Exact solution for spontaneous emission in the presence of N atoms

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N two-level "atoms" are considered in interaction with a single-mode resonant electromagnetic field. The exact solution is given nonrelativistically for all times for the case of spontaneous emission, when only one atom is initially excited. The solution is given for the general case of the N atoms in inequivalent mode positions.

I. INTRODUCTION

Dicke[1] was the first to emphasize the cooperative nature of the spontaneous emission from a system of identical atoms, where the atoms were at equivalent mode positions, e.g., located within a wavelength of each other. In the case of two atoms, the system will either decay at twice the single-atom rate (initial triplet), or it will be nonradiative (singlet), and these two are equally likely. Their equally weighted average, of course, gives the "expected" rate of the isolated atom. Stephen[2] and other[3] further studied this simplest two-atom example of cooperative decay and expressed the cooperative linewidth and line shift as a function of interatomic separation. Milonni and Knight[4] obtained improved solutions in the form of infinite series involving all the two-atom retardation times nr/c.

The present paper has the purpose of presenting an exact solution to the problem of one atom initially excited in the presence of $N-1$ unexcited atoms. All N atoms each have only two levels resonant with the single-mode, lossless electromagnetic (em) field; although the solution for the nonresonant field can also be given exactly, only a sketch of this solution will be given in Sec. III. It is further assumed that the resonant cavity is small enough that relativistic (time-of-flight) effects may be ignored, since these times are very much smaller than any transition rate. It is also assumed that the wave functions of the atoms are nonoverlapping. Unlike previous calculations, the present will consider that the atoms are in inequivalent mode positions.

Exact solutions valid for all time for N-body quantum systems are invariably interesting in their own right in view of their expository and pedagogic value. The present model is interesting in that it couples two different quantum systems, N atoms and the quantized electromagnetic (em) field.

Buley and Cummings[5] first treated the problem of N two-level atoms interacting with a classical single-mode em field, giving long-time solutions for super-radiant emission. More recently, Stroud *et al.*[6] have treated N atoms in equivalent mode positions in this semiclassical approximation; they first pointed to the interesting effect of "radiation trapping," which we will see again in the present exact and completely quantized treatment with inequivalent mode positions of the N atoms.

II. FORMULATION

The mathematical setup of the problem has been discussed often before, and is basically the extension of an earlier single-atom model[7] to the many-atom case.

The Hamiltonian for the problem is given by ($\hbar=1$)

$$H=\omega a^{\dagger}a+\omega\sum_{j=1}^{N}\sigma_j^z+\sum_j\lambda_j\sigma_j^- a^{\dagger}+\lambda_j^*\sigma_j^+ a\ . \tag{2.1}$$

The operators $a,a^{\dagger}$ satisfy the commution rule

$$[a,a^{\dagger}]=1\ , \tag{2.2}$$

and the $\sigma_j^{\pm}$ satisfy

$$[\sigma_j^+,\sigma_{j'}^-]=2\sigma_j^z\delta_{jj'}\ . \tag{2.3}$$

The λ_j are defined by

$$\lambda_j=-\vec{\mu}\cdot\vec{\mathrm{E}}(\vec{\mathrm{x}}_j)\ , \tag{2.4}$$

where $\vec{\mu}$ is the electric (induced) dipole moment ($\sim E$) [or the magnetic (permanent) dipole moment if $\vec{\mathrm{E}}$ is the magnetic field], and $\vec{\mathrm{E}}(\vec{\mathrm{x}}_j)$ is the space part of the electric field mode[7] evaluated at the position $\vec{\mathrm{x}}_j$ of the jth atom.

If we write $H=H_0+H_1$, where $H_1=0$ for $\lambda_j=0$ then we may express the matrix elements of H in terms of eigenstates of H_0. The Hamiltonian matrix then assumes a block-diagonal form, each block along the diagonal corresponding to a constant unperturbed energy. Thus, states corresponding to different energies are not connected, and we focus our attention on the simple situation where the initial energy corresponds to the presence of zero photons, and $N-1$ atoms unexcited. There are thus $N+1$ states which span the subspace of interest of constant energy we can take as zero (as it will otherwise only multiply each state by a constant phase factor).

