

Transport theory for a weakly interacting condensed Bose gas

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The Landau-Khalatnikov two-fluid hydrodynamic equations are derived for a dilute, weakly interacting, condensed Bose gas on the basis of a microscopic theory. Explicit expressions for the transport coefficients in the linearized equations are given for very low temperatures and for moderately low temperatures below the  $\lambda$  point.

The purpose of this paper is to report on a calculation of the transport properties of a dilute weakly interacting Bose gas below its  $\lambda$  point. We have been able to derive the nonlinear two-fluid equations of superfluid hydrodynamics as well as explicit expressions for the transport coefficients for this system, starting from a microscopic theory. These calculations can be viewed as the extension to nonequilibrium gases of the equilibrium calculations of Lee and Yang for a condensed Bose gas.<sup>1</sup> Further, our results can be used to compute the transport properties of spin-polarized hydrogen below its  $\lambda$  point.

As in the transport theory for classical systems,<sup>2</sup> there are two ways to derive hydrodynamic equations and expressions for the transport coefficients in terms of the microscopic properties of the particles of the systems under consideration here. One method—the distribution-function method—starts from the appropriate quantum Liouville equation and leads to a kinetic equation for the distribution function of quasiparticles in a spatially inhomogeneous dilute Bose gas below its  $\lambda$  point. We then look for a solution of this kinetic equation that describes a state close to local equilibrium and, following a procedure by Chapman and Enskog for the classical Boltzmann equation, we derive the nonlinear Landau two-fluid equations together with expressions for the transport coefficients. It should be remarked that the kinetic equation we derive has the same formal structure as the kinetic equation used by Khalatnikov and Landau in their semiphenomenological description of transport in helium.<sup>3</sup>

The other method we used—the time-correlation-function method—is based on the method first developed for a Bose fluid below its  $\lambda$  point by Hohenberg and Martin.<sup>4</sup> Here one looks for a solution of the quantum Liouville equation that describes the relaxation of a small, long-wavelength fluctuation about an equilibrium state. This procedure leads to the linearized two-fluid equations together with formal, Green-Kubo expressions for the associated transport coefficients. We have been able to evaluate these expressions for a dilute weakly interacting Bose gas and obtain results identical with those obtained by means of the distribution-function method.

To be specific, we will outline here the calculation based on the distribution-function method and indicate briefly the connection with the time-correlation-function method. Further details will be given in subsequent publications.<sup>5</sup>

We consider a system of weakly interacting spinless bosons with second-quantized Hamiltonian  $\hat{H} = \hat{H}_0 + \hat{V}$  given

by

$$\hat{H}_0 = \frac{\hbar^2}{2m} \int d\vec{r} \frac{\partial \hat{\psi}^\dagger(\vec{r})}{\partial \vec{r}} \cdot \frac{\partial \hat{\psi}(\vec{r})}{\partial \vec{r}} \quad (1a)$$

and

$$\hat{V} = \frac{1}{2} \int d\vec{r} \int d\vec{r}' V(\vec{r}') \hat{\psi}^\dagger(\vec{r} + \vec{r}') \times \hat{\psi}^\dagger(\vec{r}) \hat{\psi}(\vec{r}) \hat{\psi}(\vec{r} + \vec{r}') \quad (1b)$$

Here  $\hat{H}_0$  is the kinetic energy operator,  $\hat{V}$  is the potential-energy operator,  $\hat{\psi}(r)$  and  $\hat{\psi}^\dagger(\vec{r})$  are boson field operators,  $V(\vec{r})$  is the potential-energy function, and a circumflex denotes an operator. We consider the system to be composed of particles with mass  $m$  and they are contained in a volume  $\Omega$ , and we will eventually take the thermodynamic limit  $\Omega \rightarrow \infty$ ,  $\langle N \rangle \rightarrow \infty$ ,  $\langle N \rangle / \Omega = n$ , where  $\langle N \rangle$  denotes the average number of particles in  $\Omega$ .

