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Quantum dynamics and nonintegrability

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Results of a nonperturbative investigation of the global behavior of quantum systems with time-periodic Hamiltonians are presented. These include the proof of a theorem stating that such systems, if bounded and nonresonant, will reassemble themselves infinitely often in the course of time. In order to illustrate the applicability of this result, an analytic study of a driven harmonic oscillator is presented, together with computer simulations of quantum maps describing the dynamics of a pulsed electron in a well and a periodically kicked rotor. A quantitative study of resonance excitation also shows that in practical situations recurrence is pervasive. Several unique quantum effects are analyzed, and their relevance to the classical limit is discussed. A formula is derived for recurrence times, and computer experiments are performed to test its validity.

I. INTRODUCTION

An outstanding problem in the dynamics of quantum systems is posed by the existence of nonintegrable classical equations of motion. These equations, which in many instances are known to produce chaotic wandering of trajectories in some regions of phase space, have quantum counterparts whose behavior is largely unknown. In particular, when the Hamiltonian is time dependent and cannot be treated by perturbation theory, determining the global properties of the solutions of the corresponding Schrödinger equation is quite forbidding, and answers to simple questions such as energy growth or decay are very hard to obtain. These questions are by no means purely academic since a large amount of photochemistry research depends on their resolution.[1] Furthermore, recent advances in microelectronics are presenting challenging problems as the mean free path of electrons inside very small structures becomes larger than the size of the device.[2] At a more fundamental level, there remains the old question of finding the quantum behavior of classically nonintegrable systems.[3–7] Although for stationary Hamiltonians there exists a fairly large body of knowledge, the situation in the time-dependent case is less clear and has been the focus of recent activity.[8–12]

This paper reports the results of a fairly extensive investigation of quantum dynamical systems with time-periodic Hamiltonians without resorting to perturbative techniques. In particular, it is proved that for such cases a bounded quantum system with a discrete quasienergy spectrum will reassemble itself infinitely often in the course of time. Since the proof does not give an *a priori* prediction of the nature of the quasienergy spectrum, computer experiments are performed on a variety of quantum systems. All of the problems treated here, which range from a simple linear oscillator to the quantum version of the standard map of classical stochasticity, show recurrent behavior, an indication that the results of the theorem are widely applicable.

The paper consists of six sections and one appendix. In Sec. II the recurrence theorem for quantum systems with periodic time-dependent potentials is stated and proved. Specific examples are used to illustrate the theorem in Sec. III. Section IV discusses a number of special cases including resonances, periodicities, and their relevance to the classical limit. Section V gives an estimate of recurrence times together with the results of computer measurements which support the theoretical predictions, and in the final section we make several concluding remarks. Some technical points used in the proof of the theorem are discussed in the Appendix.

II. A RECURRENCE THEOREM

Consider any bounded quantum system described by a Hamiltonian H_0 that has a discrete spectrum, and subjected to a nonresonant time-periodic potential V with $V(t)=V(t+T)$ for an arbitrary period T, and such that $||V||$ (Ref. 12) is bounded. We will now prove that given any initial configuration of the system, both the wave function and the energy return arbitrarily close to their initial values infinitely often. More generally, if we define an almost-periodic function $f(t)$ to be a continuous, bounded function, such that for any $\epsilon>0$ there exists a relatively dense set $\{\tau_\epsilon\}$ (Ref. 14) and, for each τ_ϵ in the set, we have $|f(t+\tau_\epsilon)-f(t)|<\epsilon$ for *all* t, our theorem states that both the wave function and the energy are almost-periodic functions of time.[13]

We start by proving the almost-periodicity of the wave function. Consider the time-dependent Schrödinger equation $i\hbar\partial\Psi/\partial t=[H_0+V(t)]\Psi$. Expanding the wave function Ψ in terms of the complete orthonormal set of eigenstates of $H_0,\{u_m(r)\}$, as $\Psi(r,t)=\sum_{m=1}^{\infty}a_m(t)u_m(r)$, the coefficients of $a_m(t)$ make up a vector $a(t)$ which satisfies

$$i\hbar\dot{a}(t)=H(t)a(t) \tag{2.1}$$

and we can write

$$||\Psi(t+\tau)-\Psi(t)||^2=|a(t+\tau)-a(t)|^2 \ , \tag{2.2}$$

where we have defined $||\Psi(t)||^2 \equiv \int dr\, |\Psi(r,t)|^2$. Furthermore, if $H(t)=H(t+T)$ the wave function satisfies a Floquet theorem, i.e., $a(t)$ is of the form

$$a(t)=\sum_{k=1}^{\infty} \alpha_k \exp(iE_k t/\hbar)\Phi_k(t) \tag{2.3}$$

with $\Phi_k(t+T)=\Phi_k(t)$ and $\Phi_k^\dagger(t)\Phi_{k'}(t)=\delta_{kk'}$ for all t.[15] The set $\{E_k\}$ is called the quasienergy spectrum. Writing α_k as $\alpha_k=r_k e^{i\varphi_k}$ with r_k and φ_k real, it follows from Eq. (2.3) that

$$|a(t+NT)-a(t)|^2=2\sum_{k=1}^{\infty} r_k^2 \left[1-\cos\frac{E_k NT}{\hbar}\right] \tag{2.4}$$

for any integer N. Since the wave function is normalized, we have $\sum_{k=1}^{\infty} r_k^2 = |a(t)|^2 = ||\Psi(t)||^2=1$, an equality which implies that given $\epsilon>0$ there exists an integer n (whose value depends on ϵ) such that $\sum_{k=n+1}^{\infty} r_k^2<\epsilon/8$. This then gives the following inequality:

$$\sum_{k=n+1}^{\infty} r_k^2\left[1-\cos\frac{E_k NT}{\hbar}\right] \le 2\sum_{k=n+1}^{\infty} r_k^2 < \frac{\epsilon}{4} \ . \tag{2.5}$$

We next consider the function $f(x) =\sum_{k=1}^{n}[1-\cos(E_k xT/\hbar)]$ which is always non-negative. By our definition of nonresonance the eigenvalues E_k are discrete so that this is a finite sum of periodic functions, and so, for any positive δ, the set of integers $\{N_\delta\}$ such that $|f(x+N_\delta)-f(x)|<\delta$ for all x, is relatively dense.[16] In particular, for $\delta=\epsilon/4$ and $x=0$ there exists a relatively dense set of integers $\{N\}$ such that $f(N)<\epsilon/4$ and since each $r_k \le 1$, we have $\sum_{k=1}^{n} r_k^2[1-\cos(E_k NT/\hbar)]<\epsilon/4$. Combining this result with Eqs. (2.5), (2.4), and (2.2) we obtain

$$||\Psi(t+NT)-\Psi(t)||^2<\epsilon \tag{2.6}$$

for all times t and for a relatively dense set of times $\{NT\}$. Therefore, the wave function is indeed almost periodic.

