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Variational principle for relativistic magnetohydrodynamics

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A variational principle for relativistic magnetohydrodynamics is formulated. As an application, the properties of small-amplitude waves and their interaction with the mean flow are calculated. The generation of waves by an external current is incorporated into the formalism.

I. INTRODUCTION

It has long been realized that both the equations of hydrodynamics[1,2] and magnetohydrodynamics[3] can be derived from a variational principle. The specific form of the action used depends on whether one uses a Lagrangian or Eulerian coordinate system and the choice of fields to describe the flow. The use of variational principle is convenient for the formulation of local conservation laws associated with continuous symmetries in the system by Noether's theorem, e.g., Hill[4] and Soper.[5]

One possibility is the use of Eulerian coordinates with the "physical" variables such as the fluid velocity $\vec{v}$ and density ρ as fields. The various constraints on the flow such as conservation of mass or magnetic flux, are then treated by the method of Lagrangian multipliers. This makes this particular choice rather cumbersome since it leads to the appearance of the multipliers in the equation of motion (Euler-Lagrange equations) which then have to be eliminated algebraically in favor of the physical variables (cf. Henyey[6]).

A second possibility is the use of Lagrangian coordinates (the position of a fluid element at a given coordinate time, say $t=0$) as the basic fields, integrating the various constraint equations explicitly beforehand.[3,7] This procedure avoids the use of Lagrangian multipliers, and it is usually a simple matter to transform the resulting equations of motion to a Eulerian representation. We will therefore use this method.

In Sec. II the equations of relativistic magnetohydrodynamics (MHD) are briefly discussed. In Sec. III the variational principle is formulated. This is then used in Sec. IV to derive the properties of small-amplitude waves and their interaction with the mean flow, extending the work of Dewar[7] to the relativistic regime. The generation of waves by an external current is incorporated into the formalism.

Although I am not immediately concerned with general relativistic effects, we will allow for curvilinear coordinates. The metric will have a signature -2, reducing to $\mathrm{diag}(1,-1,-1,-1)$ in Cartesian space-time. The notation ∂_α and ∇_α is used to designate partial and covariant derivatives, respectively. We will use the notation $A\cdot B\equiv A_\alpha B^\alpha$ for the inner product of two four-vectors, with obvious generalization to contractions between tensors. The metric, as usual, has components $g_{\mu\nu}$. The four-velocity is designated by u, $u_\alpha u^\alpha=+1$.

II. ASSUMPTIONS OF RELATIVISTIC MAGNETOHYDRODYNAMICS

The equations of (general) relativistic magnetohydrodynamics have been formulated by Lichnerowicz,[8] Novikov and Thorne,[9] and Bekenstein and Oron.[10] Generally, one can define (covariant) electric and magnetic four-vectors, which reduce to the electric and magnetic fields in a comoving frame, by

$$E_\alpha=F_{\alpha\beta}u^\beta\ , \tag{1a}$$

$$B_\alpha=\tilde{F}_{\alpha\beta}u^\beta\ . \tag{1b}$$

Here $F_{\alpha\beta}$ are the components of the Maxwell tensor and $\tilde{F}_{\alpha\beta}\equiv\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$ the components of its dual. $\epsilon_{\alpha\beta\gamma\delta}\equiv(-g)^{1/2}[\alpha\beta\gamma\delta]$ is the completely antisymmetric Levi-Cività tensor, where $g=\det g_{\mu\nu}$. Note that by virtue of the antisymmetry of F and $\tilde{F}$ it follows that $E\cdot u=B\cdot u=0$ so that E and B have only three independent components.

The current density in a plasma is given by

$$J^\alpha=\epsilon u^\alpha+\sigma^{\alpha\beta}E_\beta\ . \tag{2}$$

Here ϵ is the charge density as measured by a comoving observer, and $\sigma^{\alpha\beta}$ is the conductivity tensor. The usual assumptions of magnetohydrodynamics are quasineutrality ($\epsilon\approx0$) and an isotropic medium ($\sigma^{\alpha\beta}\approx\sigma_0 g^{\alpha\beta}$), with a conductivity which is infinite ($\sigma_0\to\infty$). As was shown by Bekenstein and Oron,[10] these assumptions are justified when $\sigma_0\equiv\eta_e e^2\tau/m_s$ is large, where η_e, m_s, and τ are the electron density, mass, and collision time, respectively, while at the same time $\omega_{ce}\tau$ is small. Here ω_{ce} is the electron gyrofrequency. This last condition ensures that the gyrotropy of the plasma due to the ambient magnetic field can be ignored.

The relevant equations for the electromagnetic field now become

$$E_\alpha=F_{\alpha\beta}u^\beta=0\ , \tag{3a}$$

$$\nabla_\mu\tilde{F}^{\mu\nu}=0\ . \tag{3b}$$

The first equation ensures that the current density remains finite and says that the electric field vanishes in the comoving frame. In the "classical" limit it is just the well-known magnetohydrodynamic (MHD) condition $\vec{E}=-c^{-1}\vec{v}\times\vec{B}$. The second equation, which is usually written in the form $\nabla_{[\gamma}F_{\alpha\beta]}=0$, ensures the conservation of magnetic flux. This is most readily seen by "inverting" Eq. (1b) so that it reads (cf. Bekenstein and Oron,[10] taking into account the opposite signature of their metric)

$$\tilde{F}^{\alpha\beta}=u^{\beta}B^{\alpha}-u^{\alpha}B^{\beta} . \quad (4)$$

So, given the four-velocity of a fluid element, Eq. (3b) determines the evolution of the magnetic field. This set of equations replaces the full set of Maxwell's equations. In particular, note that the current density J^{α} does *not* appear anywhere in the equations.

III. VARIATIONAL PRINCIPLE: GENERAL THEORY

Hamilton's principle is $\delta A=0$, where the action A is given by

$$A=\int d^4x\sqrt{-g}\,\Lambda(x) . \quad (5)$$

Here $d^4x\sqrt{-g}$ is the invariant volume element and $\Lambda(x)$ a scalar, so that A is a scalar and invariant under coordinate transformations. The "Lagrangian density" Λ consists of a matter part Λ_M and a pure gravitational part Λ_G (cf. Soper[5] and Weinberg[11]). A suitable choice for Λ_M in magnetohydrodynamics is

$$\Lambda_M=-\rho(x)c^2-U(\rho)-\frac{1}{16\pi}g^{\alpha\beta}g^{\mu\nu}F_{\mu\alpha}F_{\nu\beta}$$
$$=-\rho(x)c^2-U(\rho)+\frac{1}{8\pi}B_{\alpha}B^{\alpha} , \quad (6)$$

where I have used (1) and (4). Here $U(\rho)$ is a thermodynamic potential from which the proper gas pressure P is determined by the equation

$$P=\frac{\partial U}{\partial \ln\rho}-U . \quad (7)$$

The assumption of isentropic flow (no dissipation) has been implicitly made [generally, $U=U(\rho,s)$, with s the entropy density]. The gravitational part is, as usual, given by

$$\Lambda_G=-\frac{c^4R}{16\pi G} , \quad (8)$$

where $R=R^{\mu}{}_{\mu}$ is the trace of the Ricci tensor and G the gravitational constant.

Note that, in keeping with our earlier remark, the current density J^{α} does not appear in Λ, which is a consequence of the fact that there is no irreversible exchange of energy between the electromagnetic field and the internal degrees of freedom of the plasma, i.e., the Joule heating $\sigma^{\alpha\beta}E_{\alpha}E_{\beta}$ vanishes because of (3).