We consider a weakly interacting gas, for which we can represent  $V(\vec{r}')$  in Eq. (1b) by the pseudopotential

$$V(\vec{r}') = \delta(\vec{r}') 4\pi \hbar^2 a / m,$$

where  $a$  is the  $s$ -wave scattering length for collisions between the particles.<sup>6</sup> We also introduce the thermal de Broglie wavelength  $\lambda$  by  $\lambda = (2\pi \hbar^2 / mk_B T)^{1/2}$ , where  $k_B$  is Boltzmann's constant and  $T$  the thermodynamic temperature. Since the system we consider is cooled below its  $\lambda$  point, we have the inequality  $n\lambda^3 \geq 1$ . We also suppose that the scattering length  $a \ll \lambda$ , and that the gas is dilute with respect to the scattering length, i.e.,  $na^3 \ll 1$ . In a way similar to the procedure used by Lee and Yang<sup>1</sup> for the equilibrium properties of a condensed Bose gas, we distinguish two regions: (a) a very-low-temperature region for which  $na\lambda^2 \geq 1$ , and (b) a moderately-low-temperature region for which  $na\lambda^2 \ll 1$ , but with  $n\lambda^3 \geq 1$ , and temperatures still below the  $\lambda$  point. Finally, we evaluate average values as traces with respect to a time-dependent density matrix  $\hat{\rho}(t)$ , so that  $\langle \hat{A} \rangle \equiv \text{Tr}(\hat{\rho}(t)\hat{A})$ , where  $\hat{A}$  is an operator.

Since the system is below its  $\lambda$  point, there is a condensed state that is macroscopically occupied, and we take the average values of the field operators  $\hat{\psi}(\vec{r})$  and  $\hat{\psi}^\dagger(\vec{r})$  to be nonzero and given by<sup>7,8</sup>

$$\langle \hat{\psi}(\vec{r}) \rangle = \langle \hat{\psi}^\dagger(\vec{r}) \rangle^* = n_c^{1/2}(\vec{r}, t) \exp[i\phi(\vec{r}, t)] \quad (2)$$

Here  $n_c(\vec{r}, t)$  is the local density of particles in the con-

densed state,<sup>9</sup> at time  $t$ , and  $\phi(\vec{r}, t)$  is a phase which is taken to be real. We also define a velocity  $\vec{V}_s(\vec{r}, t) = \hbar/m \vec{\nabla} \phi(\vec{r}, t)$ , determined by the phase of the condensed wave function. In the two-fluid hydrodynamic equations to be discussed below,  $\vec{V}_s(\vec{r}, t)$  is identified as the velocity of the superfluid. We then define the field operator for excited particles as  $\hat{\psi}'(\vec{r}, t) = \hat{\psi}(\vec{r}, t) - \langle \hat{\psi}(\vec{r}, t) \rangle$ .

To proceed, we find it convenient to eliminate the phase factor in Eq. (2) by a unitary transformation. As a consequence of this we perform our calculations in a local reference frame where the superfluid is at rest. For this case the condensed mode contains the lowest single particle energy state and we introduce the Wigner operator corresponding to the distribution function for particles in excited states at point  $\vec{R}$ , with velocity  $\hbar\vec{k}/m$ , by<sup>10</sup>

$$\hat{F}_p(\vec{R}, \vec{k}) = \int d\vec{r} e^{i\vec{q} \cdot \vec{r}} \hat{\psi}'\left(\vec{R} + \frac{\vec{r}}{2}\right) \hat{\psi}'\left(\vec{R} - \frac{\vec{r}}{2}\right) \quad (3a)$$

$$\equiv \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{R}} \hat{a}_{\vec{k}-\vec{q}/2}^\dagger \hat{a}_{\vec{k}+\vec{q}/2} \quad (3b)$$

where in (3b) we have used the representation  $\hat{\psi}(\vec{r}) = \Omega^{-1/2} \sum_{\vec{q}} \hat{a}_{\vec{q}} e^{i\vec{q} \cdot \vec{r}}$ , and the prime on the summation means that both  $\hat{a}_0$  and  $\hat{a}_0^\dagger$  are excluded in the sums. The subscript  $p$  denotes a "particle" Wigner operator. For the condensed Bose gas considered here it is convenient to make a transformation from a particle representation to an excitation or quasiparticle representation. Since the particles are weakly interacting, the transformation from particle operators  $\hat{a}_{\vec{k}}$  and  $\hat{a}_{\vec{k}}^\dagger$  to excitation creation and annihilation

operators  $\hat{b}_{\vec{k}}$  and  $\hat{b}_{\vec{k}}^\dagger$  is easily performed by a local, position-dependent, Bogolubov transformation.<sup>11</sup> The quasiparticle Wigner operator is defined by

$$\hat{F}(\vec{R}, \vec{k}) = \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{R}} \hat{b}_{\vec{k}-\vec{q}/2}^\dagger \hat{b}_{\vec{k}+\vec{q}/2} \quad (4)$$

where the energy of the excitation with wave number  $\vec{k}$  is given by  $E_k = [\epsilon_k^2 + 2\epsilon_k U_0 n_c(\vec{R}, t)]^{1/2}$  with  $\epsilon_k = \hbar^2 k^2 / 2m$ , and  $U_0 = 4\pi\hbar^2 a / m$ .