We now show that the energy of the system also recurs.[17] The energy is given by $E(t) =\langle\Psi|H(t)|\Psi\rangle=a^\dagger H_0 a+a^\dagger Va$. From the first part of the theorem, we know that the vector $a(t)$ is almost periodic. Furthermore, $V(t)$ is periodic and bounded so that, by the results given in the Appendix, $a^\dagger(t)V(t)a(t)$ is an almost-periodic scalar function. In particular, this product is bounded.

Since we have excluded resonant growth of the energy, $E(t)$ is itself bounded, and so $a^\dagger H_0 a$ must be bounded. If we denote the mth eigenvalue of H_0 by e_m then

$$a^\dagger H_0 a=\sum_{m=1}^{\infty} e_m |a_m(t)|^2 \ . \tag{2.7}$$

Now choose an $\epsilon>0$. Since the sum of Eq. (2.7) is bounded, there is an integer M, independent of t, such that

$$\sum_{m=M+1}^{\infty} e_m |a_m(t)|^2<\epsilon/4 \ . \tag{2.8}$$

The remaining portion of Eq. (2.7) is a finite sum of almost-periodic functions and is, therefore, almost periodic.[16] In particular, there is a relatively dense set of τ such that

$$\sum_{m=1}^{M} |e_m|\, \big|\, |a_m(t+\tau)|^2-|a_m(t)|^2 \big| <\epsilon/2 \tag{2.9}$$

for all t. For this set of τ we have from Eqs. (2.7)–(2.9) that $|a^\dagger H_0 a(t+\tau)-a^\dagger H_0 a(t)|$ is less than ϵ for all t, and so $a^\dagger H_0 a$ is almost periodic. Thus the energy $E(t)$, a sum of two almost-periodic functions, is also almost periodic.[18] This completes the proof of the theorem. Note that these techniques can also be applied to show that the expectation values of other operators are almost periodic provided they are bounded.

III. ILLUSTRATIONS

A. The driven harmonic oscillator

Consider a harmonic oscillator in a time-dependent electromagnetic field $e(t)$. Its Hamiltonian is then given by $H=H_0+\alpha x e(t)$ with

$$H_0=p^2/2m+m\omega^2x^2/2 \ , \tag{3.1}$$

where α is the polarizability of the oscillator, x its displacement operator, m its mass, and p its momentum operator. In this case the Ehrenfest theorem gives an exact solution for the time dependence of the energy $E=\langle H\rangle$.

Specifically, the time evolution of the energy is given by

$$dE/dt=\langle \partial H/\partial t\rangle=\alpha\dot{e}(t)\langle x\rangle \tag{3.2}$$

so it is sufficient to calculate the expectation value of the position operator, $\langle x\rangle$. Using the Ehrenfest theorem, we can write the following equations of motion for the expectation values of x and p:

$$d\langle x\rangle/dt=\langle p\rangle/m \ , \tag{3.3a}$$

$$d\langle p\rangle/dt=-\langle\nabla V\rangle=-m\omega^2\langle x\rangle-\alpha e(t) \ . \tag{3.3b}$$

Since the derivative of the potential is linear in x, we can combine Eqs. (3.3) to obtain

$$d^2\langle x\rangle/dt^2+\omega^2\langle x\rangle=-\alpha e(t)/m \tag{3.4}$$

whose solution is

$$\langle x\rangle=A(t)\sin\omega t+B(t)\cos\omega t \tag{3.5}$$

with

$$A(t)=-(\alpha/m\omega)\int_0^t e(t')\cos(\omega t')dt'+C_1 \tag{3.6a}$$

and

$$B(t)=(\alpha/m\omega)\int_0^t e(t')\sin(\omega t')dt'+C_2 \ , \tag{3.6b}$$

where C_1 and C_2 are constants determined by the initial conditions. The time evolution of the energy can now be obtained by integrating Eq. (3.2) and using Eqs. (3.5) and (3.6) to give, after some algebra,

$$E=E_0+(\alpha^2/m\omega)\times\int_0^t dt'\int_0^{t'} dt''\{\dot{e}(t')e(t'')\sin[\omega(t'-t'')]\}+\alpha C_1 f_1(t)+\alpha C_2 f_2(t) \tag{3.7}$$

with

$$f_1(t) \equiv \int_0^t \dot{e}(t')\sin(\omega t')dt' \quad (3.8a)$$

and

$$f_2(t) \equiv \int_0^t \dot{e}(t')\cos(\omega t')dt' \quad . \quad (3.8b)$$

As a specific example, we consider a potential consisting of a periodic string of δ functions and derive an exact expression for the evolution of the energy. For the case in which

$$e(t) = \sum_{n=1}^{\infty} \delta(t - nT) \quad (3.9)$$

we can evaluate Eq. (3.7) at times t such that $nT < t < (n+1)T$ by using the fact that

$$\sum_{k=1}^{n} e^{i\omega kT} = e^{i\omega T}(1 - e^{in\omega T})/(1 - e^{i\omega T}) \quad . \quad (3.10)$$

If at $t = 0$ we have a state with $\langle x \rangle = \langle p \rangle = 0$ (which in turn implies $C_1 = C_2 = 0$) then Eq. (3.7) becomes

$$E = E_0 + \frac{\alpha^2}{2m} \frac{\sin^2(n\omega T/2)}{\sin^2(\omega T/2)} \quad . \quad (3.11)$$

This function recurs periodically when ωT is a rational multiple of π. Otherwise we have more general almost-periodic behavior so that the energy returns arbitrarily close to its initial value infinitely often. We should also remark that when the period between pulses is such that $\omega T \ll 1$ the energy grows for short times as $E \simeq E_0 + \alpha^2 n^2/2m$.

The above analysis assumed that the initial state had $\langle x \rangle = \langle p \rangle = 0$. For the general case in which this is not true, the contributions of Eqs. (3.8) must be included. With the use of Eq. (3.10) it is easy to show that

$$f_1(t) = -\frac{\omega \cos[(n+1)\omega T/2]\sin(n\omega T/2)}{\sin(\omega T/2)} \quad (3.12a)$$

and

$$f_2(t) = \frac{\omega \sin[(n+1)\omega T/2]\sin(n\omega T/2)}{\sin(\omega T/2)} \quad , \quad (3.12b)$$

which should be added to Eq. (3.11) as indicated by Eq. (3.7) to obtain the true evolution of the energy. Note that when $\omega T \ll 1$ the energy grows linearly with n for very short times rather than quadratically, a behavior which may be wrongly identified with diffusive growth.[19]

