from an experimental signal.

To see how this quantity comes about, consider now the set of order-q Renyi entropies which are defined as follows[3]:

$$K_q=-\lim_{\tau\to 0}\lim_{\epsilon\to 0}\lim_{d\to\infty}\frac{1}{d\tau}\frac{1}{q-1}\ln\sum_{i_1,\ldots,i_d}p^q(i_1,\ldots,i_d)\quad. \tag{2}$$

By writing $p^q=p\exp(q-1)\ln p$ and expanding the exponent it is easy to see that $\lim_{q\to 1^+}K_q=K$ and $\lim_{q\to 0^+}K_q\leqslant h$. Furthermore, it is easy to see that $K_q>K_{q'}$ for every $q'>q$.

Of all the order-q quantities K_q, K_2 is singled out due to its ease of calculation from a time series. To see this, consider Eq. (2) for fixed values of d (say $d=1$) and ϵ. For $q=2$ and $d=1$, we need $\sum_i p_i^2=C(\epsilon)$, where p_i is the probability to visit the ith box and the sum i runs over all the boxes in phase space which contain a piece of the attractor. This quantity is easily calculable from a time series. Consider the time series $\{\vec{X}_i\}_{i=1}^N$ where $\vec{X}_i=\vec{X}(t=i\tau)$. Up to an ϵ independent factor,

$$C(\epsilon)=\lim_{N\to\infty}\frac{1}{N^2}\ [\text{number of pairs of points }(n,m)\text{ with }|\vec{X}_n-\vec{X}_m|<\epsilon]\quad. \tag{3}$$

It has been shown previously that[4] $C(\epsilon)$ scales like

$$C(\epsilon)\sim\epsilon^{\nu} \tag{4}$$

and ν has been called the correlation exponent. It has been proved that ν estimates the fractal dimension D of the attractor (i.e., $\nu\lesssim D$).

For any d we can consider now

$$\tilde{C}_d(\epsilon)=\lim_{N\to\infty}\frac{1}{N^2}\ \{\text{number of pairs }(n,m)\text{ with }[(\vec{X}_n-\vec{X}_m)^2+(\vec{X}_{n+1}-\vec{X}_{m+1})^2+\cdots+(\vec{X}_{n+d-1}-\vec{X}_{m+d-1})^2]^{1/2}<\epsilon\}\ , \tag{5}$$

with $d=2,3\ldots.$ Up to a factor of order unity,

$$\tilde{C}_d(\epsilon)\simeq\sum_{i_1,\ldots,i_d}p^2(i_1,\ldots,i_d)\quad.$$

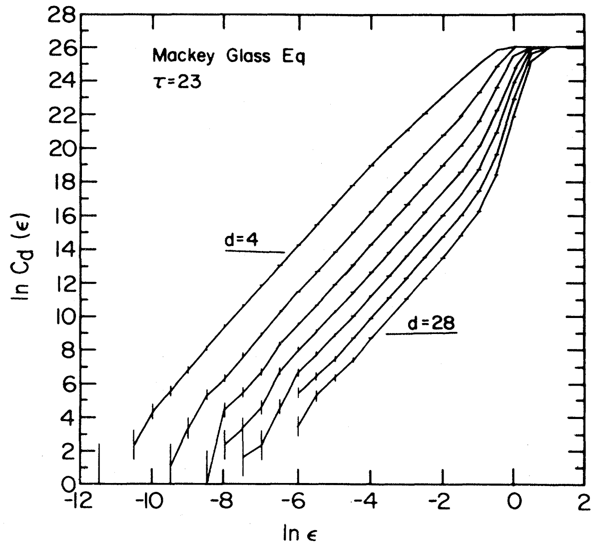

FIG. 1 Correlation integral $C_d(\epsilon)$ for the Mackey-Glass delay differential equation vs ϵ. The delay $\tau=23$. The error bars are purely statistical. Points pertaining to the same value of d are connected by lines. The values of d are $d=4$ (top curve), $8,12,\ldots,28$ (bottom curve).