These base states we define as follows:

$$|0\rangle\equiv|---\cdots-;1\rangle \tag{2.5a}$$

[Eq. (2.5a) represents one photon present, all atoms unexcited];

$$|1\rangle\equiv|+--\cdots-;0\rangle \tag{2.5b}$$

[Eq. (2.5b) represents no photons present, only first atom excited];

$$|j\rangle\equiv|----\underset{j}{+}-\cdots;0\rangle \tag{2.5c}$$

[Eq. (2.5c) represents no photons present, jth atom excited]. Since the operators $a^{\dagger}$ and σ_j^+ are stepup operators for the field and atom j, respectively, we see easily that H has the matrix elements

$$\langle 0|H|j\rangle=\begin{cases}\lambda_j & \text{for } j\neq 0\\ 0 & \text{otherwise}\ .\end{cases} \tag{2.6}$$

Thus H has zeros along the diagonal and everywhere else except along the first row and first column.

The density matrix in the Schrödinger representation has the form

$$\rho(t)=\exp(iHt)\rho(0)\exp(-iHt)\ . \tag{2.7}$$

Interest turns on the various diagonal elements $\langle 0\,|\,\rho(t)\,|\,0\rangle$, the probability that a photon has been emitted at time t, as well as the elements $\langle 1\,|\,\rho(t)\,|\,1\rangle$ and $\langle j\,|\,\rho(t)\,|\,j\rangle$, the probabilities that the first atom and the jth atom are excited, respectively.

Now

$$\langle 0\,|\,\rho(t)\,|\,0\rangle=\sum_{\alpha,\beta}U_{0\alpha}^{-1}\rho_{\alpha\beta}(0)U_{\beta 0}\ , \tag{2.8}$$

where

$$U_{j0}=[\exp(-iHt)]_{j0}=\sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\sum_{\alpha,\beta,\gamma,\ldots}H_{0\alpha}H_{\alpha\beta}H_{\beta\gamma}\cdots H_{\zeta j}\ , \tag{2.9}$$

where there are n factors of H_{lm} in the product. The $\sum_{\alpha,\beta,\gamma,\ldots}$ vanishes unless the alternate indices β, $\delta,\ldots$ are zero; also the last (nth term) index ζ must be zero for $j\neq 0$. Then U_{j0}, $j\neq 0$, reduces to the simple form

$$U_{j0}=\sum_{n=\text{odd}}^{\infty}\frac{(-it)^n}{n!}\left[\sum_{\alpha,\gamma,\epsilon}|H_{0\alpha}|^2|H_{0\gamma}|^2\cdots\right]H_{0j}\ . \tag{2.10}$$

Then

$$U_{j0}=\sum_{n=\text{odd}}^{\infty}\frac{(-it)^n}{n!}\Lambda^{n-1}\lambda_j=\frac{-i\lambda_j}{\Lambda}\sin\Lambda t\ . \tag{2.11}$$

Here Λ has been defined by

$$\Lambda^2=\lambda_1^2+\lambda_2^2+\cdots+\lambda_N^2\ . \tag{2.12}$$

If we now take for the initial density matrix in Eq. (2.8),

$$\rho_{\alpha\beta}(0)=\delta_{\alpha\beta}\delta_{\alpha 1}\ , \tag{2.13}$$

which says that the first atom is excited, all others unexcited, then from Eq. (2.8),

$$\langle 0\,|\,\rho(t)\,|\,0\rangle=|\,U_{01}\,|^2=\frac{\lambda_1^2}{\Lambda^2}\sin^2\Lambda t\ . \tag{2.14}$$

Equation (2.14) gives the transition probability, the probability of finding a photon in the field at time t. Two extreme cases are of immediate interest. First let us suppose that all $N-1$ unexcited atoms are in mode positions such that $\vec{E}(\vec{x}_j)=0$ ($j=2,3,\ldots,N$). The first atom never interacts with the other atoms since they do not interact with the field, and we have in this case the usual answer[7] for spontaneous emission from a single atom, namely, the average "photon number"

$$\langle 0\,|\,\rho(t)\,|\,0\rangle=\bar{n}(t)=\sin^2\lambda_1 t\ . \tag{2.15}$$

At the other extreme, suppose that all N atoms are in equivalent mode positions, so that $\lambda_j=\lambda$ and $\Lambda^2=N\lambda^2$. In this case