More generally, we may ask what conditions on the field $e(t)$ will give an almost-periodic function for the energy $E(t)$. For this discussion we partially integrate Eq. (3.7) for $E(t)$ to obtain

$$\begin{aligned} E = E_0 &- (\alpha^2/m)I(t) \\ &+ (\alpha^2/m\omega)e(t)[\sin(\omega t)C(t) - \cos(\omega t)S(t)] \\ &+ \alpha C_1[e(t)\sin(\omega t) - \omega C(t)] \\ &+ \alpha C_2[e(t)\cos(\omega t) - e(0) + \omega S(t)] \end{aligned} \quad (3.13)$$

with

$$I(t) \equiv \int_0^t dt' \int_0^{t'} dt'' \{e(t')e(t'')\cos[\omega(t' - t'')]\} \quad , \quad (3.14a)$$

$$S(t) \equiv \int_0^t e(t')\sin(\omega t')dt' \quad , \quad (3.14b)$$

and

$$C(t) \equiv \int_0^t e(t')\cos(\omega t')dt' \quad . \quad (3.14c)$$

In particular, suppose that $e(t)$ is an almost-periodic function of time so that it can be written as

$$e(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \quad , \quad (3.15)$$

where $c_n = c_{-n}^*$ and $\omega_{-n} = -\omega_n$ since $e(t)$ is real. $E(t)$ will be almost periodic if each of the integrals in Eqs. (3.14) is bounded.[16] By substituting Eq. (3.15) in Eqs. (3.14), a condition on the spectrum of $e(t)$ can be obtained that is sufficient to ensure that the energy will be almost periodic, namely,

$$\sum_{n=-\infty}^{\infty} \left| \frac{c_n}{\omega_n + \omega} \right| < \infty \quad (3.16a)$$

and

$$\sum_{m=-\infty}^{\infty} \sum_{\substack{n'=-\infty \\ n' \neq n}}^{\infty} \left| \frac{c_n c_{n'} \omega_{n'}}{(\omega^2 - \omega_{n'}^2)(\omega_n - \omega_{n'})} \right| < \infty \quad (3.16b)$$

provided ω does not equal any of the ω_n. Therefore, a quantum harmonic oscillator acted upon by an almost-periodic nonresonant driving force satisfying Eqs. (3.16) will recur in time infinitely often. In this case almost-periodic behavior is obtained for conditions more general than those of the theorem of Sec. II. (If ω does equal one of the ω_n, a resonant case, then the energy grows quadratically in time.)

B. Quantum maps

When the Schrödinger equation cannot be solved analytically, it is convenient to resort to quantum maps. These are obtained by taking the time-dependent potential to be a series of δ-function pulses, a process which leads to a recursion relation which can then be iterated on a computer. Although such a potential does not rigorously satisfy the requirements of the theorem, we expect that the behavior is similar to that of very narrow Gaussian pulses which, being continuous, do satisfy the conditions of the theorem. Furthermore, these maps allow us to investigate somewhat more general time behavior. In particular, we have examined the case of a quasiperiodic series of pulses which suggests that the results of the theorem hold more generally than indicated by its hypotheses on the potential $V(t)$.

Specifically, consider a system described by the Hamiltonian $H_0 + V(t)$ with

$$V(t) = v \sum_{n=-\infty}^{\infty} \delta(t/T - n) \quad , \quad (3.17)$$

where v is an operator that is independent of time. For instance v could be the position or momentum operator. A quantum map is obtained by expanding the wave function Ψ in terms of the complete orthonormal set of eigenstates of H_0, $\{u_k(r)\}$, as $\Psi(r,t) = \sum_{k=1}^{\infty} a_k(t)u_k(r)$. Between kicks the a_k's evolve as $a_k((N+1)T^-) = a_k(NT^+)e^{-iE_k T/\hbar}$, where E_k is the kth eigenvalue of H_0. During the kick, the change in the wave function is dominated by the δ function which gives

$\psi(x,(N+1)T^+)=\psi(x,(N+1)T^-)e^{-ivT/\hbar}$. Combining these results gives the map

$$a_k((N+1)T^+)=\sum_{j=1}^{\infty} a_j(NT^+)e^{-iE_jT/\hbar}M_{kj} \quad , \tag{3.18a}$$

where the matrix element M_{kj} is given by

$$M_{kj}=\int u_k^*(r)e^{-ivT/\hbar}u_j(r)dr \quad . \tag{3.18b}$$

By truncating the sum of Eq. (3.18a) at $j=J<\infty$ and specifying values for the a_k's at time $t=0$, a recurrence relation is obtained that can be iterated numerically. The results obtained from iterating the map can be used to compute various expectation values. In particular, the energy at time t is given by

$$E(t)=\sum_{n=1}^{\infty} E_n\,|a_n(t)|^2 \quad . \tag{3.19}$$

The normalization condition $\sum_{k=1}^{\infty}|a_k|^2=1$ can be used to check the error introduced by this truncation. Since the map of Eqs. (3.18) preserves the norm of the vector a, truncating the map at a finite number of states means that the norm at a given iteration will always be less than or equal to the norm of the previous iteration. Thus we examine the nondecreasing quantity

$$\Delta=1-\sum_{j=1}^{J}|a_j(NT^+)|^2 \tag{3.20}$$

after every iteration and use $\Delta_{\max}$, the largest value obtained during the run, as a check on the truncation error. A small value of $\Delta_{\max}$ indicates that enough states were retained after the truncation to accurately compute the system's time evolution.

In the remainder of this section we illustrate the rigorous result of the recurrence theorem with a set of examples using these quantum maps.

C. A pulsed electron in a well

The second system that we studied corresponds to the dynamics of a bounded electron in a pulsed field. In particular, consider an electron in an infinite square-well potential of length L which is acted upon by a set of electromagnetic pulses of strength ϵ. The Hamiltonian of the system is then given by

$$H=p^2/2m-e\epsilon x\sum_{n=-\infty}^{\infty}\delta(t/T-n) \quad , \tag{3.21}$$

where m is the electron mass, e its charge, and p its momentum.

To study the behavior of this system, we expand the wave function in terms of the eigenstates of H_0, i.e., we write

$$\psi(x,t)=(2/L)^{1/2}\sum_{n=1}^{\infty}A_n(t)\sin(n\pi x/L) \tag{3.22}$$

with the nth eigenvalue being $E_n=n^2\pi^2\hbar^2/2mL^2$. Constructing a quantum map as described by Eqs. (3.18) relates the A_n's just after the $(N+1)$st kick to the values just after the Nth kick:

$$A_n(N+1)=(4i\alpha/\pi)\sum_{r=1}^{\infty}A_r(N)e^{-ir^2\tau/2} \times[(-1)^{r+n}e^{i\alpha\pi}-1] \times\frac{nr}{[r^2-(\alpha-n)^2][r^2-(\alpha+n)^2]} \quad , \tag{3.23}$$

where $\alpha=kL/\pi$, $\tau=\pi^2\hbar T/mL^2$, and $k=e\epsilon T/\hbar$. With this map we used Eq. (3.19) to compute the energy as a function of time for various values of the parameters α and τ. In the case for which $kL=3.5$ and $\tau=1.432$, Fig. 1 shows the expectation value of the energy in units of the ground-state energy $E_g=\pi^2\hbar^2/2mL^2$ as a function of the number of pulses applied to the particle. Initially the system was in the ground state. As can be seen, the energy is both bounded and returns close to its initial value very often.