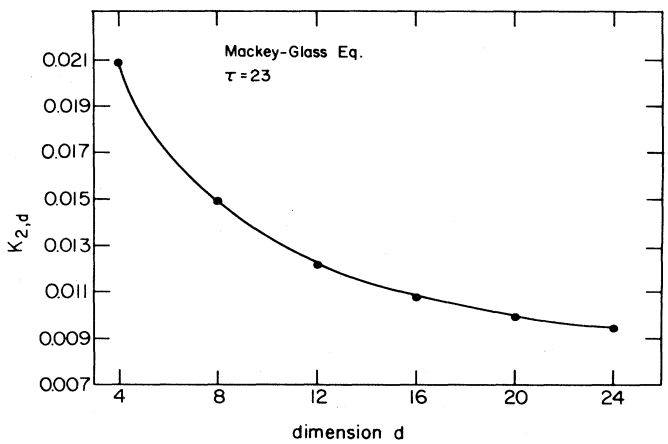

FIG. 2. Values of $K_{2,d}$ for the Mackey-Glass delay differential equations, averaged over the scaling region in ϵ. The extrapolated $(d\rightarrow\infty)$ value is $K_2=0.008\pm0.001$.

Consequently, Eqs. (2) and (4) lead to

$$\tilde{C}_d(\epsilon) \underset{\substack{d\rightarrow\infty\\ \epsilon\rightarrow 0}}{\sim} \epsilon^\nu \exp(-d\tau K_2) \ . \tag{6}$$

In practice we do not need to follow the evolution of all the degrees of freedom. Generically, the whole trajectory can be reconstructed from d measurements $(d \geqslant F)$ of any single coordinate. Taking any coordinate and denoting it by X, we consider then

$$C_d(\epsilon) = \lim_{N\rightarrow\infty} \frac{1}{N^2} \left[\text{number of pairs } (n,m) \text{ with } \left(\sum_{i=1}^{d} |X_{n+i}-X_{m+i}|^2\right)^{1/2} < \epsilon\right] \tag{7}$$

and expect it to give the same estimate $C_d(\epsilon)\sim\epsilon^\nu \times\exp(-d\tau K_2)$. At this point we should mention that a related estimate of K has been proposed by Takens.[5] He essentially replaces the Euclidean norm in Eq. (7) by the maximum norm.

The practical implementation now should be clear; if we plot $\ln C_d(\epsilon)$ as a function of $\ln\epsilon$ for a series of increasing values of d, we should get a series of straight lines with a slope ν, which are displaced from each other by the factor $\exp(-d\tau K_2)$. By looking at

$$K_{2,d}(\epsilon) = \frac{1}{\tau} \ln \frac{C_d(\epsilon)}{C_{d+1}(\epsilon)}$$

we should find

$$\lim_{\substack{d\rightarrow\infty\\ \epsilon\rightarrow 0}} K_{2,d}(\epsilon) \sim K_2 \ . \tag{8}$$

As an example to clarify the idea, we show in Fig. 1 results pertaining to the Mackey-Glass delay differential equation[2,6] with the delay constant $\tau=23$. The system is turned to be 600 dimensional by the method described in Ref. 4. It is known to possess a strange attractor which is characterized by $\nu \cong 2.4$. In Fig. 1 we plot $\ln C_d(\epsilon)$ vs $\ln\epsilon$ for $d=4,8,12,\ldots,28$. The time series consisted of $N=12\,000$ points, separated by time $\Delta t=\tau$. We see indeed a series of straight lines with a slope of 2.4 ± 0.05. In order