Following Dewar,[12] I use as the basic matter fields for the variational principle the position $X^{\mu}=X^{\mu}(x,\lambda)$ of a fluid element in space-time at a "reference state" at $t=0$, which is mapped to the actual position x^{μ} of a fluid element at time t by following the world-line of the fluid element (cf. Fig. 1). The fields X^{μ} are Lagrangian labels, which are carried by the fluid elements, and behave as scalars under coordinate transformations. The world-lines are parametrized by a parameter λ, so that the four-velocity at each point is given by

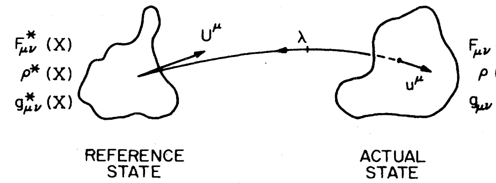

FIG. 1. Schematic representation of the mapping from the reference state at some fiducial proper time to the actual state. The mapping is a one-to-one correspondence, with points connected by fluid-element world-lines.

$$u^{\mu}=-\frac{dx^{\mu}}{d\lambda}\left[\frac{dx^{\alpha}}{d\lambda}\frac{dx_{\alpha}}{d\lambda}\right]^{-1/2} . \quad (9)$$

As I will show below, the matter part of the Lagrangian density Λ_M in the case of magnetohydrodynamics can always be represented in the form

$$\Lambda_M=\Lambda_m(X^{\mu};\partial_{\nu}X^{\mu};u^{\mu}) . \quad (10)$$

We also introduce the Eulerian variation δ at a fixed position in space-time and a Lagrangian variation Δ evaluated at a position following a fluid element along its world-line. The relation between the two is

$$\Delta=\delta+\Delta x^{\mu}\nabla_{\mu} . \quad (11)$$

[A third possible variation is $\tilde{\Delta}=\delta+L_{\Delta x}$, where $L_{\Delta x}$ is the Lie derivative along Δx (Schutz[13]). It corresponds to the change evaluated following the fluid element with respect to a coordinate system carried with the fluid, rather than a fixed system. The definitions of Δ and $\tilde{\Delta}$ coincide for scalars, but not for vectors or tensors. We will not use this particular variation here.]

The definitions of X^{μ}, Δ, and δ immediately imply

$$\Delta X^{\mu}=0;\quad \delta x^{\mu}=0;\quad \delta X^{\mu}=-\Delta x^{\alpha}\partial_{\alpha}X^{\mu} . \quad (12)$$

Using (12) and (9) and the fact that δ and ∂ commute, one can calculate

$$\delta\partial_{\mu}X^{\nu}=\partial_{\mu}(\delta X^{\nu})=-\partial_{\mu}(\Delta x^{\alpha}\partial_{\alpha}X^{\nu}) , \quad (13)$$

$$\delta u^{\mu}=h^{\mu\nu}u^{\alpha}\nabla_{\alpha}(\Delta x_{\nu})-\Delta x^{\alpha}\nabla_{\alpha}u^{\mu} . \quad (14)$$

Here $h^{\mu\nu}\equiv g^{\mu\nu}-u^{\mu}u^{\nu}$ is the projection tensor on a hyperplane perpendicular to the four-velocity. Equations (13) and (14) generalize Dewar's result to curvilinear coordinates in space-time. Note that $\partial_{\nu}X^{\mu}$ is a set of four covariant vectors, one for each value of μ, and *not* a mixed tensor, as the notation might suggest at face value.

Using (12)–(14), one can perform the variation Δx of the fluid world-lines keeping the metric $g_{\mu\nu}$ fixed. Assuming the system to be infinite and the variations Δx to vanish when $|x|\rightarrow\infty$, one obtains, after a partial integration,

$$\delta A = 0 \int d^4x \sqrt{-g}\, \Delta x^\alpha \left\{ X^\mu{}_{,\alpha} \left[\frac{1}{\sqrt{-g}} \left[\frac{\partial \Lambda_M}{\partial X^\mu{}_{,\nu}} \sqrt{-g} \right]_{,\nu} - \frac{\partial \Lambda_M}{\partial X^\mu} \right] - \nabla_\alpha u^\mu \frac{\partial \Lambda_M}{\partial u^\mu} - \frac{1}{\sqrt{-g}} \left[\sqrt{-g}\, h^\mu{}_\alpha u^\nu \frac{\partial \Lambda_m}{\partial u^\mu} \right]_{,\nu} + \Gamma^\sigma{}_{\nu\alpha} h^\mu{}_\sigma u^\nu \frac{\partial \Lambda_M}{\partial u^\mu} \right\} . \tag{15}$$

Here I have employed the notation $\varphi_{,\alpha}$ for $\partial_\alpha \varphi$ and used $\nabla_\nu(\Delta x^\alpha) = \partial_\nu(\Delta x^\alpha) + \Gamma^\alpha{}_{\nu\lambda}\Delta x^\lambda$. Since the variation Δx is arbitrary, one finds a set of Euler-Lagrange equations which describe the MHD flow, and that can be written in a manifestly covariant form

$$\partial_\alpha X^\mu \left[\nabla_\nu \left[\frac{\partial \Lambda_M}{\partial(\partial_\nu X^\mu)} \right] - \frac{\partial \Lambda_M}{\partial X^\mu} \right] - \nabla_\alpha u^\mu \frac{\partial \Lambda_M}{\partial u^\mu} - \nabla_\nu \left[h^\mu{}_\alpha u^\nu \frac{\partial \Lambda_M}{\partial u^\mu} \right] = 0 . \tag{16}$$

Here I have used $(1/\sqrt{-g})\partial_\mu(\sqrt{-g}) = \Gamma^\lambda{}_{\lambda\mu}$.

The fact that Λ_M does not *explicitly* depend on x, and the action is therefore invariant under an infinitesimal coordinate transformation corresponding to a translation $x^\mu \to x^\mu + \epsilon\alpha^\mu$, $\epsilon \ll 1$, leads by Noether's theorem to the energy-momentum conservation law

$$\nabla_\mu (T_M)^\mu{}_\nu = 0 , \tag{17}$$

where the energy-momentum tensor $(T_M)^\mu{}_\nu$ is given by

$$(T_M)^\mu{}_\nu = \partial_\nu X^\sigma \frac{\partial \Lambda_M}{\partial(\partial_\mu X^\sigma)} - h^\sigma{}_\nu u^\mu \frac{\partial \Lambda_M}{\partial u^\sigma} - \Lambda_M g^\mu{}_\nu . \tag{18}$$

A. Application to magnetohydrodynamics

We will now apply the general theory developed above to magnetohydrodynamics. In order to do so, the Lagrangian density Λ_M must be expressed in terms of X^μ, $\partial_\nu X^\mu$, and u^μ. This can be achieved by formally integrating the matter-current conservation law

$$\nabla_\mu(\rho u^\mu) = 0 , \tag{19}$$

and the conservation law (3b) for the magnetic flux, expressing the density $\rho(x)$ in terms of the density $\rho^*(X)$ in the reference state and the above fields, and similarly expressing $F^{\mu\nu}$ in terms of $F^{\alpha\beta *}(X)$.

Defining the matter current $j^\mu \equiv \rho u^\mu$, matter conservation (19) implies

$$j^\mu d\sigma_\mu = j^{*\lambda} d\Sigma_\lambda , \tag{20}$$

where $d\sigma$ is a surface element in the actual state and $d\Sigma$ its image in the reference state: j^* is the image of j in the reference state. Noting that the four-velocity U in the reference state equals

$$U^\lambda = \frac{u\cdot\partial X^\lambda}{[(u\cdot\partial)X^\sigma (u\cdot\partial)X^\tau g^*{}_{\sigma\tau}(X)]^{1/2}} , \tag{21}$$

it follows immediately that the actual density ρ can be expressed as

$$\rho = \left[\frac{g^*(X)}{g(x)} \right]^{1/2} \frac{J\rho^*(X)}{[(u\cdot\partial)X^\sigma (u\cdot\partial)X^\tau g^*{}_{\sigma\tau}(X)]^{1/2}} . \tag{22}$$

Here $J = \det(\partial_\alpha X^\beta)$, and I have used the fact that $d\sigma$ and $d\Sigma$ are related by

$$d\sigma_\mu = \frac{\sqrt{-g}}{\sqrt{-g^*}} J^{-1} \partial_\mu X^\lambda d\Sigma_\lambda .$$

$g^*{}_{\sigma\tau}(X)$ are the components of the metric in the reference state, and $g^* = \det(g^*{}_{\sigma\tau})$.