To derive the equations of two-fluid hydrodynamics and to determine the transport coefficients in them, we first derive the quantum analog of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy equations. The first hierarchy equation determines the distribution function for excitations with wave vector  $\vec{k}$ , at point  $\vec{R}$ , at time  $t$ ,

$$F(\vec{R}, \vec{k}, t) = \langle \hat{F}(\vec{R}, \vec{k}) \rangle = \text{Tr} \hat{\rho}(t) \hat{F}(\vec{R}, \vec{k}) \quad ,$$

in terms of higher-order distribution functions. One can then close the hierarchy equations by means of techniques similar to those used in classical kinetic theory to derive the Boltzmann equation for dilute gases from the Liouville equation.<sup>2</sup> In that case one obtains the Boltzmann equation as the first term in the expansion of a generalized collision operator in powers of  $n\sigma^3$ , where  $\sigma$  is the diameter of a particle. In the low-temperature quantum gas considered here, one obtains somewhat different expressions depending on the temperature range being studied:

(a) For very low temperatures where  $na\lambda^2 \gg 1$ , one can expand the collision operator in powers of the small parameter  $(n - n_c)/n_c$ . For this case one finds that  $F(\vec{R}, \vec{k}, t)$  satisfies, to lowest order, the kinetic equation<sup>12</sup>

$$\frac{\partial F(\vec{R}, \vec{k}, t)}{\partial t} + \hbar^{-1} \frac{\partial}{\partial \vec{k}} (E_k + \hbar \vec{k} \cdot \vec{V}_s) \cdot \frac{\partial F(\vec{R}, \vec{k}, t)}{\partial \vec{R}} - \hbar^{-1} \frac{\partial}{\partial \vec{R}} (E_k + \hbar \vec{k} \cdot \vec{V}_s) \cdot \frac{\partial F(\vec{R}, \vec{k}, t)}{\partial \vec{k}} = C_{12}(F) \left[ 1 + O\left(\frac{n - n_c}{n_c}\right) \right] \quad (5a)$$

where  $C_{12}$  is a collision operator that describes the process where one excitation decays into two excitations and vice versa.  $C_{12}$  is given by

$$C_{12}(F) = \frac{n_c \pi}{\hbar} U_0^2 \int \frac{d\vec{k}_1}{(2\pi)^3} \int d\vec{k}_2 \int d\vec{k}_3 \sigma^2(k_2, k_3; k_1) \delta(\vec{k}_1 - \vec{k}_2 - \vec{k}_3) \delta(E_{k_2} + E_{k_3} - E_{k_1}) [\delta(\vec{k} - \vec{k}_1) - \delta(\vec{k} - \vec{k}_2) - \delta(\vec{k} - \vec{k}_3)] \\ \times \{F(\vec{k}_3, t) F(\vec{k}_2, t) [1 + F(\vec{k}_1, t)] - F(\vec{k}_1, t) [1 + F(\vec{k}_2, t)] [1 + F(\vec{k}_3, t)]\} \quad (5b)$$

where

$$\sigma(k_2, k_3; k_1) = (u_{k_3} - v_{k_3})(u_{k_1} u_{k_2} + v_{k_1} v_{k_2}) + (u_{k_2} - v_{k_2})(u_{k_1} u_{k_3} + v_{k_1} v_{k_3}) - (u_{k_1} - v_{k_1})(u_{k_3} v_{k_2} + u_{k_2} v_{k_3}) \quad (5c)$$

with

$$u_k = [(\epsilon_k + n_c U_0) / 2E_k + \frac{1}{2}]^{1/2}; \quad v_k = [(\epsilon_k + n_c U_0) / 2E_k - \frac{1}{2}]^{1/2} \quad (5d)$$

In Eqs. (5b)–(5d), the  $F$ 's,  $\sigma$ , and  $E_k$ 's are to be evaluated at the point  $\vec{R}, t$ .

(b) For the moderately-low-temperature region, where  $na\lambda^2 \ll 1$ , but  $n\lambda^3 \gg 1$ , we can expand the collision operator in powers of  $na\lambda^2$ , or equivalently, in powers of  $a/\lambda$ , since  $an\lambda^2 = (n\lambda^3)(a/\lambda)$ . We then find that the left-hand side of Eq. (5a) is unchanged while the right-hand side is replaced by

$$[C_{12}(F) + C_{22}(F)] [1 + O(na\lambda^2)] \quad .$$

The collision operator  $C_{12}$  has the same structure as in the

very-low-temperature region, while the collision operator  $C_{22}(F)$  has the form of the Uhlenbeck-Uehling operator for bosons, and it describes processes where two excitations "collide" and produce two different excitations. In this temperature range, the excitations may be thought of as particle excitations, so that  $C_{12}$  describes collisions whereby one particle in the condensate and one excited particle collide to produce two excited particles and vice versa, while  $C_{22}$  describes collisions involving two excited particles.

