Quantum theory of laser-radiation scattering by electrons in magnetic fields

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We consider a system consisting of an electron in a static magnetic field, interacting with the quantized electromagnetic field, within the nonrelativistic and electric dipole approximations (with a cutoff in momentum space). The Heisenberg equations of motion are solved exactly and the time evolution of the electric field is determined. The power spectrum of the scattered radiation is calculated when the electromagnetic field is initially in a coherent state. Our results for the line shape of the scattered radiation are shown to be valid for magnetic fields up to 10^{12} G. The quantization of the electromagnetic field allows one to consider effects of the natural linewidth and its dependence on the magnetic field. The renormalization of the electron mass is included in our treatment, and the results remain finite when the cutoff goes to infinity.

I. INTRODUCTION

The interaction of electrons, atoms, and molecules with strong electromagnetic fields has attracted considerable interest in recent times, stimulated by the possibility of laboratory production of strong magnetic fields, by the advent of high-power lasers, by the development of plasma physics, and by discoveries in the field of astrophysics.

In fact, laser intensities up to 10^{16} W/cm^2 have been attained in recent experiments,[1,2] while magnetic fields of up to 300 kG continuous or 10^7 G pulsed have been obtained in the lab.[3] Furthermore, recent observations[4] of the x-ray binary source Her X-1 indicate that the distance between Landau levels in this case is of approximately 58 keV, which corresponds to a magnetic field of the order of 0.5×10^{13} G.

For fields of this order of magnitude, new effects show up which cannot be predicted with the usual perturbation techniques. New schemes have thus been developed, either using modified bases which include already part of the effect of the electromagnetic field,[5–7] or applying semiclassical approximations,[8] or variational methods,[9] etc.

In the present work, we study the interaction of an electron with a static and uniform magnetic field, in the presence of the quantized electromagnetic field, by means of an exactly soluble model. Exact solutions were obtained by Redmond[10] for the Klein-Gordon and Dirac equations corresponding to a charged particle moving in the presence of an electromagnetic plane wave plus a static and uniform magnetic field parallel to the wave propagation direction. On the other hand, approximate solutions for potential scattering in the simultaneous presence of a laser field and a static and uniform magnetic field have been presented by several authors.[6,11] This process seems to play a role in the problem of plasma heating by lasers.[12] The corresponding cross sections exhibit resonant behavior,[11] which has already been observed in photoionization experiments in the presence of strong magnetic fields.[13]

In all these approximate treatments, the laser field is considered to be an external field, and the transition probabilities are calculated taking the potential as a perturbation and applying Born's approximation. Consequently, these treatments do not allow the calculation of the spectrum of the scattered field, or the study of the system in the resonant region, when the Born approximation ceases to be valid and the linewidth of the Landau levels becomes important.

The model here considered allows the study of the line shape of the scattered radiation, in the resonance region, calculated from the electric field correlation function. The electric field, on the other hand, is obtained by solving the Heisenberg equations of motion.

This procedure has been frequently adopted in quantum optics,[14–17] and allows one to avoid the usual perturbation theory, as developed in the Schrödinger picture. In particular, a model similar to the one here proposed was considered in Ref. 16. It consists of a nonrelativistic isotropic harmonic oscillator interacting with the quantized electromagnetic field, in the electric dipole approximation (which is also adopted in our model, as in all of the above-mentioned approximate solutions). The electric field in the Heisenberg picture is explicitly found. Our model is, however, more realistic than the one studied in Ref. 16; besides that, we calculate the power spectrum of the scattered radiation and consider renormalization effects, which is not done in that work.

In Sec. II, we define the model and estimate its region of validity. We show then that it remains valid for magnetic fields up to 10^{12} G, close to the most intense fields already observed in astrophysics. In the Hamiltonian defining the model, we introduce a cutoff in momentum space for the quantized electromagnetic field, thus avoiding the typical divergences of quantum electrodynamics.

In Sec. III, we calculate the electric field in the Heisenberg picture, expressing it in terms of the electron renormalized mass. This mass is calculated by diagonalizing the free-electron Hamiltonian, which can be exactly done, following a treatment by Van Kampen,[18] discussed in Appendix A. After renormalization, we show that the results are practically independent of the cutoff, remaining finite in the limit when this cutoff is taken to infinity. In particular, the fact that we have here an exactly soluble model allows one to follow in detail the behavior of the "runaway" solution, which shows up frequently in problems of this kind.[18] This is done in Appendix B.

In Sec. IV, we calculate the spectrum of the scattered radiation, for the case in which the electromagnetic field is initially in a coherent state. The spectrum exhibits a resonance around the cyclotron frequency, its width increasing with the magnetic field. In Sec. V, we summarize our conclusions.

II. THE MODEL

We take the electron initially as a finite size particle, with a charge distribution $e\rho(\vec{r})$. Later on, we shall take the point-electron limit, after mass renormalization. The magnetic field is considered as an external field, described by a constant vector $\vec{B}$, taken along the z direction. The Hamiltonian of this system is given by (the cgs system is used throughout)

$$H=\left[\vec{p}-\frac{e}{c}\vec{A}(\vec{r})\right]^2\Big/2m + \sum_\lambda \int d^3k\,\hbar\omega a_\lambda^\dagger(\vec{k})a_\lambda(\vec{k})-(e/m)\vec{S}\cdot\vec{B}\ , \quad (2.1)$$

where

$$\omega=|\vec{k}|c\ , \quad (2.2)$$

$$\vec{A}(\vec{r})=\vec{A}_M(\vec{r})+\vec{A}_F^R(\vec{0})\ , \quad (2.3)$$

$$\vec{A}_M(\vec{r})=\tfrac{1}{2}\vec{B}\times\vec{r}\ , \quad (2.4)$$

and $\vec{A}_F^R(\vec{r})$ is the quantized electromagnetic vector potential regularized by the electronic charge distribution:

$$\vec{A}_F^R(\vec{r})=\int \rho(\vec{r}\,')\vec{A}_F(\vec{r}+\vec{r}\,')d^3r'\ , \quad (2.5)$$

where

$$\vec{A}_F(\vec{r})=\sum_\lambda\int\frac{d^3k}{(2\pi)^{3/2}}\left[\frac{hc^2}{\omega}\right]^{1/2}[a_\lambda(\vec{k})\vec{\epsilon}_\lambda(\vec{k})e^{i\vec{k}\cdot\vec{r}}+a_\lambda^\dagger(\vec{k})\vec{\epsilon}_\lambda^{\,*}(\vec{k})e^{-i\vec{k}\cdot\vec{r}}]\ , \quad (2.6)$$

$$[a_\lambda(\vec{k}),a_{\lambda'}^\dagger(\vec{k}\,')]=\delta(\vec{k}-\vec{k}\,')\delta_{\lambda\lambda'}\ , \quad (2.7)$$

and

$$\int\rho(\vec{r})d^3r=1\ . \quad (2.8)$$

In Eq. (2.6), λ is the polarization index, $\vec{\epsilon}_\lambda(\vec{k})$ are the polarization vectors, such that (Coulomb gauge)

$$\vec{k}\cdot\vec{\epsilon}_\lambda(\vec{k})=0\ . \quad (2.9)$$

Notice that the quantization volume is taken to be infinite from the outset.

In Eq. (2.1), the field $\vec{A}_F^R(\vec{r})$ is taken at $\vec{r}=\vec{0}$, which corresponds to making the electric dipole approximation. Consistently, we have neglected the interaction of the electron spin $\vec{S}$ with the magnetic field associated with $\vec{A}_F^R(\vec{r})$.

If $g(\vec{k})$ is the Fourier transform of $\rho(\vec{r})$,

$$g(\vec{k})=\int\rho(\vec{r})e^{i\vec{k}\cdot\vec{r}}d^3r\ , \quad (2.10)$$

then

$$\vec{A}_F^R(\vec{0})=\sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}}\left[\frac{hc^2}{\omega}\right]^{1/2} g(\vec{k})[a_\lambda(\vec{k})\vec{\epsilon}_\lambda(\vec{k})+a_\lambda^\dagger(\vec{k})\vec{\epsilon}_\lambda^{\,*}(\vec{k})] . \quad (2.11)$$

We examine now the conditions of validity of the approximations made in the present model.

The electric dipole approximation requires that the relevant wavelengths be much larger than the diameter of the region in which the electron moves. In particular, if we take the initial state of the electromagnetic field as a coherent state, the corresponding wavelength must be large enough so that this condition is satisfied. Furthermore, since the electron is not confined along the direction of the magnetic field, the propagation vector of the initial electromagnetic field must be orthogonal to $\vec{B}$.