The constraint of magnetic-flux conservation is most conveniently expressed in a coordinate-free fashion using the theory of differential forms.[13,14] To that purpose one defines as usual the Faraday two-form $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$. As is well known, the electromagnetic field can be dervied from a vector potential $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ concisely expressed as $F = dA$ and $A \equiv A_\mu dx^\mu$. We now invoke a general result for the Lie derivative $L_u F$ of a form F along a vector $u = u^\mu \partial_\mu$ (see Schutz[13])

$$L_u F = d(F\cdot u) + (dF)\cdot u . \tag{23}$$

Taking F to be the Faraday two-form and u the four-velocity, it follows that the Lie derivative $L_u F$ vanishes identically. According to the MHD condition (3a), one has $F\cdot u = E \equiv E_\alpha dx^\alpha = 0$, eliminating the first term in (23). The second term vanishes by virtue of $dF = ddA = 0$, independently of the MHD condition.

This means that the Faraday two-form is Lie dragged with the fluid. Since the reference state, with $F = \frac{1}{2}F^*{}_{\alpha\beta}dX^\alpha \wedge dX^\beta$ and the actual state with $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ are by definition connected by worldlines, this implies that

$$\tfrac{1}{2}F^*{}_{\alpha\beta}dX^\alpha \wedge dX^\beta = \tfrac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu , \tag{24}$$

or equivalently,

$$F_{\mu\nu}(x) = F^*{}_{\alpha\beta}(X)\partial_\mu X^\alpha \partial_\nu X^\beta . \tag{25}$$

The relations (22) and (25) allow us to express the Lagrangian density Λ_M as a functional of X^μ, $\partial_\nu X^\mu$, and u^μ, as was asserted above.

Introducing the differential operator

$$D^{\mu}{}_{\nu} \equiv \partial_\nu X^\sigma \frac{\partial}{\partial(\partial_\mu X^\sigma)} - u^\mu h^\sigma{}_\nu \frac{\partial}{\partial u^\sigma} , \tag{26}$$

one can write (18) in the form

$$(T_M)^\mu{}_\nu \equiv (D^\mu{}_\nu - g^\mu{}_\nu)\Lambda_M . \tag{27}$$

Using (12)—(14) (22), and (25), it follows that

$$D^\mu{}_\nu \rho = \rho h^\mu{}_\nu , \tag{28a}$$

$$D^\mu{}_\nu F_{\alpha\beta} = F_{\nu\beta} g^\mu{}_\alpha + F_{\alpha\nu} g^\mu{}_\beta . \tag{28b}$$

The definition (6) for Λ_M then immediately yields

$$(T_M)^\mu{}_\nu = [\rho c^2 + U(\rho)] u^\mu u_\nu - P h^\mu{}_\nu + \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} g^\mu{}_\nu - \frac{1}{4\pi} F^{\mu\alpha} F_{\nu\alpha} . \tag{29}$$

Writing (4) in the form $F_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} u^\gamma B^\beta$, and using the properties of the Levi-Cevità tensor (e.g., Misner, Thorne, and Wheeler[14]), one can write this in a more transparent form:

$$(T_M)^\mu{}_\nu = \left[\rho^2 c + U(\rho) + \frac{H^2}{8\pi}\right] u^\mu u_\nu - \left[P + \frac{H^2}{8\pi}\right] h^\mu{}_\nu - \frac{1}{4\pi} B^\mu B_\nu . \tag{30}$$

Here I have defined $H^2 \equiv -B_\alpha B^\alpha$, which corresponds to the magnetic field strength squared in the comoving frame.

The conservation laws $\nabla \cdot T = 0$, $\nabla \cdot (\rho u) = 0$, and $\nabla \cdot \tilde{F} = 0$, together with $u \cdot B = 0$, constitute a closed set of equations for relativistic magnetohydrodynamics. For alternative formulations, the reader is referred to Bekenstein and Oron.[10]

B. Variation of the metric

In Sec. III A I have derived the dynamical equations for the matter by varying the world-lines of the fluid elements. We will now briefly review the result of varying the metric, with the matter variables fixed.

Generally one can write

$$\delta A = \int d^4x \left[\frac{\delta A_M}{\delta g_{\mu\nu}} + \frac{\delta A_G}{\delta g_{\mu\nu}}\right] \delta g_{\mu\nu} , \tag{31}$$

where

$$\frac{\delta A_M}{\delta g_{\mu\nu}} \equiv \frac{\partial(\sqrt{-g}\,\Lambda_M)}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\lambda}\left[\frac{\partial\sqrt{-g}\,\Lambda_M}{\partial g_{\mu\nu,\lambda}}\right] + \frac{\partial^2}{\partial x^\kappa \partial x^\lambda}\left[\frac{\partial\sqrt{-g}\,\Lambda_M}{\partial g_{\mu\nu,\kappa\lambda}}\right] , \tag{32}$$

and similarly for $\delta A_g / \delta g_{\mu\nu}$. Using

$$j^\lambda = \rho u^\lambda = \frac{\sqrt{-g^*}}{\sqrt{-g}} J a^\lambda{}_\mu j^{*\mu}$$

with $a^\lambda{}_\mu$ the inverse of $\partial_\beta X^\alpha$ so that $\partial_\beta X^\alpha a^\beta{}_\gamma = g^\alpha{}_\gamma$, which follows straightforwardly from matter conservation, and writing $\rho = (g_{\alpha\beta} j^\alpha j^\beta)^{1/2}$, it follows that

$$\delta\rho = \frac{1}{2\rho}(j^\mu j^\nu - g_{\alpha\beta} j^\alpha j^\beta g^{\mu\nu}) \delta g_{\mu\nu} = -\tfrac{1}{2}\rho h^{\mu\nu} \delta g_{\mu\nu} . \tag{33}$$

Here I have used the general relation $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}$, cf. Weinberg.[11] With this result, the relation $\delta g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}$, expression (25) for $F_{\mu\nu}$, and the definition (6) for Λ_M, it follows immediately that

$$\delta(\sqrt{-g}\,\Lambda_M) = -\tfrac{1}{2}\sqrt{-g}\, T_M{}^{\mu\nu} \delta g_{\mu\nu} , \tag{34}$$

with $T_M{}^{\mu\nu}$ the matter energy-momentum tensor defined by (29).

Using the definition of $R = g^{\lambda\mu} g^{\kappa\mu} R_{\lambda\mu\nu\kappa}$ in terms of the Riemann tensor, one can write for the functional derivative $\delta A_G / \delta g_{\mu\nu}$ (cf. Weinberg[11])

$$\frac{\delta A_G}{\delta g_{\mu\nu}} = \frac{c^4\sqrt{-g}}{16\pi G}(R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu}) , \tag{35}$$

where $R^{\mu\nu}$ is the Ricci tensor. Hamilton's principle with respect to the arbitrary variation $\delta g_{\mu\nu}$ now yields Einstein's equations

$$\delta A = 0 = \int d^4x \frac{c^4\sqrt{-g}}{16\pi G}\left[R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} - \frac{8\pi G}{c^4} T_M{}^{\mu\nu}\right] \delta g_{\mu\nu} . \tag{36}$$

IV. SMALL-AMPLITUDE MHD WAVES

We will now use the Lagrangian formalism to derive the properties of small-amplitude MHD waves and the interaction between the waves and a suitably defined mean flow which supports the waves. In the nonrelativistic case, this was first done in a systematic way by Dewar.[7] He used the work of Whitham[15] who showed that in the WKB approximation, when $1/\omega T$ and $1/kL$ are small quantities (ω and k are the typical wave frequency and wave vector; T and L the typical time and length scales on which the mean flow varies appreciably), the wave properties can be derived from a variational principle employing an action averaged over the phase of the waves.