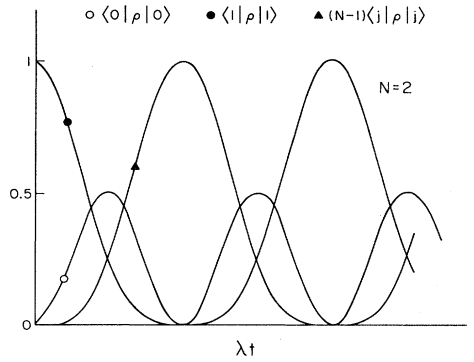

FIG. 1. Three density matrix elements as a function of time for two atoms.

$$\bar{n}(t)=N^{-1}\sin^2(N^{1/2}\lambda t)\ . \tag{2.16}$$

This surprising result says that the photon never gets a fraction greater than N^{-1} of the energy of the excited atom. At first glance one might suppose that this means that the emitted energy gets shared equally by the field and the $N-1$ atoms. The situation is more complex and interesting than this, however, for if we now compute the matrix element $\langle 1\,|\,\rho(t)\,|\,1\rangle$ for the same initial density matrix of Eq. (2.13) in a calculation exactly parallel to that leading to Eq. (2.14) we find

$$\langle 1\,|\,\rho(t)\,|\,1\rangle=[1-N^{-1}-N^{-1}\cos(N^{1/2}\lambda t)]^2 \tag{2.17}$$

when all atoms are in equivalent mode positions, $\lambda_j=\lambda$. As N gets very large, the first atom essentially never emits its energy at all. Figures 1–3 show the three transition probabilities $N=2$, 6, and 20. The presence of the other $N-1$ atoms acts to prevent the emission of the first atom, as first pointed out by Stroud *et al.*[6] based on a semiclassical analysis. Also we again encounter the enigmatic frequency

$$\Omega_N=N^{1/2}\lambda\ . \tag{2.18}$$

A physical interpretation of this frequency is not apparent. Equation (2.17) is, in the general case of inequivalent mode positions,

$$\langle 1\,|\,\rho(t)\,|\,1\rangle=\left[1-\frac{\lambda_1^2}{\Lambda^2}-\frac{\lambda_1^2}{\Lambda^2}\cos\Lambda t\right]^2\ . \tag{2.19}$$

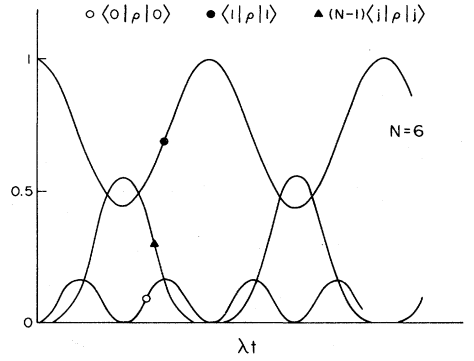

FIG. 2. Three density matrix elements as a function of time for six atoms.

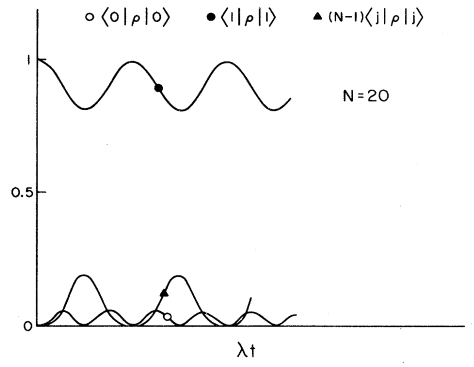

FIG. 3. Three density matrix elements as a function of time for 20 atoms.

To complete this vignette we may also compute the probability of finding the jth atom to be excited at time t, and find

$$\langle j\,|\rho(t)\,|j\rangle=(N-1)^{-1}\Lambda^{-4}\lambda_1^2(\Lambda^2-\lambda_1^2)\times(1-\cos\Lambda t)^2\,, \tag{2.20}$$

which reduces to

$$N^{-2}(1-\cos N^{1/2}\lambda t)^2 \tag{2.21}$$

for atoms in equivalent mode positions.

It is easy to verify that

$$\langle 0\,|\rho(t)\,|0\rangle+\langle 1\,|\rho(t)\,|1\rangle+(N-1)\langle j\,|\rho(t)\,|j\rangle=1 \tag{2.22}$$

as required.