On the other hand, the nonrelativistic approximation requires that the electron's speed be much smaller than c, at least for the first Landau orbits. Thus if ω_c is the cyclotron frequency and r_L the radius of the first orbit, one must have

$$v=\omega_c r_L=\frac{1}{m}\left[\frac{|e|B\hbar}{c}\right]^{1/2} \ll c , \quad (2.12)$$

so that

$$B \ll B_{\rm cr}=\frac{m^2c^3}{|e|\hbar}=4.4\times 10^{13}\ {\rm G} . \quad (2.13)$$

This restriction on B is not serious, since the critical field $B_{\rm cr}$ is much larger than the strongest fields produced in the laboratory, and still allows the treatment of the magnetic fields observed in astrophysics. On the other hand, the above condition is equivalent to

$$r_L \ll \lambda_c , \quad (2.14)$$

where λ_c is the wavelength associated with the cyclotron frequency ω_c. Equation (2.14) shows that, so long as the relevant wavelengths of the electromagnetic field be of the order of or larger than λ_c, the electric dipole approximation is valid for the first Landau orbits, if $B \ll B_{\rm cr}$. In particular, for resonant fluorescence, $\lambda \simeq \lambda_c$, and the conditions of validity of the electric dipole and nonrelativistic approximations coincide.

For more external orbits, the upper limit for B becomes more severe. However, the conditions of validity of the electric dipole and the nonrelativistic approximations remain the same: $v=\omega_c r \ll c$ or $r \ll \lambda_c$, where r is now the radius of an arbitrary Landau orbit. A treatment valid for $B \simeq B_{\rm cr}$ has been presented by Herold,[19] who used the solutions of the Dirac equation with a homogeneous magnetic field to calculate Compton and Thomson scattering in the Born approximation, considering the laser field as an external field.

Besides this condition on B, one also must impose an upper limit on the amplitude of the laser field. In order to do that, one must study with some detail the motion of the electron, which will be done in Sec. III B.

The form factor $g(\vec{k})$ should also be compatible with the approximations made so far. It should not only cut off the contribution of relativistic wave vectors in expressions like (2.11), but also of wave vectors such that $kr_L \gtrsim 1$. We shall see however that, after mass renormalization, the solution will be quite insensitive with respect to the cutoff procedure.

III. THE ELECTRIC FIELD

A. Solution of the Heisenberg equations of motion

From the Hamiltonian defined by Eq. (2.1) we derive the following Heisenberg equations of motion:

$$\dot{\vec{r}}(t)=\frac{1}{m}\left[\vec{p}(t)-\frac{e}{c}\vec{A}(\vec{r},t)\right] , \quad (3.1)$$

$$\dot{\vec{p}}(t)=\left[\vec{p}(t)-\frac{e}{c}\vec{A}(\vec{r},t)\right]\times\frac{e\vec{B}}{2mc}$$
$$=\frac{e}{2c}\dot{\vec{r}}\times\vec{B} , \quad (3.2)$$

$$\dot{a}_\lambda(\vec{k},t)=-i\omega a_\lambda(\vec{k},t)+\frac{2\pi i e}{m}\frac{1}{(h\omega)^{1/2}}g(\vec{k})\times\left[\vec{p}(t)-\frac{e}{c}\vec{A}(\vec{r},t)\right]\cdot\vec{\epsilon}_\lambda^{\,*}(\vec{k}) . \quad (3.3)$$

Because of the assumption that $\vec{B}$ is independent of $\vec{r}$, the spin term does not contribute to these equations.

Since the above equations are linear, they can be solved by the Laplace transform method. We take for simplicity

$$g(\vec{k})=\beta/(\beta^2+k^2)^{1/2} . \quad (3.4)$$

When $\beta\to\infty$, one has $g(\vec{k})\to 1$, which is the point-electron limit.

Let $x(t)$, $y(t)$, and $z(t)$ be the Cartesian components of $\vec{r}(t)$, and let $\tilde{\vec{r}}(s)$ be the Laplace transform of $\vec{r}(t)$:

$$\tilde{\vec{r}}(s)=\int_0^\infty \vec{r}(t)e^{-st}dt\ . \tag{3.5}$$

The components $\tilde{x}(s)$, $\tilde{y}(s)$, and $\tilde{z}(s)$ of $\tilde{\vec{r}}(s)$ are given by

$$\tilde{x}(s)+i\tilde{y}(s)=\vec{O}\cdot(\hat{x}+i\hat{y})\frac{s+\beta c}{Q(s)}\ , \tag{3.6}$$

$$\tilde{z}(s)=\vec{O}\cdot\hat{z}\frac{s+\beta c}{s[s+\beta c(1+\gamma)]}\ , \tag{3.7}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors corresponding to the directions x, y, and z,

$$Q(s)=s^2+[\beta c(1+\gamma)-i\omega_c]s-i\omega_c\beta c\ , \tag{3.8}$$

$$\vec{O}=\vec{r}(0)\frac{s+\beta c(1+\gamma)}{s+\beta c}-\frac{e}{2mc}\frac{\vec{r}(0)\times\vec{B}}{s}+\frac{\vec{p}(0)}{ms}+\frac{\hbar\alpha^{1/2}}{2\pi m}\sum_\lambda\int\frac{d^3k}{\sqrt{k}}g(k)\left[\frac{a_\lambda(\vec{k})\vec{\epsilon}_\lambda(\vec{k})}{s+ikc}+\frac{a_\lambda^\dagger(\vec{k})\vec{\epsilon}_\lambda^{\,*}(\vec{k})}{s-ikc}\right]\ , \tag{3.9}$$

$$\gamma=\frac{2e^2\beta}{3mc^2}\ , \tag{3.10}$$

$$\omega_c=\frac{|e|B}{mc}\ , \tag{3.11}$$

and α is the fine-structure constant.

The vector $\vec{r}(t)$ is given by the inverse Laplace transform of $\tilde{\vec{r}}(s)$:

$$\vec{r}(t)=\frac{1}{2\pi i}\int_\Gamma ds\,e^{st}\tilde{\vec{r}}(s)\ , \tag{3.12}$$

where the contour Γ is parallel to the imaginary axis, and to the right of all the singularities of $\tilde{\vec{r}}(s)$.

One gets then (in diadic notation):

$$\vec{r}(t)=\frac{1}{2\pi i}\int_\Gamma ds\,e^{st}\vec{O}\cdot\left[\hat{\epsilon}_-\epsilon_+\frac{s+\beta c}{(s-s_0^*)(s-s_1^*)}+\hat{\epsilon}_+\hat{\epsilon}_-\frac{s+\beta c}{(s-s_0)(s-s_1)}+\hat{z}\hat{z}\frac{s+\beta c}{s[s+\beta c(1+\gamma)]}\right]\ , \tag{3.13}$$

where

$$\hat{\epsilon}_+=\hat{\epsilon}_-^{\,*}=(\hat{x}+i\hat{y})/\sqrt{2}\ , \tag{3.14}$$

and s_0 and s_1 are the roots of $Q(s)$. One also gets

$$\vec{p}(t)=\frac{1}{2\pi i}\int_\Gamma ds\,e^{st}\left[\frac{\vec{p}(0)}{s}+\frac{e}{2c}[s\tilde{\vec{r}}(s)-\vec{r}(0)]\times\vec{B}\right]\ , \tag{3.15}$$

$$a_\lambda(\vec{k},t)=a_\lambda(\vec{k})e^{-i\omega t}-\frac{\alpha^{1/2}}{4\pi^2\sqrt{k}}g(k)\vec{\epsilon}_\lambda^{\,*}(\vec{k})\cdot\int_\Gamma ds\,e^{st}\frac{s\tilde{\vec{r}}(s)-\vec{r}(0)}{s+ikc}\ . \tag{3.16}$$

B. Electron operators in the point-electron limit

Up to now, we have considered the electron as having a finite size. If we now set $\beta\to\infty$, we get a point electron. However, in order to proceed to this limit, one must distinguish between the mass m in the Hamiltonian (2.1) and the measurable physical mass of the electron m_R. This distinction arises naturally when one diagonalizes the Hamiltonian corresponding to a free electron interacting with the quantized electromagnetic field, which can be done exactly, after making the nonrelativistic and electric dipole approximations. In this case, the Hamiltonian is obtained from the one given by Eq. (2.1) by setting $\vec{A}_M=0$. The diagonalization of the resulting Hamiltonian is done in Appendix A, following a procedure by Van Kampen.[18] The following spectrum is obtained:

$$E=\frac{\vec{p}^{\,2}}{2m(1+\gamma)}+\sum_{M=-1}^{+1}\int_0^\infty dk\,\hbar\omega N_M(k)\ , \tag{3.17}$$

where $N_M(k)\geq 0$. The first term on the right-hand side of Eq. (3.17) can be interpreted as the kinetic energy of the electron and the second term as the sum of the energies of the photons present in the several modes of the field. The mass which shows up in the kinetic energy term is not the original mass m, but a new value $m(1+\gamma)$, which should be identified with the physical mass m_R. We have, therefore,

$$m_R = m(1+\gamma) . \tag{3.18}$$

Since the physical mass m_R should be finite, we require that when $\beta \to \infty$, $m \to -\infty$. Our results will all be expressed in terms of the renormalized mass m_R.