We have also examined the case in which the strength of the δ-function pulses is modulated by a quasiperiodic function, with the time between pulses held constant.[20] This is illustrated in Fig. 2 which plots the energy as a function of time for the particle in an infinite square well with quasiperiodically modulated kicks, as shown in the insert. Here $\tau=1.639\,115\,3$ and the system was initially in the ground state. The value of kL used for the nth kick was $kL=1.2949[\cos(n)+\cos(2^{1/2}n)]$. The normalization

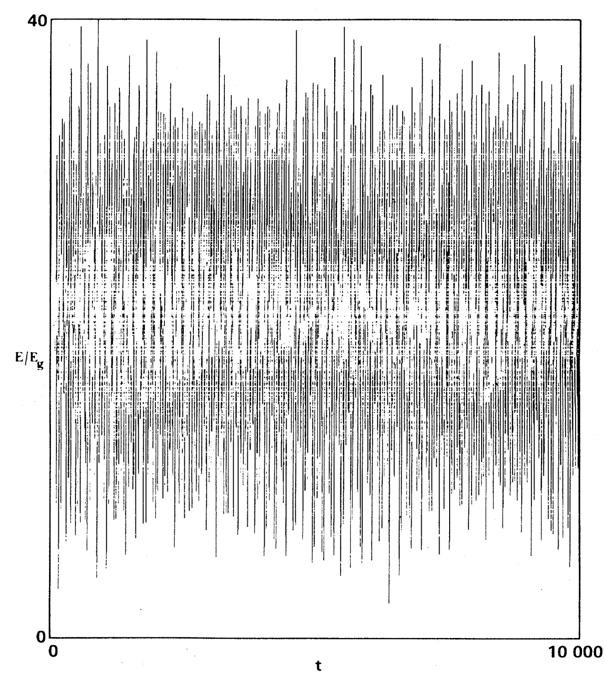

FIG. 1. Expectation value of the energy in units of the ground-state energy $E_g=\pi^2\hbar^2/2mL^2$ as a function of the number of pulses applied to the particle in an infinite square well. Initially the system was in the ground state. Parameters of the kicking potential were $kL=3.5$ and $\tau=1.432$. Normalization was checked to within 10^{-16} accuracy.

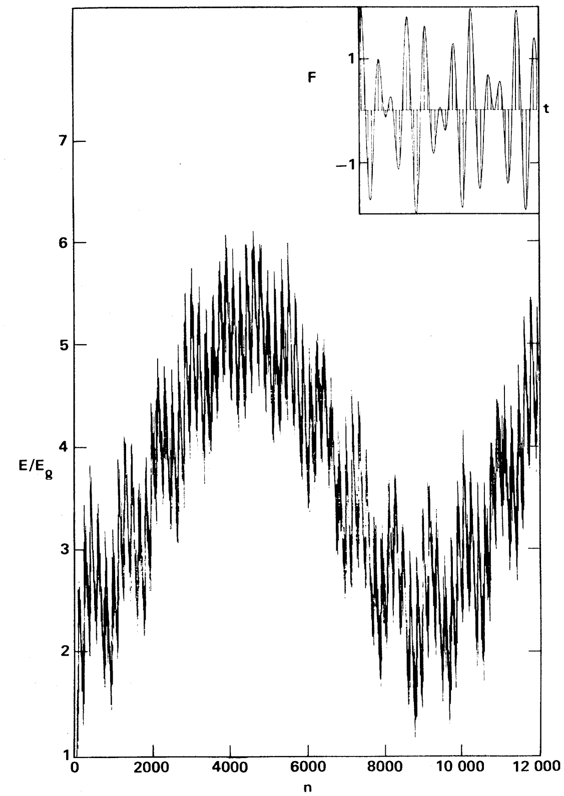

FIG. 2. Expectation value of the energy in units of the ground-state energy $E_g=\pi^2\hbar^2/2mL^2$ as a function of the number of pulses applied to the particle in an infinite square well for quasiperiodically modulated kicks. System had $\tau=1.639\,115\,3$ and was initially in the ground state. Value of kL used for the nth kick was $kL=1.2949[\cos(n)+\cos(2^{1/2}n)]$. Normalization was checked to within 3×10^{-7} accuracy. Inset: Modulating function $\cos(n)+\cos(2^{1/2}n)$ as a function of time for the first 50 kicks.

was checked to within 3×10^{-7} accuracy. The insert shows the modulating function $\cos(n)+\cos(2^{1/2}n)$ for the first 50 kicks. Because the system shows recurrent behavior, the experiment suggests that the conditions of the theorem could be relaxed to include quasiperiodic potentials.

D. A periodically kicked quantum rotor

The last problem that we will discuss, the quantum version of a periodically kicked rotor, still poses many intriguing questions on the issue of quantum chaotic behavior. Its classical counterpart is one of the paradigms of chaotic behavior in nondissipative dynamical systems and as such it has been thoroughly studied.[21] Therefore, the quantum rotor is a testing ground for ideas on quantum stochasticity and on the issue of the classical limit of nonintegrable systems.

The Hamiltonian for the periodically kicked rotor is

$$H=P_\theta^2/2I-\omega_0^2 I\cos(\theta)\sum_{n=-\infty}^{\infty}\delta(t/T-n) \ , \qquad (3.24)$$

where θ is the angle, P_θ is the angular momentum, and I is the moment of inertia. Following the technique of Casati *et al.*,[9] we expand Ψ in terms of the eigenstates of $H_0=P_\theta^2/2I$ as $\Psi(\theta,t)=(2\pi)^{-1}\sum_{n=-\infty}^{\infty}a_n(t)e^{in\theta}$ to obtain the map

$$a_n(t+T^+)=\sum_{r=-\infty}^{\infty}a_r(t)b_{n-r}(k)e^{-ir^2\tau/2} \ , \qquad (3.25)$$

where $k=\omega_0^2 IT/\hbar$, $\tau=\hbar T/I$, and $b_s(k)=i^s J_s(k)$ with J_s as the ordinary Bessel function of the first kind and order s. Using this map we have computed the energy $E(t)=\sum_{n=-\infty}^{\infty}(n^2/2I)\hbar^2|a_n(t)|^2$ for several values of k and τ while checking the normalization condition $\sum|a_n|^2=1$ to 16 digits at every iteration. Figure 3 shows the time evolution of the energy for the case where $k=2.871$, $\tau=2.532$, and the system was initially in the ground state. Once again, and in analogy with the problem of the electron in the quantum well, we observe that the excursions in the energy are bounded and recur many times.