The starting point is the assumption that it is possible to divide the motion of a fluid element into a slowly varying component and a nearly periodic wave train

$$\tilde{x}^\mu = x^\mu + \epsilon\xi^\mu , \quad \xi^\mu = a^\mu e^{iS} + \text{c.c.} \tag{37}$$

Here ϵ is a dummy parameter which will be set equal to unity in the end results, and c.c. denotes complex conjugation. The congruence of world-lines $\tilde{x}$ constitutes the exact motion of the fluid, whereas the congruence of world-lines $x^\mu(\lambda)$ of (fictitious) fluid elements constitute the "background state," which evolves according to a selfconsistent set of dynamical equations which follow from the variational principle.

The action for the system is expanded in powers of ϵ (for simplicity we assume Cartesian space-time, limiting this discussion to special relativity):

$$A=\int d^4\tilde{x}\,\tilde{\Lambda}_M(\tilde{x})$$
$$=\int d^4x\,\tilde{J}[\Lambda_0(x)+\epsilon\Lambda_1(x)+\epsilon^2\Lambda_2(x)+O(\epsilon^3)]\ . \tag{38}$$

Here $\tilde{J}=\det(\partial\tilde{x}^\mu/\partial x^\nu)$ is the determinant of the transformation $\tilde{x}\to x$, which is given by

$$\tilde{J}=1+\epsilon\partial_\alpha\xi^\alpha+\tfrac{1}{2}\epsilon^2[(\partial_\alpha\xi^\alpha)^2-\partial_\beta\xi^\alpha\partial_\alpha\xi^\beta]+O(\epsilon^3)\ . \tag{39}$$

Likewise $\Lambda_M(x)$ has been expanded in powers of ϵ.

The work of Whitham[15] and Dewar[7] now states that the action A can be replaced by the *average* action $\langle A\rangle$ with $\langle\cdots\rangle\equiv(1/2\pi)\int_0^{2\pi}ds(\cdots)$. Performing the average, one gets

$$\langle A\rangle=\int d^4x\langle L\rangle(x)=\int d^4x(L_0+\epsilon^2L_2)\ . \tag{40}$$

Here

$$L_0(x)=\Lambda_0(x)\ , \tag{41a}$$
$$L_2(x)=\langle\partial_\alpha\xi^\alpha\Lambda_1+\Lambda_2\rangle\ . \tag{41b}$$

To get the explicit expressions in terms of the background fluid variables such as ρ, B^μ, etc., one must consider the mapping $X\to\tilde{x}=x+\epsilon\xi$ to calculate Λ_1 and Λ_2, by expansion of $\tilde{\Lambda}_M(\tilde{x})$. This is done in the Appendix. Defining the wave vector k_μ by

$$k_\mu\equiv-\frac{\partial S}{\partial x^\mu}\ , \tag{42}$$

the result reads

$$L_0=-\rho c^2-U(\rho)+\frac{1}{8\pi}B_\alpha B^\alpha\ , \tag{43a}$$
$$L_2=-(\rho c^2+P+U)(k\cdot u)^2(a_\perp\cdot a^*{}_\perp)-\rho\frac{\partial P}{\partial\rho}(k_\perp\cdot a)(k_\perp\cdot a^*)$$
$$-\frac{1}{4\pi}(k\cdot u)^2[(B\cdot a)(B\cdot a^*)-(B\cdot B)(a_\perp\cdot a^*{}_\perp)]$$
$$+\frac{1}{4\pi}\Omega^\mu\Omega^*{}_\mu\ . \tag{43b}$$

Here I have used the notation $A_\perp{}^\alpha\equiv h^{\alpha\beta}A_\beta$ for any four-vector A, which satisfies $A_\perp\cdot u=0$, and defined

$$\Omega^\mu\equiv(k_\perp\cdot a)B^\mu-(k\cdot B)a_\perp{}^\mu\ . \tag{44}$$

We note here that, since $a_\perp\cdot u=k_\perp\cdot u=B\cdot u=0$, the average Lagrangian $\langle L\rangle$ is invariant under the "gauge" transformation

$$a^\mu\to a^\mu+\lambda u^\mu\ , \tag{45}$$

which corresponds to a relabeling of proper time on the world-lines of the background fluid elements. This means that one can always choose a gauge with

$$a\cdot u=0\ . \tag{46}$$

The average Lagrangian $\langle L\rangle$ is now to be considered as a functional:

$$\langle L\rangle=\langle L\rangle(X^\mu;\partial_\alpha X^\mu;u^\mu;a^\mu;a^{*\mu};S;k_\mu)\ . \tag{47}$$

Here $X^\mu(x,\lambda)$ is the mapping from the reference state to the *background* state, u^μ the four-velocity of the fluid elements in the background state, and a^μ, $a^{*\mu}$, and S are to be regarded as fields which can be varied independently.

A. Properties of the waves

The properties of the small-amplitude waves follow from the variation of $\langle L\rangle$ with respect to $a^*{}_\mu$ (or a_μ) and S: Varying $a^*{}_\mu$ yields

$$\frac{\partial L_2}{\partial a^*{}_\mu}=0\ . \tag{48}$$

This can be written in the form

$$\left[\left[\rho c^2+P+U+\frac{H^2}{4\pi}\right](k\cdot u)^2-\frac{(k\cdot B)^2}{4\pi}\right]a_\perp{}^\mu+\left[\left[\rho\frac{\partial P}{\partial\rho}+\frac{H^2}{4\pi}\right](k_\perp\cdot a)+\frac{(k\cdot B)(a\cdot B)}{4\pi}\right]k_\perp{}^\mu$$
$$+\left[\frac{1}{4\pi}(k\cdot u)^2(B\cdot a)+\frac{1}{4\pi}(k\cdot B)(k_\perp\cdot a)\right]B^\mu=0\ . \tag{49}$$

This set of equations determines the wave frequencies and the polarization characteristics of the wave amplitudes a^μ. Note that there are only *three* independent equations (and, therefore, three independent kinds of waves) since $u_\mu(\partial L_2/\partial a^*{}_\mu)=0$. This can be seen directly from (49), or can be interpreted as a consequence of Noether's theorem due to the invariance of $\langle L\rangle$ under the gauge transformation (45). Furthermore, L_2 is bilinear in a and a^*, so that $a^*{}_\mu(\partial L_2/\partial a^*{}_\mu)=L_2=0$. One can show with considerable algebra that this set of equations is equivalent with $\partial_\mu T_1{}^{\mu\nu}=-ik_\mu T_1^{\mu\nu}=0$. $T_1^{\mu\nu}$ is the order ϵ perturbation in the energy-momentum tensor $[\tilde{T}^{\mu\nu}(\tilde{x})=T_0{}^{\mu\nu}(x)+\epsilon T_1{}^{\mu\nu}(x)+O(\epsilon^2)]$ induced by the waves. This once again proves the equivalence of the average Lagrangian method with the full dynamics of the system when the perturbations can be represented as traveling waves.