In particular, after replacing m by its expression in terms of m_R, one gets (see Appendix B)

$$\lim_{\beta\to\infty} \beta c(1+\gamma) = -\frac{3m_R c^3}{2e^2} , \tag{3.19}$$

$$\lim_{\beta\to\infty} s_0 = +i\omega_c'[1+O(a^2\alpha^2)] - \Gamma , \tag{3.20}$$

$$\lim_{\beta\to\infty} s_1 = \frac{3m_R c^3}{2e^2}[1+O(a\alpha)] , \tag{3.21}$$

where ω_c' is the physical cyclotron frequency, that is,

$$\omega_c' = \frac{|e|B}{m_R c} , \tag{3.22}$$

$$a = \hbar\omega_c'/m_R c^2 = B/B_{\text{cr}} , \tag{3.23}$$

and

$$\Gamma = \frac{2e^4 B^2}{3m_R^3 c^5}[1+O(a^2\alpha^2)] \tag{3.24}$$

will be identified later on with the spectral linewidth. For $B \ll B_{\text{cr}}$, one has $a \ll 1$.

Using these results, and neglecting terms of $O(a\alpha)$, one gets

$$\lim_{\beta\to\infty} \vec{r}(t) = \int_\Gamma \frac{ds}{2\pi i} e^{st} \left\{ \vec{r}(0) + \lim_{\beta\to\infty} \frac{1}{m} \frac{s+\beta c}{s - 3m_R c^3/2e^2} \times \left[\frac{\vec{p}(0) - (e/2c)\vec{r}(0)\times\vec{B}}{s} + \frac{\hbar\alpha^{1/2}}{2\pi m_R} \sum_\lambda \int \frac{d^3k}{\sqrt{k}} \left[\frac{a_\lambda(\vec{k})\vec{\epsilon}_\lambda(\vec{k})}{s+ikc} + \frac{a_\lambda^\dagger(\vec{k})\vec{\epsilon}_\lambda^{\,*}(\vec{k})}{s-ikc} \right] \right] \right\} \cdot \left[\frac{\hat{\epsilon}_+\hat{\epsilon}_-}{s-s_0} + \frac{\hat{\epsilon}_-\hat{\epsilon}_+}{s-s_0^*} + \frac{\hat{z}\hat{z}}{s} \right] . \tag{3.25}$$

The integrand of Eq. (3.25) has poles at $s = s_0$, s_0^*, $\pm i\omega$, O, and $3m_R c^3/2e^2$. This last one is much larger than the moduli of the others, since $3m_R c^3/2e^2 \simeq 10^{23}$ s^{-1}. Its contribution to $\vec{r}(t)$ is of the form $\exp(3m_R c^3 t/2e^2)$, giving rise therefore to a "runaway solution," which is always eliminated in the literature, in order to get physical results.[18] In Appendix B we discuss in detail the origin of this term and justify its elimination.

For the contributions of the other poles, since $|s| \ll 3m_R c^3/2e^2, \beta c$, one can write

$$\frac{1}{m}\frac{s+\beta c}{s - 3m_R c^3/2e^2} \simeq \frac{1}{m_R} , \tag{3.26}$$

and therefore one gets, in the limit $\beta \to \infty$,

$$\vec{r}(t) = \vec{r}(0)\cdot(\hat{\epsilon}_+\hat{\epsilon}_- e^{s_0 t} + \hat{\epsilon}_-\hat{\epsilon}_+ e^{s_0^* t} + \hat{z}\hat{z}) + \tilde{\vec{v}}(0)\cdot \left[\frac{e^{s_0 t}-1}{s_0}\hat{\epsilon}_+\hat{\epsilon}_- + \frac{e^{s_0^* t}-1}{s_0^*}\hat{\epsilon}_-\hat{\epsilon}_+ + t\hat{z}\hat{z} \right] + \frac{\hbar\alpha^{1/2}}{2\pi m_R} \sum_\lambda \int \frac{d^3k}{\sqrt{k}} \left[a_\lambda(\vec{k})\vec{\epsilon}_\lambda(\vec{k})\cdot \left[\frac{\hat{\epsilon}_+\hat{\epsilon}_-}{s_0+ikc}(e^{s_0 t} - e^{-ikct}) + \frac{\hat{\epsilon}_-\hat{\epsilon}_+}{s_0^*+ikc}(e^{s_0^* t} - e^{-ikct}) + \frac{\hat{z}\hat{z}}{ikc}(1-e^{-ikct}) \right] + \text{H.c.} \right] , \tag{3.27}$$

where we have set

$$\tilde{\vec{v}}(0)=\frac{1}{m_R}\left[\vec{p}(0)-\frac{e}{2c}\vec{r}(0)\times\vec{B}\right]. \quad (3.28)$$

The operator $\tilde{\vec{v}}(0)$ does not coincide with the initial velocity $\vec{v}(0)$ of the electron, since the quantized electromagnetic field is not present in (3.28). One has, in fact, the relation

$$\vec{v}(0)=\tilde{\vec{v}}(0)-\frac{e\vec{A}_F(0)}{m_R c}, \quad (3.29)$$

which is easily seen to be satisfied by $d\vec{r}/dt$ obtained from (3.27).

From Eqs. (3.20) and (3.27), we can see that $1/\Gamma$ is the lifetime of the system. When $t \gg 1/\Gamma$, the terms in Eq. (3.27) which do not depend on the electromagnetic field operators become

$$\vec{r}=[z(0)+\tilde{v}_z(0)t]\hat{z}-\frac{1}{\omega_c'^2+\Gamma^2}[\tilde{\vec{v}}(0)\times\omega_c'\hat{z}-\Gamma\tilde{\vec{v}}_\perp(0)], \quad (3.30)$$

where $\tilde{\vec{v}}_\perp=\tilde{v}_x\hat{x}+\tilde{v}_y\hat{y}$. For $\Gamma\to 0$, one gets precisely the classical expression for the motion of an electron with the physical mass m_R in a static and homogeneous magnetic field. On the other hand, for $B=0$, one gets from (3.27), as expected,

$$\vec{r}(t)=\vec{r}(0)+\tilde{\vec{v}}(0)t-\frac{e}{m_R c}\int_0^t \vec{A}_F^0(\vec{0},t')dt', \quad (3.31)$$

where $\vec{A}_F^0(\vec{r},t)$ is obtained from Eq. (2.6) by multipling each $a_\lambda(\vec{k})$ by $e^{-i\omega t}$, corresponding to the free evolution of the electromagnetic field.

The terms in Eq. (3.27) which depend on the electromagnetic field operators represent the forced motion of the electron under the action of the laser and the vacuum field. Although the average of these terms in the vacuum state is zero, they contribute to the fluctuations in the position of the electron, even when the field is in the vacuum state. These fluctuations turn out to be logarithmically divergent in the point-electron limit, even after mass renormalization. The spectrum will not exhibit this divergence, however, because it will be defined in terms of normally ordered operators.

If the electromagnetic field is initially in a coherent state $|v_{\vec{k}_0,\lambda_0}\rangle$, with $v_{\vec{k}_0,\lambda_0}=|v_{\vec{k}_0,\lambda_0}|e^{i\phi}$, one may introduce the amplitude $E_{\rm cl}$ of the corresponding classical field through the relation (assuming the light is linearly polarized)

$$a_\lambda(\vec{k})|v_{\vec{k}_0,\lambda_0}\rangle=-\frac{\pi E_{\rm cl}e^{i\phi}}{(\hbar\omega_0)^{1/2}}\delta(\vec{k}-\vec{k}_0)\delta_{\lambda\lambda_0}\times|v_{\vec{k}_0,\lambda_0}\rangle \quad (3.32)$$

with $\omega_0=k_0c$.