Even when the atoms are at inequivalent mode positions, the surprising effect of "radiation trapping"[6] remains. The difference, as we see from examination of Eqs. (2.19) and (2.20), is that the "bare" N must be replaced by $N_{\rm eff}$ by the definition

$$\Lambda^2\equiv N_{\rm eff}\lambda_1^2 \tag{2.23}$$

in which case Eqs. (2.19) and (2.20) reduce to those for the equivalent mode positions with $N\rightarrow N_{\rm eff}$. We may expect that for an ensemble of measurements with atoms in random positions that

$$\overline{\Lambda^2}=\sum_j\overline{\lambda_j^2}=\frac{N}{2}\lambda^2\,,$$

and

$$N_{\rm eff}=N/2\,.$$

It is perhaps worth noting that had we taken as our initial starting point the information that only one atom was excited, but we did not know which one, then

$$\rho_{\alpha\beta}(0)=N^{-1}\delta_{\alpha\beta}\quad(\alpha,\beta\neq 0)\,. \tag{2.24}$$

This gives again the result of Eq. (2.16) in the equivalent mode position case, as expected, since the field does not care which atom is initially excited in this case.

Consider now the case of two atoms, and let the initial matrix be of the (nondiagonal) "triplet" form

$$\rho_{\alpha\beta}(0)=\tfrac{1}{2},\quad \alpha,\beta=1,2$$
$$\rho_{\alpha\beta}(0)=0,\quad \alpha,\beta=0\,. \tag{2.25}$$

A straightforward calculation via Eq. (2.8) gives that

$$\bar{n}(t)=\sin^2\sqrt{2}\lambda t\,. \tag{2.26}$$

For short times, the transition probability is

$$\bar{n}(t)\simeq 2(\lambda t)^2\,, \tag{2.27}$$

in other words, twice the rate for one atom only. This is the simplest result of Dicke,[1] who first pointed out that the presence of an unexcited neutron could double the transition rate of an excited neutron if the initial state is the triplet corresponding to Eq. (2.25), giving rise to "super-radiance" in this case, as contrasted with "subradiance" as described above.

III. EXTENSION TO CAVITY DETUNING

If we return to examination of the Hamiltonian relevant to the present problem, i.e., Eq. (2.6), we see that it has the interesting property that all powers of H can be expressed in terms of H and H^2 by

$$H^n=\begin{cases}\Lambda^{n-1}H & (n\text{ odd}) \qquad (3.1)\\ \Lambda^{n-2}H^2 & (n\text{ even})\,. \qquad (3.2)\end{cases}$$

This fact gives an alternate way to compute $\rho(t)$ in Eq. (2.7). When the atoms all have energy level differences of Ω and the (nonresonant) field has frequency ω, then we can also find exact solutions, but we do not carry out the details here, but note the following. Again we can express H^n as a linear combination of only H and H^2, where the coefficients are functions of the detuning parameter $\Delta=(\omega-\Omega)/2$ and Λ, viz.;

$$H^n=a_n(\Delta,\Lambda)H+b_n(\Delta,\Lambda)H^2\,, \tag{3.3}$$

and

$$H^3=\Lambda^2H+2\Delta H^2\,. \tag{3.4}$$

The a_n and b_n satisfy the linear difference equations

$$b_{n+1}=a_n+2\Delta b_n \tag{3.5}$$

and

$$a_{n+1}=\Lambda^2 b_n\,, \tag{3.6}$$

or

$$a_{n+2}-2\Delta a_{n+1}-\Lambda^2 a_n=0\,. \tag{3.7}$$

This last difference equation can be solved easily as a linear combination of α_+^n and α_-^n where

$$\alpha_\pm=\Delta\pm(\Delta^2+\Lambda^2)^{1/2}\equiv\Delta\pm\Gamma\,, \tag{3.8}$$

and the coefficients are chosen to give the correct values for a_1 and a_2, i.e.,

$$a_n=\Lambda^2/2\Gamma[(\Delta+\Gamma)^{n-2}-(\Delta-\Gamma)^{n-2}]\,, \tag{3.9}$$

with b_n given by Eq. (3.6). Then $U(t)$ [Eq. (2.9)] can be summed exactly.

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