The three independent wave modes can be projected out by contracting (49) with $\Sigma_\mu=\epsilon_{\mu\nu\kappa\lambda}u^\kappa B^\lambda k^\nu \equiv F_{\mu\nu}k^\nu$, k_μ, and B_μ, the first being orthogonal to the second two. The resulting equation can be written as a matrix equation $\underline{D}\,\underline{Z}=0$, where

$$\underline{Z}\equiv\begin{bmatrix}(\Sigma\cdot a)\\(k_\perp\cdot a)\\(B\cdot a)\end{bmatrix},\quad \underline{D}\equiv\begin{bmatrix}D_{11}&0&0\\0&D_{22}&D_{23}\\0&D_{32}&D_{33}\end{bmatrix}. \tag{50}$$

The elements of the matrix $\underline{D}$ are given by

$$\begin{aligned}
D_{11}&=\left[\rho c^2+P+U+\frac{H^2}{4\pi}\right](k\cdot u)^2-\frac{1}{4\pi}(k\cdot B)^2,\\
D_{22}&=\left[\rho c^2+P+U+\frac{H^2}{4\pi}\right](k\cdot u)^2+\left[\rho\frac{\partial P}{\partial\rho}+\frac{H^2}{4\pi}\right]k_\perp^2,\\
D_{23}&=k^2\frac{1}{4\pi}(k\cdot B),\\
D_{32}&=\rho\frac{\partial P}{\partial\rho}(k\cdot B),\\
D_{33}&=(\rho c^2+P+U)(k\cdot u)^2.
\end{aligned} \tag{51}$$

The dispersion relation corresponds to the requirement that there are nontrivial solutions of $\underline{D}\,\underline{Z}=0$:

$$\det\underline{D}=D_{11}(D_{22}D_{33}-D_{23}D_{32})=0. \tag{52}$$

The solution $D_{11}=0$ corresponds to the Alfvén mode, which has the polarization $k_\perp\cdot a=B\cdot a=0$, and a frequency $\Omega\equiv ck\cdot u$ in the plasma rest frame given by

$$\Omega_A=\pm ck_\mu\beta_A{}^\mu,\quad \beta_A{}^\mu\equiv\frac{B^\mu}{\sqrt{4\pi}\left[\rho c^2+P+U+\dfrac{H^2}{4\pi}\right]^{1/2}}. \tag{53}$$

The second term in the dispersion relation (52) corresponds to the fast and slow MHD modes:

$$\begin{aligned}
\Omega^2=&\tfrac{1}{2}[l^2c^2\beta_M{}^2+\beta_S{}^2(ck\cdot\beta_A)^2]\\
&\pm\tfrac{1}{2}\{[l^2c^2\beta_M{}^2+\beta_S{}^2(ck\cdot\beta_A)^2]^2\\
&-4l^2c^2\beta_S{}^2(ck\cdot\beta_A)^2\}^{1/2}.
\end{aligned} \tag{54}$$

Here I have defined $l^2=-(k_\perp\cdot k_\perp)$, the length of the wave vector squared in a comoving frame, and

$$\beta_M\equiv\left[\frac{\rho\dfrac{\partial P}{\partial\rho}+\dfrac{H^2}{4\pi}}{\rho c^2+P+U+\dfrac{H^2}{4\pi}}\right]^{1/2}, \tag{55a}$$

$$\beta_S\equiv\left[\frac{\rho\dfrac{\partial P}{\partial\rho}}{\rho c^2+P+U}\right]^{1/2}, \tag{55b}$$

which are the relativistic (proper) fast magnetosound and the sound speed, respectively, in units of c. If one chooses the gauge $(a\cdot u)=0$, these waves will have their amplitude a^μ in the hyperplane containing $k_\perp$ and B.

Varying the phase $S(x)$ of the waves leads to a conservation law for the wave-action current density

$$\partial_\mu N^\mu=0,\quad N^\mu\equiv\frac{\partial L_2}{\partial k_\mu}. \tag{56}$$

The wave-action current density can be expanded as

$$N^\mu=N\left[cu^\mu-\frac{\partial\Omega}{\partial k_{\perp\mu}}\right]. \tag{57}$$

Here I have used that for a frequency $\Omega(k_\perp,x)$ which satisfies the dispersion relation (52), one has $L_2(\Omega,k_\perp)=0$, so that for any quantity φ, one has

$$\frac{\partial L_2}{\partial\varphi}=\frac{\partial L_2}{\partial\Omega}\frac{\partial\Omega}{\partial\varphi}+\left[\frac{\partial L_2}{\partial\varphi}\right]_\Omega=0,$$

$$\frac{\partial}{\partial k_\mu}=cu^\mu\frac{\partial}{\partial\Omega}+h^{\alpha\mu}\frac{\partial}{\partial k_\perp{}^\alpha}.$$

We also defined the wave-action density in the comoving frame by

$$N=\frac{\partial L_2}{\partial\Omega}=c^{-1}u_\mu N^\mu.$$

B. Interaction between the waves and the background

We now turn to the question of the interaction between the waves and the background flow. By Noether's theorem the invariance of $\langle L\rangle$ under the (infinitesimal) translation $x^\mu\to x^\mu+d^\mu$ (no explicit dependence on x) implies the energy-momentum conservation law $\partial_\mu T^{\mu\nu}=0$, with $T^{\mu\nu}$ being the total energy-momentum tensor of the system of background flow and waves which is given by

$$T^{\mu\nu}=(\tilde{D}^{\mu\nu}-g^{\mu\nu})\langle L\rangle, \tag{58}$$

where

$$\begin{aligned}
\tilde{D}^{\mu\nu}&=\partial^\nu X^\sigma\frac{\partial}{\partial X^\sigma{}_\mu}-u^\mu h^{\rho\mu}\frac{\partial}{\partial u^\rho}+k^\nu\frac{\partial}{\partial k_\mu}\\
&\equiv D^{\mu\nu}+k^\nu\frac{\partial}{\partial k_\mu}.
\end{aligned} \tag{59}$$

Here I have used the Euler-Lagrange equations (16), (48), and (56).

This allows one to write

$$\begin{aligned}
T^{\mu\nu}&=T_{BG}{}^{\mu\nu}+T_W{}^{\mu\nu},\\
T_{BG}{}^{\mu\nu}&\equiv\left[\rho c^2+U(\rho)+\frac{H^2}{8\pi}\right]u^\mu u^\nu-\left[P+\frac{H^2}{8\pi}\right]h^{\mu\nu}\\
&\quad-\frac{1}{4\pi}B^\mu B^\nu.
\end{aligned} \tag{60}$$

$$T_W{}^{\mu\nu}\equiv N^\mu k^\nu+D^{\mu\nu}L_2.$$

This split up of the total energy-momentum tensor into a "background part" and a "wave part" is rather arbitrary and is motivated by the fact that the background part is unchanged by the presence of the waves. This corresponds to the so-called physical split up as defined by Dewar.[7,12] Other divisions are, of course, equally valid. For instance, one could assign the canonical energy-momentum tensor

$$T_C{}^{\mu\nu} \equiv \frac{\partial L_2}{\partial k_\mu} k^\nu - L_2 g^{\mu\nu} = N^\mu k^\nu$$

to the waves and the remainder of $T^{\mu\nu}$ to the background. This arbitrariness is the basis for the well-known Abrahams-Minkovski controversy about the "correct" form for the energy-momentum tensor associated with electromagnetic waves in a medium,[12,16] which shows up here in the MHD limit.

One can associate a *ponderomotive force* with the waves to describe their interaction with the background flow as a simple consequence of overall energy-momentum conservation:

$$\partial_\mu T_{BG}{}^{\mu\nu} \equiv f^\nu_{\rm pond} = -\partial_\mu T_W{}^{\mu\nu} \, . \tag{61}$$

The precise form of the ponderomotive force, of course, does depend on the particular split up chosen.