This relation is easily derived by requiring that

$$\langle v_{\vec{k}_0,\lambda_0}|\vec{E}(\vec{r})|v_{\vec{k}_0,\lambda_0}\rangle=\vec{\epsilon}_{\vec{k}_0\lambda_0}E_{\rm cl}\times\sin(\vec{k}\cdot\vec{r}+\phi), \quad (3.33)$$

with

$$\vec{E}(\vec{r})=-\frac{1}{c}\left.\frac{\partial\vec{A}_F^0(\vec{r},t)}{\partial t}\right|_{t=0}. \quad (3.34)$$

One gets then, for the field-dependent part $\vec{r}_F(t)$ of $\vec{r}(t)$, using Eq. (3.27):

$$\langle v_{\vec{k}_0,\lambda_0}|\vec{r}_F(t)|v_{\vec{k}_0,\lambda_0}\rangle=\frac{eE_{\rm cl}}{m_R\omega_0}\vec{\epsilon}_{\vec{k}_0\lambda_0}\cdot{\rm Im}\left[\frac{\hat{\epsilon}_+\hat{\epsilon}_-}{\omega_0-\omega_c'+i\Gamma}(e^{-i\omega_c't-\Gamma t}-e^{-i\omega_0t})+\frac{\hat{\epsilon}_-\hat{\epsilon}_+}{\omega_0+\omega_c'+i\Gamma}(e^{i\omega_c't-\Gamma t}-e^{-i\omega_0t})+\frac{\hat{z}\hat{z}}{\omega_0}(1-e^{-i\omega_0t})\right]. \quad (3.35)$$

For $B=0$, one gets

$$\langle v_{\vec{k}_0,\lambda_0}|\vec{r}_F(t)|v_{\vec{k}_0,\lambda_0}\rangle=\frac{eE_{\rm cl}}{m_R\omega_0^2}\vec{\epsilon}_{\vec{k}_0\lambda_0}\sin(\omega_0t), \quad (3.36)$$

which is just the classical expression for the motion of an electron in an oscillating field $E_{\rm cl}\vec{\epsilon}_{\vec{k}_0\lambda_0}\sin(\omega_0t)$, with the initial conditions $\vec{r}_F(0)=0$ and $\dot{\vec{r}}_F(0)=eE_{\rm cl}/m_R\omega_0$. From (3.27) and (3.29) we see that this value of $\dot{\vec{r}}_F(0)$ yields $\dot{\vec{r}}(0)=\vec{v}(0)$, as it should be.

For $B\neq 0$, expression (3.35) exhibits a resonant term (as well as an antiresonant one), centered around $\omega_0=\omega_c'$ and with width Γ. From this expression, one can extract a condition on the laser field amplitude, so that the nonrelativistic and electric dipole approximations hold. In the most critical situation, when $\omega_0=\omega_c'$, one has, except for oscillating factors

$$\left|\left\langle\frac{d\vec{r}_F}{dt}\right\rangle\right|\simeq\frac{|e|E_{\rm cl}}{m_R\Gamma}=\frac{\omega_c'}{\Gamma}\frac{E_{\rm cl}}{B}c\ , \tag{3.37}$$

so that, in order that $|\langle d\vec{r}_F/dt\rangle|\ll c$, one must have

$$E_{\rm cl}\ll\frac{\Gamma}{\omega_0}B\ . \tag{3.38}$$

This is also the condition for the electric dipole approximation to hold at resonance. For $\omega_0\neq\omega_c'$, the condition on $E_{\rm cl}$ becomes less severe.

If E_0 is expressed in V/cm and B in G, the above relation becomes

$$E_{\rm cl}\ll 3\times10^6\frac{\Gamma}{\omega_c'}B\simeq3\times10^{-10}B^2\ , \tag{3.39}$$

so that, for $B=10^{12}$ G, one should have $E_0\ll3\times10^{14}$ V/cm. For smaller magnetic fields, the resonance becomes sharper, and the maximum allowed laser amplitude is consequently reduced.

C. Electric field in the point-electron limit

The vector potential at an arbitrary position $\vec{R}$ is given by

$$\vec{A}(\vec{R},t)=\sum_\lambda\int\frac{d^3k}{(2\pi)^{3/2}}\left[\frac{hc^2}{\omega}\right]^{1/2}\times[a_\lambda(\vec{k},t)\vec{\epsilon}_\lambda(\vec{k})e^{i\vec{k}\cdot\vec{R}}+\text{H.c.}]\ . \tag{3.40}$$

Replacing $a_\lambda(\vec{k},t)$ by the expression in Eq. (3.16), one gets

$$\vec{A}(\vec{R},t)=\sum_\lambda\int\frac{d^3k}{(2\pi)^{3/2}}\left[\frac{hc^2}{\omega}\right]^{1/2}\times[a_\lambda(\vec{k})\vec{\epsilon}_\lambda(\vec{k})e^{i(\vec{k}\cdot\vec{R}-\omega t)}+\text{H.c.}]+\vec{A}_S(\vec{R},T)\ . \tag{3.41}$$

The first term on the right-hand side of the above equation stands for the incoming free field, while $\vec{A}_S(\vec{R},t)$ represents the scattered part.

Using that

$$\sum_\lambda\vec{\epsilon}_\lambda(\vec{k})\vec{\epsilon}^{\,*}_\lambda(\vec{k})=1-\hat{k}\hat{k} \tag{3.42}$$

and that

$$\int d\Omega_{\vec{k}}(\overleftrightarrow{1}-\hat{k}\hat{k})e^{i\vec{k}\cdot\vec{R}}=4\pi\left[(\overleftrightarrow{1}-\hat{R}\hat{R})\frac{\sin(kR)}{kR}+(\overleftrightarrow{1}-3\hat{R}\hat{R})\left[\frac{\cos(kR)}{k^2R^2}-\frac{\sin(kR)}{k^3R^3}\right]\right]\ , \tag{3.43}$$

where $\hat{k}=\vec{k}/|\vec{k}|$, $\hat{R}=\vec{R}/|\vec{R}|$, $R=|\vec{R}|$, and $k=|\vec{k}|$, one has, after integrating over $\vec{k}$,

$$\vec{A}_S(\vec{R},t)=\frac{e}{Rc}\int\frac{ds}{2\pi i}[s\tilde{\vec{r}}(s)-\vec{r}(0)]\cdot\left[(\overleftrightarrow{1}-\hat{R}\hat{R})e^{s(t-R/c)}+c^2(\overleftrightarrow{1}-3\hat{R}\hat{R})\left[\frac{e^{s(t-R/c)}}{cRs}-\frac{e^{st}-e^{s(t-R/c)}}{R^2s^2}\right]\right]\ . \tag{3.44}$$

In terms of $\vec{r}(t)$, one gets, from this expression,

$$\vec{A}_S(\vec{R},t)=\frac{e}{Rc}(\overleftrightarrow{1}-\hat{R}\hat{R})\cdot\dot{\vec{r}}\left[t-\frac{R}{c}\right]+\frac{e}{R^2}(\overleftrightarrow{1}-3\hat{R}\hat{R})\cdot\left[\vec{r}\left[t-\frac{R}{c}\right]-\frac{c}{R}\int_{t-R/c}^{t}\vec{r}(t')dt'\right]\ . \tag{3.45}$$

The scattered electric field is given by

$$\vec{E}_S(\vec{R},t)=-\vec{\nabla}\phi(\vec{R},t)-\frac{1}{c}\frac{\partial\vec{A}_S(\vec{R},t)}{\partial t} . \tag{3.46}$$

Since we have adopted the Coulomb gauge, we must use, for $\phi(\vec{R},t)$, the instantaneous Coulomb potential

$$\phi(\vec{R},t)=\frac{e}{|\vec{r}(t)-\vec{R}|} . \tag{3.47}$$

Furthermore, in order to be consistent with the electric dipole approximation, one should expand $-\vec{\nabla}\phi(\vec{R},t)$ in multipoles, keeping only the monopole and dipole terms:

$$-\vec{\nabla}\phi(\vec{R},t)\simeq\frac{e\hat{R}}{R^2}+\frac{(3\hat{R}\hat{R}-\overleftrightarrow{1})}{R^3}\cdot\vec{d}(t) , \tag{3.48}$$

where $\vec{d}(t)$ is the electric dipole moment at time t:

$$\vec{d}(t)=e\vec{r}(t) . \tag{3.49}$$

We have then

$$\vec{E}_S(\vec{R},t)=-\frac{\ddot{\vec{d}}(t-R/c)}{Rc^2}\cdot(\overleftrightarrow{1}-\hat{R}\hat{R})+\left[\frac{\dot{\vec{d}}(t-R/c)}{R^2c}+\frac{\vec{d}(t-R/c)}{R^3}\right]\cdot(3\hat{R}\hat{R}-\overleftrightarrow{1})+\frac{e\hat{R}}{R^2} . \tag{3.50}$$

This equation should be compared with the classical expression for the electric field produced by an oscillating dipole.[20] Except, of course, for the monopole term, the two expressions coincide. This is not surprising, since this result was obtained from Eqs. (3.1) and (3.3), which are linear; therefore, the noncommutability of the operators involved does not have any consequence in getting Eq. (3.50). One should also notice that this result does not depend on the potential acting on the electron, since this information appears only in Eq. (3.2). As in the classical case, the noncausal contributions coming from $\vec{A}(\vec{R},t)$ and $\phi(\vec{r},t)$ exactly cancel.