Since a plot of the energy versus time as shown in Fig. 3 does not distinguish between almost-periodic behavior and the bounded chaos common to classical chaotic systems below the stochasticity limit, we computed the power spectrum for these energy values. This power spectrum was then compared with the corresponding spectrum for the classical kicked rotor displaying both quasiperiodic and bounded chaotic motion. Specifically, the classical behavior of the Hamiltonian of Eq. (3.24) is given by the map

$$x_{n+1}=x_n-(r/2\pi)\sin(2\pi y_n) \ , \qquad (3.26a)$$

$$y_{n+1}=y_n+x_{n+1} \ , \qquad (3.26b)$$

where $x_n=2\pi P_\theta T/I$ and $y_n=\theta/2\pi$ evaluated at time $t=nT$, and $r=(\omega_0 T)^2$. This map was iterated 20 000 times to remove transients and then the next 8192 values of x with $r=0.85$ were used to compute the spectrum. The initial conditions were chosen so as to produce both a Komolgorov-Arnold-Moser (KAM) surface ($x_0=0.472$, $y_0=0.378$), which is an example of almost-periodic behavior, and bounded chaotic motion ($x_0=0.9$, $y_0=0.5$).

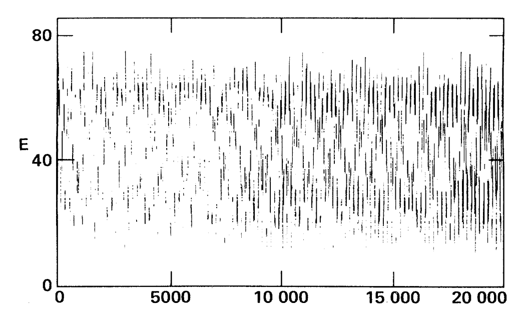

FIG. 3. Expectation value of the energy in units of $\hbar^2/I$ as a function of the number of pulses applied to the quantum rotor. Initially the system was in the ground state. Parameters used were $k=2.871$ and $\tau=2.532$. A total of 201 states were used and normalization was checked to within 10^{-16} accuracy.

The resulting power spectra are plotted in Fig. 4, where we show them for (a) the KAM surface, (b) classical bounded chaos, and (c) the quantum case of Fig. 3. As the figure clearly shows, the quantum case corresponds to an almost-periodic function and not random excursions among a finite set of energy levels.

This behavior for the energy indicates that the periodically kicked rotor might satisfy the conditions of our theorem and is therefore not chaotic. (Note that these results were obtained for parameter values such that $k\tau > 1$, a situation which in the classical problem leads to erratic wandering in phase space.) Indeed, recent work connecting the periodically kicked rotor to the localization problem indicates that its quasienergy spectrum is discrete.[22]

IV. PERIODICITIES AND RESONANCES

We now consider a number of special cases of the quantum maps studied in the previous section. First, note that

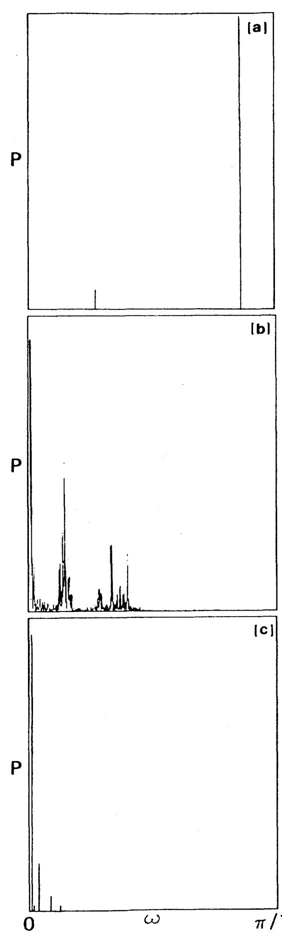

FIG. 4. Power spectra for the following: (a) a quasiperiodic KAM surface of the classical standard map, (b) bounded chaotic motion for the standard map at $r=0.85$, and (c) the quantum system of Fig. 3. In each case 8192 iterates were used for the computation and the dc component was removed from the spectrum. Angular frequency varies from 0 to π/T, where T is the time between kicks.

for a particular value of the kicking period, namely, $\tau=2\pi$, the systems that we studied exhibit simple periodic behavior as a function of time. Specifically, no matter how large the kicking potential is, the system returns to its initial state after every other kick.

This behavior can be easily seen for the rotor described by the map of Eq. (3.25). For $\tau=2\pi$ this map becomes

$$A_n((N+1)T)=\sum_{r=-\infty}^{\infty} A_r(NT)i^{n-r}J_{n-r}(k)(-1)^r \quad (4.1)$$

since $e^{-in^2(2\pi)/2}=(-1)^r$. Iterating the map twice gives

$$A_n((N+2)T)=\sum_{r'=-\infty}^{\infty} A_{r'}(NT)i^{n-r'}(-1)^{r'} \times \sum_{r=-\infty}^{\infty} J_{r-r'}(k)J_{n-r}(k)(-1)^r \ . \quad (4.2)$$

Setting $s=r'-r$ in the r sum gives

$$\sum_{s=-\infty}^{\infty} J_{-s}(k)J_{n-r'+s}(k)(-1)^{r'-s} \quad (4.3)$$

which equals $(-1)^{r'}\sum_{s=-\infty}^{\infty} J_s(k)J_{(n-r')+s}(k)$. Finally, we use the identity

$$J_n(u\pm v)=\sum_{s=-\infty}^{\infty} J_k(u)J_{n\pm k}(v) \quad (4.4)$$

to obtain

$$A_n((N+2)T)=\sum_{r'=-\infty}^{\infty} A_{r'}(NT)i^{n-r'}J_{n-r'}(0) = A_n(NT) \ . \quad (4.5)$$

Thus we see that any initial state will repeat after every other kick in spite of the fact that very many states are mixed in between as can be seen from Eq. (4.1). Note that this holds no matter how large the strength of the kicking potential is, i.e., for all values of k.[23] This behavior is also seen for the particle in the infinite square well and the periodically driven harmonic oscillator.

Another unique quantum effect is that the maps we studied are periodic in the parameter τ. In particular, the behavior for a given value of k and τ is identical to that for k and $\tau+4\pi$ even though the corresponding classical parameter (the product of k and τ) is quite different. So, for instance, a system with $k=2$, $\tau=38$, and $k\tau=76$, which is much greater than one, has the same behavior as a system with $k=2$, $\tau=38-12\pi=0.30$, and $k\tau=0.60$, which is less than one. This implies that the value of the classical parameter, $k\tau$, does not uniquely determine the quantum dynamics of the system and, in particular, there is no sharp distinction in the behavior of systems with $k\tau$ greater than one and those where it is less than one. This observation is particularly relevant to several analyses of the classical limit of the quantum rotor which have been published, and which seem to have overlooked this effect.[9]

Finally, we should mention that an interesting problem is posed by the existence of resonances, i.e., special values of the parameters that give an unbounded growth in the energy. For the particular case of the periodically kicked rotor, Izraelev and Shepelyanskii have shown that at resonance, which occurs whenever τ is a rational multiple of π, the quasienergy spectrum is continuous and the energy grows quadratically in time,[24] a situation which does not satisfy the conditions of our theorem.