Using the relations

$$\begin{aligned} &\tilde{D}^{\alpha\beta}\rho = \rho h^{\alpha\beta} \, , \\ &\tilde{D}^{\alpha\beta}B^\mu = u^\alpha u^\mu B^\beta - h^{\mu\beta}B^\alpha + h^{\alpha\beta}B^\mu \, , \\ &\tilde{D}^{\alpha\beta}\Omega = u^\alpha u^\beta \Omega \, , \\ &\tilde{D}^{\alpha\beta}k_\perp{}^\mu = k_\perp{}^\beta g^{\mu\alpha} + (k\cdot u)(h^{\mu\alpha}u^\beta + h^{\mu\beta}u^\alpha) \, , \end{aligned} \tag{62}$$

one gets for the wave part of the energy-momentum tensor in the gauge $a\cdot u = 0$, taking into account that $L_2[\Omega(k_\perp,B,x,\rho);k_\perp,B,x,\rho]=0$, and the only dependence of L_2 on the four-velocity is through Ω:

$$T_W{}^{\mu\nu} = N\Omega \left\{ u^\mu u^\nu - \left[h^{\mu\nu}\frac{\partial}{\partial \ln\rho} + (u^\mu u^\alpha B^\nu - h^{\alpha\nu}B^\mu + h^{\mu\nu}B^\alpha)\frac{\partial}{\partial B^\alpha} + \left[k_\perp{}^\nu g^{\mu\alpha} + \frac{\Omega}{c}(u^\mu h^{\alpha\nu} + u^\mu h^{\alpha\mu}) \right] \frac{\partial}{\partial k_\perp{}^\alpha} \right] \ln\Omega \right\} . \tag{63}$$

Note that energy density of the waves as measured by a comoving observer equals $W = u_\mu T_W{}^{\mu\nu} u_\nu = N\Omega$.

As an example, for Alfvén waves the energy-momentum tensor becomes

$$\begin{aligned} T_A{}^{\mu\nu} = N\Omega_A [&u^\mu u^\nu - \tfrac{1}{2}(1-\beta_M{}^2)h^{\mu\nu} + \beta_A{}^\mu \beta_A{}^\nu \\ &- u^\mu \beta_A{}^\nu - u^\nu \beta_A{}^\mu] \, . \end{aligned} \tag{64}$$

Here Ω_A and β_A are defined in Eq. (53), choosing the plus sign in the solution of Ω_A. (The equivalent tensor with the minus sign follows by replacing β_A by $-\beta_A$.) β_M is the fast magnetosound velocity defined in (55a).

One can always go to a three-vector notation by realizing that $T^\mu{}_\nu$ can be thought of as being built out of the following "components" (indices i take the values 1 to 3):

$$\begin{aligned} &T^0{}_0 = W \, , \\ &T^i{}_0 \equiv -c^{-1} S_i \, , \\ &T^0{}_i \equiv -cG_i , \\ &T^i{}_j \equiv -T_{ij} \, , \end{aligned} \tag{65}$$

where W is the energy density, S is the energy flux G is the momentum flux, and T is the physical stress tensor.

C. Wave turbulence and wave generation

So far I have assumed essentially monochromatic waves. All the results derived above can be extended in a simple manner to the case of wave turbulence where a wide range of frequencies and wave vectors is present. We write

$$\xi^\mu = \sum_l a_l e^{iS_l} + \text{c.c.} \tag{66}$$

Here the $\sum_l$ indicates a sum over mode numbers or, equivalently, over initial wave vectors. Expression (42) generalizes to $k_l{}^\mu = -\partial S_l / \partial x_\mu$. The averaging procedure used to find the average action $\langle A \rangle$ can now be reinterpreted as an ensemble average with the assumption of random phases, i.e., $\langle a_l{}^\mu a_{l'}{}^\nu \rangle = a_l{}^\mu a^*{}_l{}^\nu \delta_{l'-l}$. All the equations involving wave quantities derived above remain valid, provided one includes the subscript l' on each quantity indicating the mode under consideration. For instance, one can write $L_2 = \sum_l (L_2)_l$, and the variation of this Lagrangian with respect to the phase S_l gives the conservation of wave action for each model l, analogous to (56):

$$\partial_\mu N_l{}^\mu = 0 \, . \tag{67}$$

In this section, I will use a "3 + 1" notation rather than a covariant notation (roman indices run from 1 to 3). The dispersion relation is now used in the form $\omega = ck^0 = \omega(\vec{k},\vec{x},t)$, and (67) can be written as $\partial n_l / \partial t + (\partial/\partial x^i)[(\partial\omega_l/\partial k_l{}^i)n_l] = 0$. Here I have defined $n_l \equiv \partial (L_2)_l / \partial \omega_l$. Note that n_l is not a Lorentz invariant but the time component of the four-vector $N_l{}^\mu = [cn_l, (\partial\omega_l/\partial\vec{k}_l)n_l]$, where I used $(L_2)_l = 0$. In geometrical optics it is customary to define the wave-action density with respect to the local wave vector $\vec{k}$ rather than the mode number, which corresponds to a Eulerian rather than a Lagrangian parametrization in space. This can be done straightforwardly by defining

$$n_k(\vec{x},t) \equiv \sum_l n_l(\vec{x},t)\delta(\vec{k}-\vec{k}_l) \, ,$$

cf. Katou.[17] Using the well-known result

$$\left[\frac{\partial}{\partial t} + \frac{\partial \omega_l}{\partial k_j}\frac{\partial}{\partial x_j} \right] k_i = -\frac{\partial \omega_l}{\partial x_i} \, ,$$

the conservation of wave action becomes

$$\frac{\partial n_k}{\partial t}+\frac{\partial}{\partial \vec{x}}\cdot\left[\frac{\partial \omega_k}{\partial \vec{k}}n_k\right]-\frac{\partial}{\partial \vec{k}}\cdot\left[\frac{\partial \omega_k}{\partial \vec{x}}n_k\right]=0 \ . \tag{68}$$

The total energy-momentum tensor associated with the ensemble of waves is simply the sum of all contributions in k space, formally defined as

$$T_w^{\mu\nu}\equiv\int\frac{d^3k\,d\omega}{(2\pi)^4}\sum_l(T_W^{\mu\nu})_l\delta(\vec{k}-\vec{k}_l)2\pi\delta(\omega-\omega_l(\vec{k}_l,\vec{x},t)) \ . \tag{69}$$

We now turn to the question of wave generation and/or wave damping. A variational principle, in general, cannot handle dissipative (irreversible) effects in a system. However, when the generation or damping of the waves is due to an "external" current which could, for instance, describe the resonant generation of waves by a small fraction of the particles in the plasma which are considered as a separate component in the system, it can be included in the variational principle.