The explicit time dependence of $\vec{E}_S(\vec{R},t)$ in the point-electron limit is obtained by replacing $\vec{r}(t)$ in Eq. (3.50) by expression (3.27). For simplicity, we assume that the detector is sufficiently far from the source so that only the radiation part of $\vec{E}_S(\vec{R},t)$ is relevant:

$$\vec{E}_{\rm rad}(\vec{R},t)=-\frac{e}{Rc^2}\ddot{\vec{r}}\left(t-\frac{R}{c}\right)\cdot(\overleftrightarrow{1}-\hat{R}\hat{R}) . \tag{3.51}$$

We get, finally,

$$\begin{aligned}\vec{E}_{\rm rad}(\vec{R},t)=-\frac{e}{Rc^2}(\overleftrightarrow{1}-\hat{R}\hat{R})\cdot\Bigg\{&\vec{r}(0)\cdot[s_0^2\hat{\epsilon}_+\hat{\epsilon}_-e^{s_0(t-R/c)}+(s_0^*)^2\hat{\epsilon}_-\hat{\epsilon}_+e^{s_0^*(t-R/c)}]\\&+\vec{v}(0)\cdot(s_0\hat{\epsilon}_+\hat{\epsilon}_-e^{s_0(t-R/c)}+s_0^*\hat{\epsilon}_-\hat{\epsilon}_+e^{s_0^*(t-R/c)})\\&+\frac{\hbar\alpha^{1/2}}{2\pi m_R}\sum_\lambda\int\frac{d^3k}{\sqrt{k}}\Bigg[a_\lambda(\vec{k})\vec{\epsilon}_\lambda(\vec{k})\\&\cdot\Bigg[\frac{\hat{\epsilon}_+\hat{\epsilon}_-}{s_0+i\omega}(s_0^2e^{s_0(t-R/c)}+\omega^2e^{-i\omega(t-R/c)})\\&+\frac{\hat{\epsilon}_-\hat{\epsilon}_+}{s_0^*+i\omega}[(s_0^*)^2e^{s_0^*(t-R/c)}+\omega^2e^{-i\omega(t-R/c)}]\\&-i\omega\hat{z}\hat{z}e^{-i\omega(t-R/c)}\Bigg]+\text{H.c.}\Bigg]\Bigg\} .\end{aligned} \tag{3.52}$$

This expression greatly simplifies when $t\gg 1/\Gamma$. In this limit, one gets

$$\vec{E}_{\rm rad}(\vec{R},t)=\frac{e^2\hbar^{1/2}}{2\pi m_R c^{1/2}R}(\overleftrightarrow{1}-\hat{R}\hat{R})\cdot\sum_\lambda\int d^3k\,k^{3/2}\left[a_\lambda(\vec{k})\hat{\epsilon}_\lambda(\vec{k})e^{-ikc(t-R/c)}\cdot\left[\frac{\hat{\epsilon}_+\hat{\epsilon}_-}{s_0+ikc}+\frac{\hat{\epsilon}_-\epsilon_+}{s_0^*+ikc}+\frac{\hat{z}\hat{z}}{ikc}\right]+\text{H.c.}\right]. \tag{3.53}$$

These results will now be used to calculate the power spectrum of the scattered radiation.

IV. THE POWER SPECTRUM

As usual, we define the power spectrum in terms of the counting rate $w(t)$ of a photoelectric detector.[21] For a broad-band detector, one has[21]

$$w(t)=\text{Re}\sum_{\mu,\nu}s_{\nu\mu}G^{(1)}_{\mu\nu}(\vec{R},t;\vec{R},t)\ , \tag{4.1}$$

where the coefficients $s_{\nu\mu}$ depend on the characteristics of the detector and $G^{(1)}_{\mu\nu}(\vec{R},t;\vec{R}\,',t')$ is the correlation tensor for the electromagnetic field:

$$G^{(1)}_{\mu\nu}(\vec{R},t;\vec{R}\,',t')=\text{Tr}[\rho E^{(-)}_\mu(\vec{R},t)E^{(+)}_\nu(\vec{R}\,',t')]\ . \tag{4.2}$$

In Eq. (4.2), ρ is the density matrix for the electromagnetic field, and $E^{(+)}$ and $E^{(-)}$ stand for the positive and negative frequency part of E, respectively.

From (3.52), (4.1), and (4.2), one can explicitly calculate the time-dependent spectrum of the radiation, for any given ρ, by convoluting the electric field with an appropriate filter function.[22] For brevity, however, we assume that $t\gg 1/\Gamma$, so that the transient terms become unimportant. We also assume that the field is initially in a coherent state, so that $\rho=|v_{\vec{k}_0,\lambda_0}\rangle\langle v_{\vec{k}_0,\lambda_0}|$. For definiteness, we take it to be linearly polarized. The circular polarization case can be treated along similar lines.

Under these conditions, one gets, using Eq. (3.32),

$$G^{(1)}_{\mu\nu}(\vec{R},t';\vec{R},t)=\left[\frac{e^2\omega_0}{2m_Rc^2R}\right]^2E^2_{\rm cl}\hat{x}_\mu\cdot(\overleftrightarrow{1}-\hat{R}\hat{R})\cdot\left[\frac{\hat{\epsilon}_+\hat{\epsilon}_-}{s_0^*-i\omega_0}+\frac{\hat{\epsilon}_-\hat{\epsilon}_+}{s_0-i\omega_0}+\frac{\hat{z}\hat{z}}{-i\omega_0}\right]\cdot\vec{\epsilon}^{\,*}_{\vec{k}_0\lambda_0}\hat{x}_\nu\cdot(\overleftrightarrow{1}-\hat{R}\hat{R})\cdot\left[\frac{\hat{\epsilon}_-\hat{\epsilon}_+}{s_0+i\omega_0}+\frac{\hat{\epsilon}_+\hat{\epsilon}_-}{s_0^*+i\omega_0}+\frac{\hat{z}\hat{z}}{i\omega_0}\right]\cdot\vec{\epsilon}_{\vec{k}_0\lambda_0}e^{i\omega_0(t'-t)}\ . \tag{4.3}$$

The correlation tensor is seen to depend on time only through the difference $t'-t$, which is characteristic of a stationary process. Besides that, one sees that only one Fourier component is present in the above expression, with frequency ω_0 equal to the frequency of the incident field, which is characteristic of an elastic process.

In order to parametrize the polarization vectors which appear in Eq. (4.3), we choose the vectors $\hat{x}_\mu$ to coincide with the vectors $\hat{\epsilon}_1$, $\hat{\epsilon}_2$, and $\hat{R}$, where $\hat{R}$ is the unit vector corresponding to $\vec{R}$, $\hat{\epsilon}_2$ is a unit vector orthogonal to $\hat{R}$ and contained in the plane determined by the vectors $\vec{R}$ and $\vec{B}$, and $\hat{\epsilon}_1$ is such that $\hat{\epsilon}_1\times\hat{\epsilon}_2=\hat{R}$ (see Fig. 1). Since $\vec{k}_0$ was assumed from the outset to be orthogonal to $\vec{B}$, one can choose the x axis parallel to $\vec{k}_0$ and the z axis parallel to $\vec{B}$, without any further lack of generality. We denote by θ the angle between $\vec{\epsilon}_{\vec{k}_0\lambda_0}$ and the y axis.

Our results will be expressed in terms of the fol-

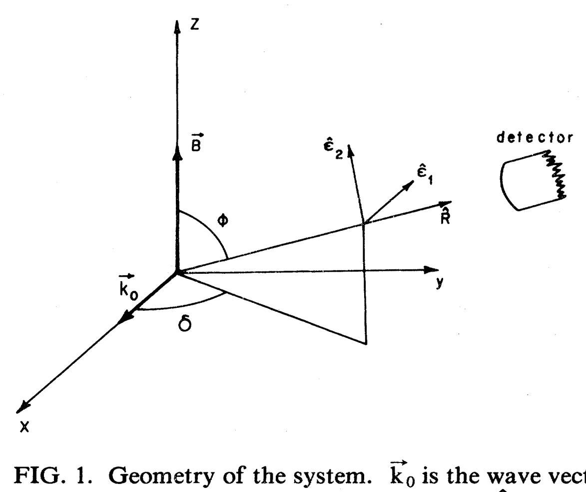

FIG. 1. Geometry of the system. $\vec{k}_0$ is the wave vector of the incident field, and $\hat{\epsilon}_2$ is orthogonal to $\hat{R}$ and contained in the plane determined by $\hat{R}$ and $\vec{B}$, while $\hat{\epsilon}_1=\hat{\epsilon}_2\times\hat{R}$.

lowing functions:

$$I_1(\omega)=\frac{\omega^2}{\Gamma^2+(\omega-\omega_c')^2}+\frac{\omega^2}{\Gamma^2+(\omega+\omega_c')^2} \tag{4.4}$$

and

$$I_2(\omega)=\frac{\omega(\omega_c'-\omega)}{\Gamma^2+(\omega-\omega_c')^2}-\frac{\omega(\omega+\omega_c')}{\Gamma^2+(\omega+\omega_c')^2} . \tag{4.5}$$

We shall consider three different cases. In the first two, we assume there is a polarizer in front of the detector, with polarization direction along $\hat{\epsilon}_1$ or $\hat{\epsilon}_2$. In the third case, we assume there is no polarizer and the detector is isotropic, that is, $s_{12}=s_{21}=0$ and $s_{11}=s_{22}$ in Eq. (4.1).