Nevertheless, numerical experiments indicate that the existence of this set of resonances (which is of measure zero), will not, in practice, prevent the system from reassembling itself infinitely often. This is because minute departures from the numerical values which determine resonance will eventually produce a dephasing of the wave function and set in motion the mechanism of recurrence.

These effects are illustrated in Figs. 5 and 6, where we display the time evolution of the energy of the rotor for parameter values which are near resonance, namely, $k=0.5$ and $\tau=8\pi/5\pm\delta$, where δ is small. Each of the curves in the two figures corresponds to a different value of δ. Although the energy initially grows almost quadratically, it eventually deviates from this resonant growth after a time that depends on how close to resonance the system is. This indicates that a high-precision specification of the parameters is needed to observe these resonances for fairly long times. Furthermore, we observe that the times required to depart from the initial quadratic growth differ depending on whether one is below or above resonance. In particular, defining n to be the number of iterations required to reach the first maximum in the energy we find a power-law behavior given by

$$n=a\delta^{-b} . \tag{4.6}$$

From the data shown in Fig. 5 we have determined that $a=3.2$ and $b=0.55$, whereas above resonance Fig. 6 gives $a=8.9$ and $b=0.48$.

The above remarks are important in actual experimental situations since the observation of resonant energy growth in a quantum system subjected to laser radiation would require extremely good frequency stability. Notice that the values of δ used in producing the data of Figs. 5 and 6 imply a frequency stability of about one part in a million.

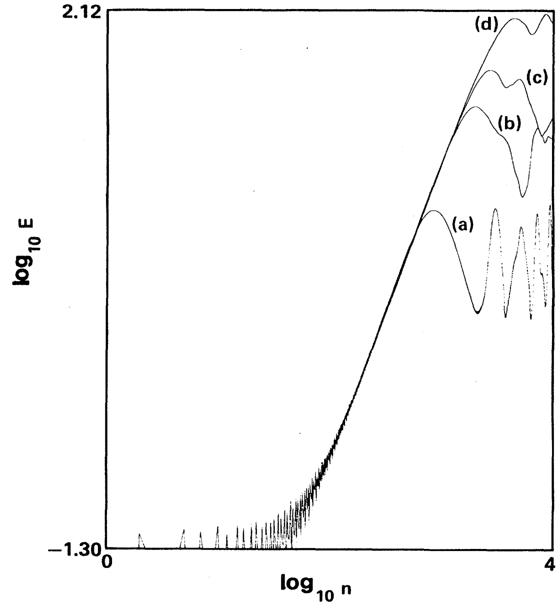

FIG. 6. A log-log plot of the energy in units of $\hbar^2/I$ as a function of time for the quantum rotor slightly above a resonance starting in the ground state. Parameters are $k=0.5$ and $\tau=8\pi/5+\delta$ with the following choices of δ: (a) 8.0×10^{-5}, (b) 1.5×10^{-5}, (c) 8.0×10^{-6}, and (d) 3.0×10^{-6}. Figure represents 10 000 pulses and the plotted energy values range from 0.05 to 131.5. Normalization was checked to within 10^{-15} accuracy for each of the four curves.

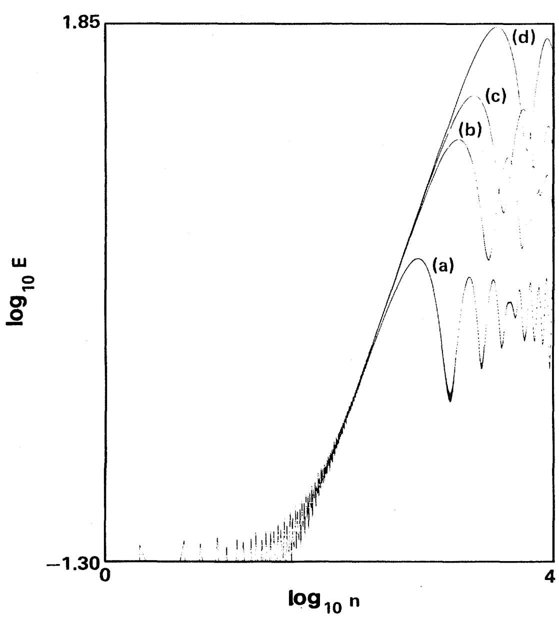

FIG. 5. A log-log plot of the energy in units of $\hbar^2/I$ as a function of time for the quantum rotor slightly below a resonance starting in the ground state. Parameters are $k=0.5$ and $\tau=8\pi/5-\delta$ with the following choices of δ: (a) 8.0×10^{-5}, (b) 1.5×10^{-5}, (c) 8.0×10^{-6}, and (d) 3.0×10^{-6}. Figure represents 10 000 pulses and the plotted energy values range from 0.05 to 70. Normalization was checked to within 10^{-15} accuracy for each of the four curves.

V. RECURRENCE TIMES

It is possible to make a rough estimate of the recurrence time for a kicked system described by $H_0+V(t)$ by assuming that the state vector $\vec{a}(t)$ moves uniformly among N eigenstates of H_0.[25] In this section we derive such an estimate and discuss its applicability to the systems we have considered in this paper. Here N is the number of states that are significantly excited by the action of $V(t)$. We consider the recurrence of the wave function, while noting that the methods developed here can also be applied to other quantities of interest. Specifically, we examine the behavior of

$$\Delta(t)=\min_{0\leq\varphi(t)<2\pi}||e^{i\varphi(t)}\Psi(t)-\Psi(0)||^2$$
$$=\min|e^{i\varphi(t)}a(t)-a(0)|^2 \tag{5.1}$$

and give the derivation for a system starting in a single eigenstate of H_0, though it can be generalized to other initial conditions.

Let $a_m(t)=r_m(t)\exp[i\Theta_m(t)]$, where $r_m=|a_m|$ and $0\le\Theta_m<2\pi$ and suppose that $a_m(0)=\delta_{km}$, i.e., the system starts in the kth eigenstate of H_0. Then Eq. (5.1) becomes

$$\Delta(t)=2[1-r_k(t)] \tag{5.2}$$

since $\sum_m |r_m|^2=||\Psi||^2=1$.

Now we assume that the wave function moves uniformly among N states so that the vector $(r_1,\ldots,r_N)$ moves uniformly on the first "quadrant" of an N-dimensional unit sphere. If each kick moves the state vector randomly around the sphere, the average number of kicks to a recurrence within ϵ, $n_{\rm rec}$, will be the reciprocal of f, the fraction of time spent in states with $\Delta<\epsilon$. By Eq. (5.2) this constraint becomes $r_k>1-\epsilon/2$ and f is the ratio of the area on the sphere satisfying this constraint to the total surface area of the first quadrant of the N-dimensional unit sphere.