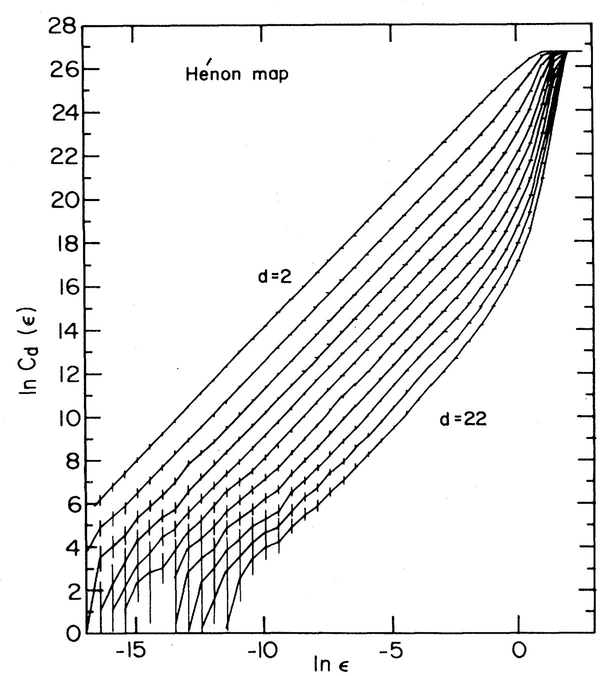

FIG. 3. Same as Fig. 1, but for the Hénon map. The values of d are $d=2$ (top curve), $4,6,8,\ldots,22$ (bottom curve).

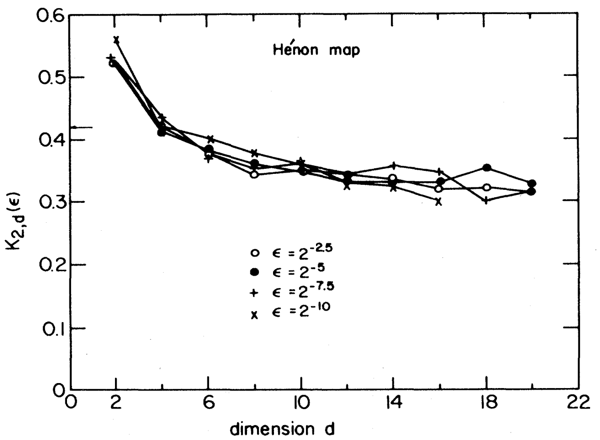

FIG. 4 Values of $K_{2,d}(\epsilon)$ for the Hénon map. For $d \to \infty$ all the curves tend to the extrapolated common value of $K_2 = 0.325 \pm 0.02$.

to estimate the Kolmogorov entropy, we compute the quantities

$$K'_{2,d}(\epsilon) = \ln[C_d(\epsilon)/C_{d+4}(\epsilon)]/4\tau$$

averaged over the scaling region in ϵ (see Fig. 2). The extrapolated value $K_2 = 0.008 \pm 0.01$ is indeed lower than the sum of positive Lyapunov exponents,[2,7] and it is in perfect agreement with the indirect estimate of K_2 from the asymptotic behavior of the equations for the tangent vectors.[7]

Another example is the Hénon map,[8] with $a = 1.4$, $b = 0.3$. In Fig. 3 we show $\ln C_d(\epsilon)$ vs $\ln \epsilon$ calculated from a series of $N = 15\,000$ points, for $d = 2, 4, 6, \ldots, 22$. The common slope in the scaling region is $\nu = 1.22 \pm 0.01$ in agreement with Ref. 4. Figure 4 shows

$$K''_{2,d}(\epsilon) = \frac{1}{2} \ln \frac{C_d(\epsilon)}{C_{d+2}(\epsilon)}$$

for various values of ϵ. Indeed for $d \to \infty$ these curves tend to a common value $K_2 = 0.325 \pm 0.02$. This value is again *lower* than the Kolmogorov entropy ($K \cong 0.42$). But again it agrees with calculations[7] based on the behavior of the tangent map.

Summarizing, we might thus conclude that a very good lower bound on the metric entropy of a strange attractor can be obtained from an experimental time series, using essentially the same algorithm which gives also a good lower bound on its dimensions. We hope that this will find applications in the important task of characterizing experimental deterministic chaos.

ACKNOWLEDGMENT

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