To that purpose, I add an interaction term to the action (38):

$$A=\int d^4x\,\tilde{J}[\Lambda_0(x)+\epsilon\Lambda_1(x)+\epsilon^2\Lambda_2(x)]+\epsilon(1/c)\int d^4x\,J_e{}^{\alpha}A_\alpha \ . \tag{70}$$

Here the external current density J_e is explicitly given by a superposition of plane waves

$$J_e{}^{\mu}(x)=\int\frac{d^4q}{(2\pi)^4}j_e{}^{\mu}(q)e^{-iq\cdot x} \ . \tag{71}$$

As usual, $A^\alpha=A^\alpha{}_0+\epsilon A^\alpha{}_1+\cdots$ is the vector potential. The fact that $J_e(x)$ is real requires $j_e(-q)=j_e{}^*(q)$, whereas charge conservation $\partial_\mu J_e{}^\mu=0$ implies $q_\mu j_e{}^\mu(q)=0$. I will assume that the external current $j_e(q)$ is small in the sense that the change $\delta\omega$ in the wave frequency that it induces is small, i.e., $|\delta\omega/\omega|\ll 1$. This means that to lowest order the quantities associated with the waves, such as the first-order change $(F^{\mu\nu})_1$ in the Maxwell tensor, are unaffected ("rigid") by the presence of the external current. Performing the variation in the total action and taking an (ensemble) average of the result yields for each mode l (suppressing the l index)

$$\langle\delta A\rangle=0=\int d^4x\left\{\Delta x^\alpha\frac{\delta\langle L\rangle}{\delta X^\alpha}+\sum_l\left[\delta a^\lambda\left[\frac{\partial L_2}{\partial a^\lambda}+\frac{1}{c}F_{\lambda\alpha}j_e{}^{*\alpha}(k)\right]+\text{c.c.}\right]+\sum_l\delta S_l\left[\frac{\partial}{\partial x^\mu}\left[\frac{\partial L_2}{\partial k_\mu}\right]+\left[i\frac{1}{c}F_{\lambda\alpha}a^\lambda j_e{}^{*\alpha}(k)+\text{c.c.}\right]\right]\right\} \ . \tag{72}$$

Here the random-phase approximation was used to calculate $\langle j_e{}^\alpha a^\beta\rangle$ and the fact that one locally can always choose $S\approx -ik^\mu x_\mu$. In this equation I have defined

$$\langle L\rangle\equiv L_0+\sum_l\left[(L_2)_l+\frac{1}{c}[j_e{}^{*\alpha}(k_l)(A_\alpha)_1(k_l)+\text{c.c.}]\right] \ ,$$

with L_0 and L_2 given by Eqs. (43a) and (43b) and used the fact that the perturbing vector potential $A_1(k)$ due to the waves can be expressed in terms of a^μ and the (unperturbed) Maxwell tensor $F_{\mu\nu}$ as $(A_\mu)_1=a^\lambda F_{\lambda\mu}e^{iS}+\text{c.c.}$, up to an arbitrary of electromagnetic gauge $(A_\mu\to A_\mu+ik_\mu X)$, which does not change the result since current requires $k_\mu j_e{}^\mu(k)=0$. This expression for A_1 can straightforwardly be derived from the definition of F in terms of A and the perturbation expansion (A4) in the Appendix. The functional derivative $\delta\langle L\rangle/\delta X^\alpha$ is a shorthand notation for the left-hand side of Eq. (16) with $\langle L\rangle$ instead of Λ_M and covariant derivatives replaced by ordinary derivatives since I am assuming flat space-time. It corresponds to the functional derivative with respect to the variation of the fluid-element orbits in the background and not to any variation in x^α, $\delta x^\alpha=0$, as the notation might suggest. (This procedure does not introduce any spurious degrees of freedom; instead of the four fields ξ^μ describing the waves, there now are the three independent components of a^μ, since the invariance of the equations under the transformation $a^\mu\to a^\mu+\lambda u^\mu$ is still satisfied as is demonstrated below, and the phase S.)

In the absence of any external current, the variations in Δx^α, $\delta a_l{}^\lambda$, and δS_l yield the Eqs. (16), (48), and (56) derived in the previous sections. With the current, the last two equations are modified for each mode l to

$$\frac{\partial L_2}{\partial a^{*\mu}}=-\frac{1}{c}F_{\mu\alpha}j_e{}^\alpha(k_l) \ , \tag{73a}$$

$$\frac{\partial N_l{}^\mu}{\partial x^\mu}=\frac{i}{c}F_{\lambda\alpha}a^{*\lambda}j_s{}^\alpha(k_1)+\text{c.c.} \tag{73b}$$

The first equation describes the dispersion relation of the MHD waves modified by the external current. Note that there are still only three independent equations. $u^\mu(\partial L_2/\partial a^{*\mu})=0$ because of the MHD condition (3a). In the classical limit $(c\to\infty)$ this set of equations can be shown to be equivalent to the equations derived by Akhiezer *et al.*[18] for this case.

We are interested in the case where the external current is the linear response of some (external) system to the wave-electromagnetic field. In that case one can always write[19]

$$j_e{}^\mu(k)=\alpha_e{}^{\mu\nu}(A_\nu)_1 \ . \tag{74}$$

The linear response tensor $\alpha_e{}^{\mu\nu}$ can be expressed in terms of the conductivity tensor σ_{ij} in ordinary space which connects the spatial components of the current density to the wave electric field by $j_i=\sigma_{ij}E_j$:

$$\alpha^{\mu\nu}\equiv -i\frac{c}{\omega}\begin{bmatrix}\sigma_{rs}k_rk_s & \sigma_{ir}k_r\dfrac{\omega}{c}\\ \sigma_{rj}k_r\dfrac{\omega}{c} & \sigma_{ij}\dfrac{\omega^2}{c^2}\end{bmatrix} \ . \tag{75}$$

It can be easily checked that the response tensor α constructed in this way satisfies $k_\nu \alpha^{\mu\nu} = \alpha^{\mu\nu} k_\nu = 0$ which guarantees charge conservation and the invariance of the current with respect to gauge transformations of the vector potential. The generation of waves is due to the *Hermitian* part $\sigma^H{}_{ij} \equiv \frac{1}{2}(\sigma_{ij} + \sigma^*{}_{ji})$ of the conductivity tensor, and consequently to the *anti-Hermitian* part of $\alpha^{\mu\nu}$, as follows from its definition. In the following I will neglect the contribution of the anti-Hermitian part to the external current, which leads to a small shift in the real part of the wave frequency. Defining the tensor $M_{\mu\lambda} \equiv (i/c)\alpha_e{}^{\sigma\tau} F_{\mu\sigma} F_{\lambda\tau}$, we can write $\partial L_2 / \partial a^{*\mu} = i M^H{}_{\mu\lambda} a^\lambda$.

Note that $u^\mu M_{\mu\lambda} = M_{\mu\lambda} u^\lambda = 0$ which ensures the invariance of the equations under the gauge transformation $a^\mu \to a^\mu + \lambda u^\mu$. Using the procedure used in (50) project out three different MHD modes, I now find $\underline{D}\,\underline{Z} = i\underline{S}$, where the column vector $\underline{S}$ has the components $-M^H{}_{\lambda\tau} a^\tau [\Sigma^\lambda, k_\perp{}^\lambda, 0]$. Here I have used $B^\lambda F_{\lambda\alpha} = 0$. Since I assumed the external current to be small, one can expand this dispersion relation by putting $\omega = \omega_0 + i\gamma$, with $|\gamma/\omega_0| \ll 1$ and $\omega_0(\vec{k}, \vec{x}, t)$ corresponding to the solution of the "zero-order" dispersion relation $\underline{D}\,\underline{Z} = 0$. Once the frequency ω_0 and the corresponding vector $\underline{Z}$ for all three modes is determined, the first-order correction due to J_e for each of the modes follows from $\gamma[(\partial/\partial\omega_0)\underline{D}]\underline{Z} = \underline{S}$, which yields

$$\gamma = \frac{\underline{Z} \cdot \underline{S}}{\left[\dfrac{\partial}{\partial \omega_0} \underline{D} \right] : \underline{Z}\,\underline{Z}} . \tag{76}$$

Equation (73b) describes the generation of wave quanta by the external current. This can be most easily seen by employing the relation

$$a^{*\mu} \frac{\partial L_2}{\partial a^{*\mu}} = L_2 = -\frac{1}{c} F_{\mu\sigma} a^{*\mu} j_e{}^\sigma = i M^H{}_{\mu\lambda} a^{*\mu} a^\lambda$$

and expanding this equation with respect to $\omega_0 + i\gamma$. To lowest order one has $L_2 = 0$, whereas to next order one finds

$$i\gamma \partial L_2 / \partial \omega_0 = i\gamma n_l = i M^H{}_{\mu\lambda} a^{*\mu} a^\lambda (k^l) .$$

This allows one to write (73b) in the form

$$\left[\frac{\partial}{\partial t} + \frac{\partial \omega_0}{\partial \vec{k}} \cdot \frac{\partial}{\partial \vec{x}} - \frac{\partial \omega_0}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{k}} \right] n_k = 2\gamma_k n_k . \tag{77}$$

Here I have transformed again to Eulerian representation in wave-number space.