By introducing N-dimensional polar coordinates $\theta_1,\ldots,\theta_{N-1}$ with r_k as the main axis so that $r_k=\cos(\theta_1)$, the total surface area of the first quadrant is

$$S=\int_0^{\pi/2}d\theta_1\sin^{N-2}\theta_1 \times\int_0^{\pi/2}d\theta_2\sin^{N-3}\theta_2\cdots\int_0^{\pi/2}d\theta_{N-1} \ . \tag{5.3}$$

For the constrained area we must have $r_k>1-\epsilon/2$ or $\sin^2(\theta_1)<\epsilon(1-\epsilon)$. Assuming that $\epsilon\ll1$, this becomes simply $\theta_1<\epsilon^{1/2}$ and the constrained area is

$$S_\epsilon=\int_0^{\epsilon^{1/2}}d\theta_1\theta_1^{N-2} \times\int_0^{\pi/2}d\theta_2\sin^{N-3}\theta_2\cdots\int_0^{\pi/2}d\theta_{N-1} \ . \tag{5.4}$$

Thus we get from Eqs. (5.3) and (5.4) that

$$n_{\rm rec}=\frac{S}{S_\epsilon}=\frac{N-1}{\epsilon^{(N-1)/2}}\,\frac{\pi^{1/2}\Gamma((N-1)/2)}{2\Gamma(N/2)} \ , \tag{5.5}$$

where $\Gamma(x)$ is the gamma function. Note that this estimate is independent of the initial state k. When N is large we can use Stirling's formula to get

$$n_{\rm rec}=\frac{(\pi N/2)^{1/2}}{\epsilon^{(N-1)/2}} \ . \tag{5.6}$$

For example, when $N=7$, Eq. (5.5) gives $n_{\rm rec}=16/5\epsilon^3=3200$ for $\epsilon=0.1$. It should be pointed out that even for this small a value for N, the approximation of Eq. (5.6) gives essentially the same result, e.g., 3300 for $\epsilon=0.1$.

We have examined the recurrence time as a function of ϵ numerically for the quantum rotor starting in the ground state and have found that it does obey a power-law relation. Figure 7 is a log-log plot of the average recurrence times for various values of ϵ as well as a weighted least-squares fit. The indicated error bars are based on the variance of the measured recurrence times for each value of ϵ. These variances were comparable to the average value, as would be expected for a uniform distribution. Thus a measurement over many recurrences was necessary to give accurate measurements of their average values. The fit gives $n_{\rm rec}=a\epsilon^b$ with $a=0.23$ and $b=-2.54$. Based on Eq. (5.6), this exponent yields a value for N of about 6 and so one would expect the coefficient a to be about 3. The measured value of a is much smaller. This is because the probability for the system to visit the various states is not uniform but instead peaked in lower states so that the system returns close to the ground state more frequently than would be expected from a uniform distribution.

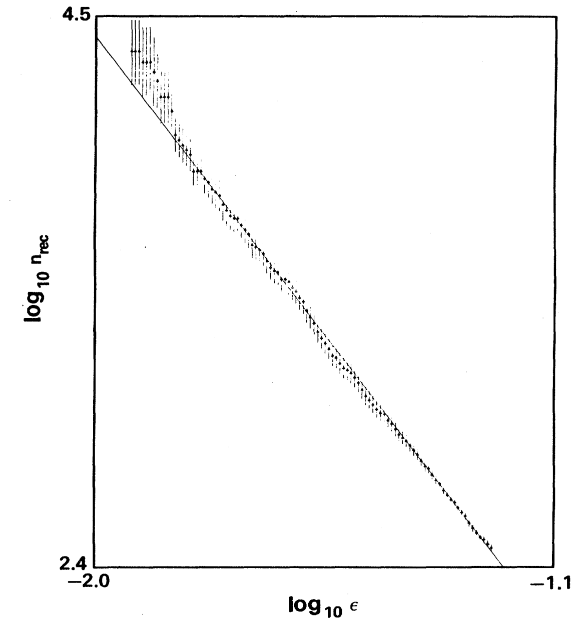

FIG. 7. A log-log plot of the average recurrence times measured for various values of ϵ between 0.0117 and 0.06 for the quantum rotor with $k=1.5$, $\tau=2.5$, and starting in the ground state. Error bars indicate the variance in the mean values estimated from the standard deviation of the recurrence times. Each point represents the average of between 10 (for the smallest ϵ values) and 400 (for the largest) recurrences measured over 250 000 iterations of the map. A weighted least-squares fit to the data is also shown. We used 101 states and the norm remained equal to one to within 10^{-16} accuracy.

It is also possible to use this technique to estimate recurrence times for other quantities such as $||\Psi(t)-\Psi(0)||^2$ or the energy $E(t)$. A similar power-law behavior is obtained. Note that these recurrence times are not the same as the ϵ almost-periods discussed in the derivation of the theorem. For instance, in order for τ to be an ϵ almost-period of Ψ, the quantity $||\Psi(t+\tau)-\Psi(t)||$ must be less than ϵ for *all* times t and not just for $t=0$.

Although we cannot determine an appropriate value for N *a priori* for a given kicking potential, especially since small changes in parameter values can cause a considerable change in the recurrence time due to resonances, the number of states significantly involved can usually be determined from the numerical data long before the system recurs. So this gives an estimate for the average recurrence time without the need to carry out lengthy computations to actually measure the recurrences.

A further example is given by the recurrence of the energy for the driven harmonic oscillator discussed in Sec. III. For the case in which the initial state has $\langle x \rangle = \langle p \rangle = 0$, the evolution of the energy is given by Eq. (3.11) so that

$$\Delta E_n \equiv E(nT) - E(0) = A \sin^2(n\tau) \quad , \tag{5.7}$$

where $A = \alpha^2/(2m \sin^2\tau)$ and $\tau = \omega T/2$. When τ is a noninteger rational multiple of π, i.e., when the kicking period T is commensurate with the natural period of the oscillator, this gives periodic behavior for the energy. An irrational value of τ/π, on the other hand, produces almost-periodic behavior for which an average recurrence time can be computed as follows. For $|\Delta E_n|$ to be less than some small $\epsilon > 0$, it is necessary that $n\tau$ be near a multiple of π. Specifically, let $\delta = (\epsilon/|A|)^{1/2}$ which we take to be much less than 1. Then the condition $|\Delta E_n| < \epsilon$ requires that $n\tau$ be within δ of an integer multiple of π. When τ/π is irrational, on the average one integer in every $\pi/2\delta$ will be in an allowed interval $(k\pi - \delta, k\pi + \delta)$ for some integer k. Thus the averge recurrence time for the energy will be

$$n_{\rm rec} = \pi/2\delta = \pi |A|^{1/2}/2\epsilon^{1/2} \quad , \tag{5.8}$$

which becomes exact in the limit that ϵ/A goes to zero.