To derive hydrodynamic equations from our kinetic equations, we follow Khalatnikov<sup>3</sup> and solve them by a generalization of the Chapman-Enskog procedure used in the theory

of the classical Boltzmann equation. That is, we look for solutions  $F(\vec{R}, \vec{k}, t)$  of our kinetic equations that have the form

$$F(\vec{R}, \vec{k}, t) = F_l(\vec{R}, \vec{k}, t) \{1 + [1 + F_l(\vec{R}, \vec{k}, t)] \times \psi^{(1)}(\vec{R}, \vec{k}, t) + \dots\}, \quad (6)$$

where  $F_l(\vec{R}, \vec{k}, t)$  is the local equilibrium distribution function given by

$$F_l(\vec{R}, \vec{k}, t) = (\exp\{\beta(\vec{R}, t)[E_k(\vec{R}, t) + \hbar \vec{k} \cdot \vec{A}(\vec{R}, t)]\} - 1)^{-1},$$

where  $\vec{A}(\vec{R}, t)$  is a space- and time-dependent vector, which is eventually identified with  $\vec{V}_s(\vec{R}, t) - \vec{V}_n(\vec{R}, t)$  in the two-fluid equations, with  $\vec{V}_n(\vec{R}, t)$  the local velocity of the normal component of the fluid. The correction term  $\psi^{(1)}(\vec{R}, \vec{k}, t)$  is proportional to the gradients of the local thermodynamic variables. As in the theory for the Boltzmann equation, one first derives conservation laws by taking moments of the kinetic equations with respect to the dynamically conserved variables of mass, momentum, and energy. For the case of interest here, the conservation equations do not arise solely from the kinetic equation, which describes only excitations of the system, but in order to obtain conservation laws one has to supplement those moments by equations we have derived for the condensed state.<sup>5</sup> If one then evaluates the expressions in the conservation laws using only the local equilibrium distribution function, one obtains the ideal two-fluid equations, together with expressions for the local thermodynamic functions. For a superfluid at rest the derived thermodynamic functions are identical to those calculated by Lee and Yang<sup>1</sup> in the appropriate temperature regions, and for  $\vec{V}_n - \vec{V}_s \neq 0$  the thermodynamic functions satisfy the usual phenomenological superfluid thermodynamic relations.<sup>3</sup> The ideal two-fluid equations are needed to determine  $\psi^{(1)}(\vec{R}, \vec{k}, t)$  in Eq. (6). This function is then used to compute the first correction to the expressions in the conservation laws beyond the local equilibrium result. This procedure then leads to the dissipative Landau-Khalatnikov two-fluid equations, with expressions for the transport coefficients in terms of  $\psi^{(1)}$ .

As mentioned above, one can use the time-correlation-function method expressions for the transport coefficients that appear in the linearized, dissipative two-fluid equations. However, these expressions are given in terms of the quantum Liouville operator for the system. We have been able to obtain explicit expressions for these transport coefficients for the case of interest here by showing that the time correlation functions can be evaluated in terms of the solutions of kinetic equations that are the linearized versions of the kinetic equations discussed above. We find that, for the transport coefficients appearing in the linearized two-fluid equations, both the kinetic equation method and the time-correlation-function method give identical results.

Here we present the explicit values for the transport coefficients appearing in the linearized equations. The additional coefficients appearing in the nonlinear equations will be discussed in a subsequent publication. There are six coeffi-

cients: the shear viscosity  $\eta$ , the thermal conductivity  $\kappa$ , and four bulk viscosities  $\zeta_1, \zeta_2, \zeta_3$ , and  $\zeta_4$ , where we use the notation as given by Khalatnikov.<sup>3</sup> We find that kinetic theory predicts  $\zeta_1 = \zeta_4$ , in accord with the Onsager symmetry principle. To evaluate these quantities we had to solve an integral equation for  $\psi^{(1)}$ , and we here present values obtained by keeping only the first nonzero term in the solution of this equation in terms of orthogonal polynomials. This corresponds to the first Enskog approximation in the classical theory.<sup>2</sup> Higher terms in the polynomial expansion will modify the numerical values in the expressions given below, probably by a few percent, but we have not yet pursued this point.