If the polarizer is oriented along $\hat{\epsilon}_1$, one gets, in terms of the angles θ, δ, and ϕ (see Fig. 1)

$$w_1(t)=s_{11}\left[\frac{e^2}{4m_Rc^2R}\right]^2 E_{\rm cl}^2\cos^2\theta \times\left[I_1(\omega_0)\left[1+\frac{\Gamma}{\omega_c'}\sin 2\delta\right] - I_2(\omega_0)\left[\cos 2\delta-\frac{\Gamma}{\omega_c'}\sin 2\delta\right]\right] . \tag{4.6}$$

For a polarizer oriented along $\hat{\epsilon}_2$, one has

$$w_2(t)=s_{22}\left[\frac{e^2}{4m_Rc^2R}\right]^2 E_{\rm cl}^2\left\{\cos^2\theta\cos^2\phi\left[I_1(\omega_0)\left[1-\frac{\Gamma}{\omega_c'}\sin 2\delta\right]+I_2(\omega_0)\left[\cos 2\delta-\frac{\Gamma}{\omega_c'}\sin 2\delta\right]\right] + \sin 2\theta\sin 2\phi\left[I_1(\omega_0)\frac{\Gamma}{\omega_c'}\cos\delta+I_2(\omega_0)\left[\sin\delta-\frac{\Gamma}{\omega_c'}\cos\delta\right]\right]+4\sin^2\phi\sin^2\theta\right\} . \tag{4.7}$$

If no polarizer is present, and if the detector is isotropic, $w(t)$ will be given by the sum of the two expressions above, with $s_{11}=s_{22}$:

$$w(t)=s_{11}{\rm Tr}G^{(1)}=w_1(t)+w_2(t) . \tag{4.8}$$

For $B\to 0$ or $\omega_0\gg\omega_c'$, one gets from (4.6)—(4.8)

$$w(t)=\frac{s_{11}}{4}\left[\frac{e^2}{m_Rc^2R}\right]^2 E_{\rm cl}^2\sin^2\xi , \tag{4.9}$$

where ξ is the angle between $\vec{\epsilon}_{\vec{k}_0\lambda_0}$ and $\hat{R}$. This expression is proportional to the classical expression for the time-averaged power per unit solid angle irradiated by an electron in the presence of a plane electromagnetic wave, in the electric dipole approximation[20]:

$$\left\langle\frac{dP}{d\Omega}\right\rangle=\left[\frac{e^2}{m_Rc^2}\right]^2\frac{cE_{\rm cl}^2}{8\pi}\sin^2\xi . \tag{4.10}$$

One should also notice that in Eq. (4.7) there is a term independent of the frequency ω_0. Its contribution to $w_2(t)$ is equal to

$$\frac{s_{22}}{4}\left[\frac{e^2}{m_Rc^2R}\right]^2 E_{\rm cl}^2\sin^2\phi\sin^2\theta .$$

Comparison of this expression with (4.9) and (4.10) makes it clear that it originates from the oscillations of the electron along the z direction, for which the magnetic field plays no role. In this case, the field amplitude to be used in (4.9) should be $E_{\rm cl}\sin\theta$, the z component of the applied electric field, and ξ should be replaced by ϕ, the angle between the detector and the z axis.

Expressions (4.6) and (4.7) exhibit a resonance for $\omega_0=\omega_c'$ with linewidth Γ. Close to resonance, when $|\omega_0-\omega_c'|\lesssim\Gamma$, the dominant contribution is the one proportional to $I_1(\omega_0)$, which has approximately a Lorentzian shape.

The functions $I_1(\omega)$ and $I_2(\omega)$ are plotted in Figs. 2 and 3, for magnetic fields ranging from 10^3 to 10^{12} G, close to resonance. By choosing convenient units, one can make all graphs for $I_1(\omega)$ to coincide, within graphical precision, for all magnetic fields considered, and so long as one is close to resonance. The same is valid for $I_2(\omega)$. In Table I, we display some typical values of B and the corresponding values of ω_c', Γ and Γ/ω_c'.

V. CONCLUSION

Even though the model here considered is a very simplified one, it still has some relation with physical reality and it allows a detailed analysis of the resonant scattering process. The model is also interesting in that it allows an exact renormalization procedure, after which the calculated spectrum remains finite, in the point-electron limit. In partic-

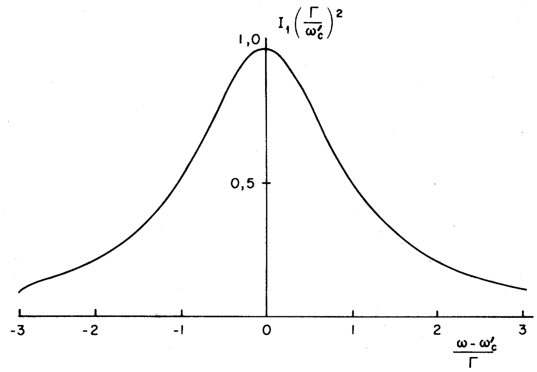

FIG. 2. Plot of $I_1(\Gamma/\omega_c')^2$ in the resonance region, as a function of $(\omega-\omega_c')/\Gamma$, for magnetic fields from 10^3 to 10^{12} G. Within graphical precision, all plots coincide.

ular, we have explicitly calculated the electron and the field operators in the Heisenberg picture, and found the dependence of the linewidth and the energy shift on the magnetic field B. Both the linewidth and the line shift go to a finite value when the cutoff goes to infinity, after mass renormalization. Our analysis of the spectrum has been limited to the case of a coherent field, and a stationary situation. Consideration of other states of the field or of the transient regime does not present any essential difficulty in the framework of the present model.

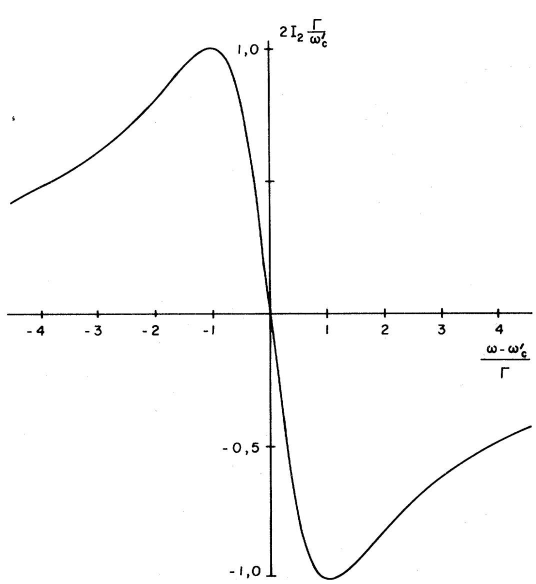

FIG. 3. Plot of $2I_2\Gamma/\omega_c'$ in the resonance region, as a function of $(\omega-\omega_c')/\Gamma$, for magnetic fields from 10^3 to 10^{12} G. Within graphical precision, all plots coincide.

TABLE I. Some typical values of B and the corresponding values of ω_c', Γ, and Γ/ω_c'. Both ω_c' and Γ are expressed in inverse seconds. $\omega_c' = |e|B/m_R c$ and $\Gamma = 2e^4B^2/3m_R^3c^5$.

B (G)	ω_c' (s^{-1})	Γ (s^{-1})	Γ/ω_c'
10^3	1.76×10^{10}	1.94×10^{-3}	1.10×10^{-13}
10^7	1.76×10^{14}	1.94×10^{5}	1.10×10^{-9}
10^{10}	1.76×10^{17}	1.94×10^{11}	1.10×10^{-6}
10^{12}	1.76×10^{19}	1.94×10^{15}	1.10×10^{-4}

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APPENDIX A

In this appendix we diagonalize the free-electron Hamiltonian, following a procedure similar to the one adopted by Van Kampen,[18] except for the fact that we treat here the infinite-volume case. The expression "free electron" should, of course, be understood as referring to the electron interacting with the vacuum field, which is always present.

The corresponding Hamiltonian is obtained from (2.1) by setting $\vec{B}=\vec{0}$, that is (we set, in this appendix, $\hbar=c=1$),

$$H=[\vec{p}-e\vec{A}_F^R(\vec{0})]^2/2m + \sum_\lambda \int d^3k\, k a_\lambda^\dagger(\vec{k}) a_\lambda(\vec{k}) , \tag{A1}$$

with $\vec{A}_F^R(\vec{0})$ given by Eq. (2.11).