Finally, I write the equation of energy-momentum conservation in the case with an external current. The fact that $\langle L \rangle$ does not explicitly depend on x, i.e.,

$$\frac{\partial \langle L \rangle}{\partial x^\alpha} - \frac{\partial \langle L \rangle}{\partial X^\beta} X^\beta{}_{,\alpha} - \frac{\partial \langle L \rangle}{\partial X^\beta{}_{,\gamma}} X^\beta{}_{,\alpha\gamma} - \frac{\partial \langle L \rangle}{\partial u^\beta} u^\beta{}_{,\alpha} - \frac{\partial \langle L \rangle}{\partial k^\beta} k^\beta{}_{,\alpha} = 0$$

can be written in the form

$$\partial_\gamma T^{\gamma\alpha} = \frac{i}{c} \sum_l k_l{}^\alpha F_{\lambda\mu} j_e{}^\mu a^*{}_l{}^\lambda + \text{c.c.} = \sum_l 2\gamma_l k_l{}^\alpha n_l . \tag{78}$$

The right-hand side of this equation is the mean Lorentz force density $-1/c j_e{}^{*\mu}(F_{\mu\alpha})_1 + \text{c.c.}$ due to the external current on the system. Here I used Eq. (A4) from the Appendix, and charge conservation. The energy-momentum tensor $T^{\mu\nu}$ is given by the analog of (60), $T^{\mu\nu} = T_{bg}{}^{\mu\nu} + T_W{}^{\mu\nu}$, with the background part unchanged, and the wave part is given by $T_W{}^{\mu\nu} = \sum_l [N_l{}^\mu k_l{}^\nu + D^{\mu\nu}(L_2)_l]$. Here I have neglected the terms due to the external current in $T^{\mu\nu}$, since they only enter into the equations as derivatives and are, therefore, higher-order terms in the WKB approximation.

The fact that the energy-momentum tensor is no longer conserved is physically obvious, since the external current does work on the system, which leads to the nonconservation of the wave-action current. Mathematically it can be traced to the explicit dependence on x of the external current, as given in (71).

V. CONCLUSIONS

In this paper I have formulated a variational principle for relativistic magnetohydrodynamics from a Lagrangian point of view. I have employed that variational principle to calculate the properties of small-amplitude MHD waves and their interaction with the mean flow. The formalism was extended to include the generation of waves by an "external" current. A simple transformation law for the Maxwell tensor in MHD was found, linking it to its values at some initial proper time to the actual value.

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APPENDIX

We briefly describe the expansion of $\tilde{\Lambda}_M(\tilde{x})$ in powers of ϵ due to the transformation $\tilde{x}^\mu = x^\mu + \epsilon\xi^\mu$, writing

$$\tilde{\Lambda}_M(\tilde{x}) = \Lambda_0(x) + \epsilon\Lambda_1(x) + \epsilon^2\Lambda_2(x) + O(\epsilon^3) , \tag{A1}$$

and likewise for all other quantities, e.g.,

$$\tilde{\rho}(\tilde{x}) = \rho_0(x) + \epsilon\rho_1(x) + \epsilon^2\rho_2(x) + \cdots ,$$
$$\tilde{F}_{\mu\nu}(\tilde{x}) = (F_0)_{\mu\nu}(x) + \epsilon(F_1)_{\mu\nu}(x) + \epsilon^2(F_2)_{\mu\nu}(x) + \cdots . \tag{A2}$$

This can be most straightforwardly achieved by considering the mapping $X \to \tilde{x}$ to the exact fluid state and the mapping $X \to x$ to the background fluid state and expanding the first in terms of the latter. The relations (22) and (25) of the main paper then immediately yield

$$\rho_1 = -\rho h^{\alpha\beta}\partial_\beta \xi_\alpha \,,$$
$$\rho_2 = \rho\left(\tfrac{1}{2}\{[(u\cdot\partial)\xi]^2 - (uu:\partial\xi)^2 + (\partial\cdot\xi)^2 + \partial\xi\cdot\partial\xi\} - (uu:\partial\xi)(\partial\cdot\xi)\right) , \qquad \text{(A3)}$$

$$(F_{\mu\nu})_1 = F_{\lambda\mu}\partial_\nu\xi^\lambda - F_{\lambda\nu}\partial_\mu\xi^\lambda \,,$$
$$(F_{\mu\nu})_2 = F_{\mu\lambda}\partial_\tau\xi^\lambda\partial_\nu\xi^\tau + F_{\lambda\nu}\partial_t\xi^\lambda\partial_\mu\xi^\tau + F_{\kappa\lambda}\partial_\mu\xi^\kappa\partial_\nu\xi^\lambda \,. \qquad \text{(A4)}$$

Here I have dropped the subscripts 0 on ρ_0 and $(F_0)_{\mu\nu}$. Making a Taylor expansion of $\widetilde{\Lambda}_M(\widetilde{x})$ as given by (6) with a tilde substituted on all variables, using (A3) and (A4), one finds after some algebra, using $F_{\mu\nu} = \epsilon_{\mu\nu\kappa\lambda}u^\kappa B^\lambda$:

$$\Lambda_1 = [\rho c^2 + P + U(\rho)]h^{\alpha\beta}\partial_\alpha\xi^\beta + \frac{1}{4\pi}[B^\alpha B^\beta - (B\cdot B)h^{\alpha\beta}]\partial_\alpha\xi_\beta \,,$$
$$\Lambda_2 = -\tfrac{1}{2}[\rho c^2 + P + U(\rho)][(u\cdot\partial\xi)^2 - (uu:\partial\xi)^2 + (\partial\cdot\xi)^2 + (\partial\xi:\partial\xi) - 2(uu:\partial\xi)(\partial\cdot\xi)] - \tfrac{1}{2}\rho\frac{\partial P}{\partial\rho}(h^{\alpha\beta}\partial_\beta\xi_\alpha)^2$$
$$-\frac{1}{8\pi}[B_\alpha B_\beta - (B\cdot B)h_{\alpha\beta}](\partial^\kappa\xi^\alpha + \partial^\alpha\xi^\kappa)\partial_\kappa\xi^\beta - \frac{1}{8\pi}F_\mu{}^\lambda F^\nu{}_\kappa\partial_\nu\xi^\mu\partial_\lambda\xi^\kappa - \frac{1}{16\pi}F^{\mu\nu}F_{\kappa\lambda}\partial_\mu\xi^\kappa\partial_\nu\xi^\lambda \,. \qquad \text{(A5)}$$

Equations (43) of the main paper now follow straightforwardly by putting $\xi = ae^{iS} + \text{c.c.}$ and averaging over the phase S, according to Eq. (40) of the main paper. Owing to the antisymmetry of $F_{\mu\nu}$, the last term in Λ_2 vanishes identically, whereas the second but last term can be eliminated using the identity

$$F^{\alpha\beta}F_{\gamma\delta}k_\alpha k^\delta a_\beta a^{*\gamma} + \text{c.c.} = -2\Omega^\alpha\Omega^*{}_\alpha - 2k_\perp{}^2[B^\alpha B^\beta - (B\cdot B)h^{\alpha\beta}]a_\alpha a^*{}_\beta \,. \qquad \text{(A6)}$$

Ω_α is defined by Eq. (44) of the main paper.

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