For very low temperatures, where  $na\lambda^2 \gg 1$ , we find

$$\begin{aligned} \eta &\cong (1.2 \times 10^{-3}) m (na\lambda^2)^7 [a^2 \pi^{3/2} (2\beta m)^{1/2} n \lambda^3]^{-1} \\ \kappa &= 0.24 k_B [a^2 \pi^{3/2} (2\beta m)^{1/2} n \lambda^3]^{-1}, \\ \zeta_1 = \zeta_4 &\cong 1.3 \times 10^{-2} \Delta, \\ \zeta_2 &\cong 4.4 \times 10^{-3} mn \Delta, \\ \zeta_3 &\cong 4 \times 10^{-2} \Delta / mn, \end{aligned}$$

with

$$\Delta = [\pi^{3/2} (2\beta m)^{1/2} a^2 (n \lambda^3) (na\lambda^2)]^{-1}.$$

For moderate temperatures ( $na\lambda^2 \ll 1$ ) we find that all four bulk viscosities  $\zeta_i \approx 0, i = 1, \dots, 4$ , as expected for a dilute gas, and that  $\eta$  and  $\kappa$  are given by

$$\eta \cong \frac{m}{\pi a^2 (2\beta m)^{1/2}} \frac{(0.02)}{1 + (0.32) n_s \lambda^3}, \quad (7a)$$

$$\kappa \cong \frac{k_B}{\pi a^2 (2\beta m)^{1/2}} \frac{0.06}{1 + (0.11) n_s \lambda^3}, \quad (7b)$$

where  $n_s$  is the superfluid number density. It is worth noting that while  $\eta$  and  $\kappa$  have the same temperature dependence in the moderately-low-temperature region, for very low temperatures  $\eta$  is proportional to  $T^{-5}$  while  $\kappa$  is proportional to  $T^2$ . The difference is due to the fact that the processes contributing to  $C_{12}$  lead to differing mean free paths for energy transport and for momentum transport. The mean free path for the transport of momentum needed for a shear viscosity becomes infinite as  $T \rightarrow 0$ . This as well as other related points will be discussed elsewhere.<sup>5</sup>

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<sup>1</sup>T. D. Lee and C. N. Yang, *Phys. Rev.* **112**, 1419 (1958); **113**, 1406 (1959); **117**, 897 (1960).

<sup>2</sup>For a review see J. R. Dorfman and H. van Beijeren, in *Statistical Mechanics, Part B*, edited by B. Berne (Plenum, New York, 1977), p. 79.

<sup>3</sup>See, for example, I. M. Khalatnikov, *Theory of Superfluidity* (Benjamin, New York, 1965).

<sup>4</sup>P. C. Hohenberg and P. C. Martin, *Ann. Phys. (N.Y.)* **34**, 291 (1965).

<sup>5</sup>T. R. Kirkpatrick and J. R. Dorfman (unpublished).

<sup>6</sup>The ensuing ultraviolet divergence that occurs in the evaluation of the thermodynamic functions discussed below should be removed in a way discussed by Lee and Yang (Ref. 1).

<sup>7</sup>The averages in Eq. (2) must be interpreted as quasiaverages [N. Bogoliubov, *Physica* **26**, S1 (1960)] or  $\eta$ -ensemble averages (Hohenberg and Martin, Ref. 4).

<sup>8</sup>Bogoliubov, Ref. 7.

<sup>9</sup>In our final results we eliminate the condensate density in terms of the macroscopic variables since  $n_c$  is not an independent hydrodynamic variable.

<sup>10</sup>For the slightly inhomogeneous Bose gas considered here, the condensed state involves wave numbers  $\leq (L_\nabla)^{-1}$ , where  $L_\nabla$  is a macroscopic gradient length, so that the sum in Eq. (3b) should exclude the  $\hat{a}_{\vec{k}}$  and  $\hat{a}_{\vec{k}}^\dagger$  for  $k < (L_\nabla)^{-1}$ . For the weakly interacting Bose gas only the removal of  $\hat{a}_0$  and  $\hat{a}_0^\dagger$  are important.

<sup>11</sup>See, for example, A. Fetter and J. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971), p. 313.

<sup>12</sup>To obtain Eq. (5a) a gradient expansion is also used. As discussed elsewhere (Kirkpatrick and Dorfman, Ref. 5), Eq. (5a) can be used to consistently derive two-fluid hydrodynamics for a weakly interacting Bose gas.