The calculations are greatly simplified by expanding $\vec{A}_F^R(r)$ in terms of multipole waves.[23] In the electric dipole approximation, only electric dipole waves will couple with the electron. In terms of electric dipole annihilation and creation operators, one has[23]

$$\vec{A}_F^R(\vec{0})=\left[\frac{2}{3\pi}\right]^{1/2}\sum_{M=-1}^{+1}\int_0^\infty dk\, g(k)\sqrt{k}\,[a_M(k)\hat{\epsilon}_M+a_M^\dagger(k)\hat{\epsilon}_M^*]\ , \tag{A2}$$

where

$$[a_M(k),a_{M'}^\dagger(k')]=\delta_{MM'}\delta(k-k') \tag{A3}$$

and

$$\hat{\epsilon}_M=\begin{cases}-(\hat{x}+i\hat{y})/\sqrt{2}, & M=1\\ \hat{z}, & M=0\\ (\hat{x}-i\hat{y})/\sqrt{2}, & M=-1\ ,\end{cases} \tag{A4}$$

satisfying

$$\hat{\epsilon}_M\cdot\hat{\epsilon}_{M'}^*=\delta_{MM'}\ . \tag{A5}$$

In terms of electric dipole wave operators, the free field Hamiltonian can be written as

$$H_F=\sum_{M=-1}^{+1}\int_0^\infty dk\, k a_M^\dagger(k)a_M(k)\ , \tag{A6}$$

neglecting the higher-multipole modes, since they do not couple with the electron.

We will now apply to H a unitary transformation which will decouple the field and electronic variables. Let U be given by

$$U\equiv\exp(-ie\vec{p}\cdot\vec{H}/m_R)\ , \tag{A7}$$

where

$$m_R=m+\frac{4e^2}{3\pi}\int_0^\infty g^2(k)dk\ , \tag{A8}$$

and $\vec{H}$ is the Hertz vector:

$$\vec{H}=\left[\frac{2}{3\pi}\right]^{1/2}\sum_{M=-1}^{+1}\int_0^\infty \frac{dk}{\sqrt{k}}g(k)[a_M(k)\hat{\epsilon}_M-a_M^\dagger(k)\hat{\epsilon}_M^*]\ . \tag{A9}$$

The transformation (A7) is the generalization of the so called "space translation transformation"[24] to the quantized field case. It is easy to see that $\vec{p}\,'\equiv U^\dagger\vec{p}U=\vec{p}$ and that

$$a_M'(k)=a_M(k)-\left[\frac{2}{3\pi}\right]^{1/2}\frac{eg(k)}{m_R\sqrt{k}}\vec{p}\cdot\hat{\epsilon}_M^*\ . \tag{A10}$$

From (A8) and (A10) one can show that, when expressed in terms of the new canonical variables, H becomes

$$H=\frac{\vec{p}\,'^2}{2m_R}+\frac{e^2}{2m}\vec{A}_F^{R'}(\vec{0})^2+H_F'\ , \tag{A11}$$

where $\vec{A}_F^{R'}(\vec{0})$ and H_F' have the same form as in Eqs. (A2) and (A6), respectively, but with $a_M'(k)$ in place of $a_M(k)$.

In Eq. (A11), the field and electron coordinates are already decoupled. The Hamiltonian is not yet diagonalized in the field coordinates, however.

Let us now introduce the canonically conjugate variables

$$\vec{q}(k)\equiv\sum_{M=-1}^{+1}[a_M'(k)\hat{\epsilon}_M+a_M'^\dagger(k)\hat{\epsilon}_M^*]/(2k)^{1/2} \tag{A12}$$

and

$$\vec{p}(k)\equiv -i\sum_{M=-1}^{+1}[a_M'(k)\hat{\epsilon}_M-a_M'^\dagger(k)\hat{\epsilon}_M^*](k/2)^{1/2} \tag{A13}$$

so that

$$\begin{aligned}&[q_i(k),q_j(k')]=0,\ \ [p_i(k),p_j(k')]=0\ ,\\ &[q_i(k),p_j(k')]=i\delta_{ij}\delta(k-k')\ .\end{aligned} \tag{A14}$$

Then

$$\vec{A}_F^{R'}(\vec{0})=\frac{2}{\sqrt{3\pi}}\int_0^\infty dk\, kg(k)\vec{q}(k)\ , \tag{A15}$$

$$H_F'=\tfrac{1}{2}\int_0^\infty dk[\vec{p}\,^2(k)+k\vec{q}\,^2(k)]\ , \tag{A16}$$

and

$$H=\frac{(\vec{p}\,')^2}{2m_R}+\tfrac{1}{2}\int_0^\infty dk\int_0^\infty dk' A(k,k')\vec{q}(k)\cdot\vec{q}(k')+\tfrac{1}{2}\int_0^\infty dk\,\vec{p}\,^2(k)\ , \tag{A17}$$

where

$$A(k,k')=k^2\delta(k-k')+\alpha(k)\alpha(k')\ , \tag{A18}$$

with

$$\alpha(k)=2ekg(k)/(3\pi m)^{1/2}\ . \tag{A19}$$

The problem of diagonalizing H is therefore reduced to finding the principal axes of the quadratic form in $\vec{q}(k)$ which appears in (A17). In order to do that, we must diagonalize the continuous matrix $A(k,k')$. Its eigenvalue equation is in fact a Fredholm integral equation of the second kind, homogeneous and with degenerate kernel,[25] which can be solved exactly:

$$sx(k,s)=\int_0^\infty dk' A(k,k')x(k',s)$$
$$=k^2x(k,s)+\alpha(k)\int_0^\infty dk'\,\alpha(k')x(k',s)\ , \tag{A20}$$

where $x(k,s)$ is the eigenvector corresponding to the eigenvalue s.

The solutions of (A20) are given by

$$x(k,s)=\alpha(k)\beta\left[\mathscr{P}\frac{1}{s-k^2}+\lambda\delta(s-k^2)\right]\ , \tag{A21}$$

where $\beta=\int_0^\infty dk'\alpha(k')x(k;s)$ can be considered as a normalization constant, determined by the requirement that

$$\int_0^\infty dk\,x(k,s)x(k,s')=\delta(s-s')\ , \tag{A22}$$

while λ is an arbitrary real constant. In Eq. (A21), $\mathscr{P}$ stands for "the principal part thereof."

Insertion of (A21) into the expression for β yields

$$1=\mathscr{P}\int_0^\infty \frac{dk\,\alpha^2(k)}{s-k^2}$$
$$+\lambda\int_0^\infty dk\,\alpha^2(k)\delta(s-k^2)\ , \tag{A23}$$

which determines the eigenvalues s.

For $s<0$, the second term on the right-hand side of (A23) does not contribute. One gets, then, setting $s=-K_0^2$,

$$\frac{4e^2}{3\pi m}\int_0^\infty \frac{dk\,k^2 g(k)}{K_0^2+k^2}=1\ . \tag{A24}$$

Expressing m in terms of m_R, as in Eq. (A8), one gets

$$\int_0^\infty \frac{dk\,K_0^2 g(k)}{K_0^2+k^2}=\frac{3\pi m_R}{4e^2}\ . \tag{A25}$$

In the point-electron limit $[g(k)\to 1]$, one gets

$$K_0=3m_R/2e^2 \tag{A26}$$

and therefore

$$s=-K_0^2=-(3m_R/2e^2)^2\ . \tag{A27}$$

One has therefore only one negative eigenvalue. For $s>0$, setting $s=K^2$ and using (A8), one finds

$$\mathscr{P}\int_0^\infty \frac{dk\,K^2 g^2(k)}{K^2-k^2}$$
$$+\frac{\lambda}{2K}\int_0^\infty dk\,k^2 g^2(k)\delta(k-K)=3\pi m_R/4e^2\ . \tag{A28}$$

In the point-electron limit, the first term on the left-hand side of (A28) vanishes, and one gets

$$K=3\pi m_R/2\lambda e^2\ , \tag{A29}$$

that is,

$$s=K^2=(3\pi m_R/2\lambda e^2)^2\ . \tag{A30}$$

Since λ is arbitrary, we conclude that any $s>0$ is allowed: For positive s, one gets a continuous spectrum.