VI. CONCLUSION

Throughout this study we have seen that recurrence is a pervasive phenomenon in bounded quantum systems acted upon by periodic time-dependent potentials. This is to be contrasted with corresponding classical systems that show mixing behavior, i.e., asymptotic decay of correlation functions. This difference in behavior cannot be attributed solely to the finite value of Planck's constant, which leads to a coarse grained phase space. Recurrence requires more, namely, a subtle cancellation among the phases of a wave function that has spread among an infinity of energy levels. As we have shown, these cancellations appear to take place even in systems having a dense set of resonances.

Previous studies of the quantum rotor have claimed the existence of quantum chaos by observing initial linear growth of the energy.[9] However, the almost-periodic behavior of the energy which we have shown to exist in these problems prevents one from making definite statements about its long-time monotonic growth or decay. Since the choice of initial condition is completely arbitrary, the system could start in a state for which the energy initially *decreases* and then oscillates instead of growing linearly in time. Therefore, little can be deduced from the short-time behavior of such systems.

We conclude by mentioning some unsolved problems. The first concerns the nature of the quasienergy spectrum of bounded quantum systems. At present there is no simple analytic criterion which can be used to decide whether a general potential will have a discrete or continuous quasienergy spectrum. This would certainly be useful for problems which cannot be simply expressed in terms of quantum maps. Second, there remains the old problem of obtaining mixing behavior in the classical limit of a recurrent quantum system. Although, in principle, one could start with a narrow wave packet, take first the limit $\hbar \to 0$ and then let the time run to infinity, this prescription is hard to implement starting from a quantum map. For example, the 4π periodicity in τ that we encountered in Sec. IV implies a serious ambiguity in finding the corresponding classical system for the quantum rotor. A possible way out of this dilemma is to find simple equations for the expectation values of observables using the quantum maps, but we have found them hard to express in closed form. Finally, we should mention the problem of damping, which plays an important role in realistic physical systems. Although it can be studied numerically, it is difficult at present to draw conclusions about general behavior from specific examples.

ACKNOWLEDGMENTS

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APPENDIX: SOME PROPERTIES OF ALMOST-PERIODIC VECTORS

The properties of almost-periodic vectors in a Hilbert space were used in the proof given in Sec. II. We establish the necessary lemmas in this appendix.

First, a vector function $\vec{x}(t)$ is almost periodic (ap) provided it is continuous and such that for any $\epsilon > 0$, the set of τ such that $|\vec{x}(t+\tau) - \vec{x}(t)| < \epsilon$ for all t is relatively dense.[14] Each such τ is called an "ϵ almost-period" of $\vec{x}(t)$. Such a function is bounded if there is a constant M such that $|\vec{x}(t)| \leq M < \infty$ for all t.

Lemma 1. If $\vec{x}(t)$ and $\vec{y}(t)$ are bounded almost-periodic vectors, then their scalar product $\alpha(t) = \vec{x}(t) \cdot \vec{y}(t)$ is an almost-periodic scalar function.

Proof. Let M and N be the finite bounds on $\vec{x}(t)$ and $\vec{y}(t)$, respectively. For any τ, we have

$$\alpha(t+\tau) - \alpha(t) = [\vec{x}(t+\tau) - \vec{x}(t)] \cdot \vec{y}(t+\tau) - \vec{x}(t) \cdot [\vec{y}(t+\tau) - \vec{y}(t)]$$

so that

$$|\alpha(t+\tau) - \alpha(t)| \leq |\vec{x}(t+\tau) - \vec{x}(t)| N + M |\vec{y}(t+\tau) - \vec{y}(t)| \quad . \tag{A1}$$

Now pick any $\epsilon > 0$. Since $\vec{x}$ and $\vec{y}$ are ap, the sets $S_x = \{\tau \text{ such that } |\vec{x}(t+\tau) - \vec{x}(t)| < \epsilon/2N \text{ for all } t\}$ and $S_y = \{\tau \text{ such that } |\vec{y}(t+\tau) - \vec{y}(t)| \leq \epsilon/2M \text{ for all } t\}$ are relatively dense. It can be shown that their intersection is also relatively dense[16] and for any $\tau \in S_x \cap S_y$, we have from Eq. (A1) that $|\alpha(t+\tau) - \alpha(t)| < \epsilon$ for all t. Thus $\alpha(t)$ is indeed almost periodic.

Lemma 2. If $\underline{A}(t)$ is a periodic, bounded matrix and $\vec{x}(t)$ is an almost-periodic vector, then the vector $\vec{y}(t) = \underline{A}(t)\vec{x}(t)$ is almost periodic.

Proof. Let $\underline{A}$ have period $T > 0$, i.e, $\underline{A}(t+T) = \underline{A}(t)$ for each t. Then for any integer N,

$$\vec{y}(t+NT) - \vec{y}(t) = \underline{A}(t)[\vec{x}(t+NT) - \vec{x}(t)] \quad .$$

Now let M be a bound on $\underline{A}$, i.e., $||\underline{A}(t)|| \leq M < \infty$ for all

t.[13] Then we have

$$|\vec{y}(t+NT)-\vec{y}(t)| \leq M\,|\vec{x}(t+NT)-\vec{x}(t)| \quad . \tag{A2}$$

The question now is whether there is a relatively dense set of *integer* multiples of T for which $|\vec{x}(t+NT)-\vec{x}(t)|$ can be made arbitrarily small for all t. To see that this is indeed the case, let $f(s)=\vec{x}(sT)$. Then $f(s)$ is almost periodic because given any $\delta>0$ there is a relatively dense set of τ such that $\sup|\vec{x}(t+\tau)-\vec{x}(t)|<\delta$, and hence,

$$\sup_s |f(s+\tau/T)-f(s)| = \sup_t |\vec{x}(t+\tau)-\vec{x}(t)| < \delta \quad .$$

Thus $\{\tau/T\}$ forms a relatively dense set of δ almost-periods for $f(s)$.

It can be shown[16] that for any $\delta>0$, the set of δ almost-periods that are integers is also relatively dense, i.e., there is a relatively dense set of integers $\{N_\delta\}$ such that $\sup|f(s+N_\delta)-f(s)|<\delta$. For this set of integers we also have

$$\sup_t |\vec{x}(t+N_\delta T)-\vec{x}(t)| = \sup_s |f(s+N_\delta)-f(s)| < \delta \quad .$$

Finally, choosing $\delta<\epsilon/M$ gives a relatively dense set of integers such that $|\vec{y}(t+NT)-\vec{y}(t)|<M(\epsilon/M)=\epsilon$ for all t. Thus $\vec{y}(t)$ is an almost-periodic vector.

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