In terms of $x(k,s)$, one can write the following relation between the coordinates $\vec{q}(k)$, $\vec{p}(k)$ and the new coordinates $\vec{q}\,'(k),\vec{p}\,'(k)$ which diagonalize H:

$$\vec{q}(k)=\int_0^\infty dK\,x(k,K^2)\vec{q}\,'(K)$$
$$+x(k,-k_0^2)\vec{q}\,'(K_0)\ , \tag{A31}$$

$$\vec{p}(k)=\int_0^\infty dK\,x(k,K^2)\vec{p}\,'(K)$$
$$+x(k,-K_0^2)\vec{q}\,'(K_0)\ . \tag{A32}$$

It is easy to see, by using (A20) and (A22), that H indeed becomes diagonal when expressed in terms of $\vec{p}\,'$ and $\vec{q}\,'$. One gets, in fact,

$$H=\frac{(\vec{p}\,')^2}{2m_R}+\tfrac{1}{2}\int_0^\infty dK[(\vec{p}\,')^2(K)+K^2(\vec{q}\,')^2(K)]$$
$$+\tfrac{1}{2}[(\vec{p}\,')^2(K_0)-K_0^2(\vec{q}\,')^2(K_0)]\ . \tag{A33}$$

The condition for $\vec{p}\,'$ and $\vec{q}\,'$ to satisfy the same commutation relations as $\vec{p}$ and $\vec{q}$ is

$$\int_0^\infty dK\,x(k,K^2)x(k',K^2)$$
$$+x(k,-K_0^2)x(k',-K_0^2)=\delta(k-k')\ , \tag{A34}$$

which is recognized as the completeness relation for the eigensolutions $x(k,s)$.

In (A33), one gets a normal mode with negative energy. It shows up because, in the point-electron limit, the mathematical (or "bare") mass m goes to $-\infty$, in order to keep m_R finite, so that H is no longer positive definite. The usual approach to this difficulty is to discard this mode, on the grounds that it does not represent a physical mode and therefore would never be excited.

If one follows this procedure, then one gets from (A33) the following spectrum for H:

$$E=\frac{\vec{P}^2}{2m_R}+\sum_{M=-1}^{+1}\int_0^\infty dk\, kN_M(k)\ , \qquad (A35)$$

where $\vec{P}$ is a real vector and $N_M(k)\geq 0$. The quantity $N_M(k)$ is the eigenvalue of the operator $\tilde{a}^\dagger_M(k)\tilde{a}_M(k)$, where $\tilde{a}_M(k)$ and $\tilde{a}^\dagger_M(k)$ are defined in terms of $\vec{q}\,'(k)$ and $\vec{p}\,'(k)$ as in Eqs. (A12) and (A13). From (A35) it is quite natural to interpret m_R as the physical mass of the electron.

APPENDIX B

The solutions obtained for $\vec{r}(t)$ and $\vec{E}(\vec{R},t)$ contain time-dependent terms proportional to $e^{s_1 t}$ and $e^{s_0 t}$, where s_0 and s_1 are the roots of the equation

$$s^2+[\beta c(1+\gamma)-i\omega_c]s-i\omega_c\beta c=0\ . \qquad (B1)$$

We analyze now the dependence of s_0 and s_1 with respect to the cutoff β, after mass renormalization.

Rewriting Eq. (B1) in terms of m_R, we get

$$\left[m_R-\frac{2e^2\beta}{3c^2}\right]s^2+\left[\beta m_R c+\frac{ieB}{c}\right]s+ieB\beta=0\ , \qquad (B2)$$

which has as solutions

$$S=\left[2\left[m_R-\frac{2e^2\beta}{3c^2}\right]\right]^{-1}\left\{-\left[m_R\beta c+\frac{ieB}{c}\right]\pm\left[\left[m_R\beta c-\frac{ieB}{c}\right]^2+\frac{8ie^3\beta^2B}{3c^2}\right]^{1/2}\right\}, \qquad (B3)$$

with the square root defined so that it is positive when the radicand is real and positive.

Let us first analyze the root corresponding to the plus sign in (B3), which we shall call s_0. Defining a parameter y by

$$y=\omega_c'/\beta c\ , \qquad (B4)$$

where $\omega_c'=|e|B/m_R c$, we see that $y\ll 1$ so long as $\beta c\gg\omega_c'$, which should be valid for any reasonable cutoff. Of course, one must still have $\hbar\beta c\lesssim m_R c^2$, which can be satisfied so long as $\hbar\omega_c'/m_R c^2\ll 1$. This last condition will hold if $B\ll B_{\rm cr}$.

In terms of y, one has, from (B3),

$$s_0=+2i\omega_c'\{1-iy+[(1+iy)^2-\tfrac{8}{3}ia\alpha]^{1/2}\}^{-1}\ , \qquad (B5)$$

where

$$a=\hbar\omega_c'/m_R c^2=B/B_{\rm cr} \qquad (B6)$$

and α is the fine-structure constant.

Expanding (B5) in terms of the parameter $a\alpha(1+iy)^{-2}$, one gets

$$s_0=+i\omega_c'\left[1+\frac{2a\alpha y}{3(1+y^2)}-\frac{4a^2\alpha^2(2+y^2)}{9(1+y^2)^3}\right]-\frac{2a\alpha\omega_c'}{3(1+y^2)}\left[1-\frac{2a^2\alpha^2 y}{3(1+y^2)^2}\right]+O(\omega_c' a^3\alpha^3)\ . \qquad (B7)$$

The real part of s_0 corresponds to the linewidth Γ. Since $y\ll 1$, one can write

$$\Gamma=\tfrac{2}{3}a\alpha\omega_c'[1+O(y^2,a^2\alpha^2 y)]=\frac{2e^4B^2}{3m_R^3c^5}[1+O(y^2,a^2\alpha^2y)]\ , \qquad (B8)$$

so the linewidth changes quadratically with B.

One also gets from (B7) a frequency shift which is given by

$$\Delta\omega_c'\simeq-\frac{2a\alpha\omega_c'}{3(1+y^2)}\left[y-\frac{2a\alpha(2+y^2)}{3(1+y^2)^2}\right]. \qquad (B9)$$

It is interesting to notice that, as $\beta\to\infty$, $y\to 0$ and $\Delta\omega_c'\simeq 8\omega_c' a^2\alpha^2/9$. Therefore, after mass renormalization, one does not get a divergent energy shift, when the cutoff goes to infinity. This is in contrast with the perturbation theory result for the nonrelativistic hydrogen atom, which yields a logarithmically divergent energy shift.[26] On the other hand, $\Gamma\simeq 2e^4B^2/3m_R^3c^5[1+O(a^2\alpha^2)]$ when $\beta\to\infty$.

It should be clear from the above expressions that s_0 is rather insensitive to the value of β, as it ranges from a value much larger than ω_c'/c to infinity.

Let us now study the behavior of s_1, the other root of Eq. (B1), which is given by

$$s_1=-\frac{m_R c^2 a/2\hbar}{y-\frac{2a\alpha}{3}}\left[1-iy+\left[(1+iy)^2-\frac{8ia\alpha}{3}\right]^{1/2}\right]. \qquad (B10)$$

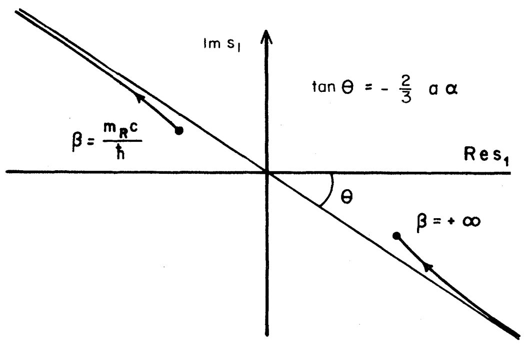

FIG. 4. Dependence of s_1 on the cutoff β, as it ranges from $m_R c/\hbar$ to infinity.

For a relativistic cutoff, $\beta_R \simeq m_R c/\hbar$, one has $y \simeq a$ and

$$\mathrm{Re}s_1 \simeq -m_R c^2/\hbar \simeq -10^{21}\ \mathrm{s}^{-1}\ , \tag{B11}$$

corresponding to a decay time of the order of 10^{-21} s. On the other hand, for $\beta \to \infty$, we get

$$s_1 = \frac{3m_R c^2}{2\hbar\alpha}[1 + \tfrac{4}{9}a^2\alpha^2 - \tfrac{2}{3}ia\alpha + O(a^3\alpha^3)]\ , \tag{B12}$$

so that $\mathrm{Re}s_1 \simeq 10^{23}\ \mathrm{s}^{-1}$. In this case, s_1 gives rise to the "runaway solution," which should therefore be identified with the solution which for $\beta_R = m_R c/\hbar$ decayed rapidly to zero. In Fig. 4 we sketch the trajectory of s_1 as β changes from $m_R c/\hbar$ to infinity.

The above analysis suggests a more consistent approach towards tackling the runaway problem. One should first discard the corresponding term when $\beta \lesssim \beta_R$, since it decays rapidly to zero, and only then should one set $\beta \to \infty$, since the remaining solution is rather insensitive to the value of β in the range β_R to ∞.

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