Photon statistics of the free-electron-laser startup

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We show that, for the high electron currents used in present-day free-electron lasers, spontaneous radiation is distributed according to thermal statistics.

In a previous publication[1] (henceforth to be referred to as I), we developed a fully quantized many-particle theory of the free-electron laser (FEL) suitable for the description of the small signal, cold beam regime. The approach does not contain space-charge effects. Yet many-particle effects, i.e., terms proportional to $N_e(N_e-1)$, with N_e the total number of electrons, appeared because the radiation emitted by one electron affects the behavior of a second electron. As a general tendency we have found in I that in the regime of stimulated emission, when the initial laser field (with N photons, $N \gg 1$) is strong, the previously mentioned many-particle contributions are negligible, i.e., leading terms in gain and spread are proportional to NN_e, as commonly assumed. On the other hand, for spontaneous emission (no photons initially present, $N=0$) many-particle effects turned out to be very important if the electron current were sufficiently strong. They significantly affect the spectrum of spontaneous emission through amplified spontaneous emission and completely dominate the laser field fluctuations of the startup, suppressing the quantum-mechanical zero-point field fluctuations which are only proportional to N_e. The expressions derived in I for gain and spread already suggested that the photon statistics for $N=0$ and $N_e \gg 1$ should be close to thermal, in contrast to the one-particle case ($N_e=1$) where they are well known to be almost Poissonian.[2,3]

In this Brief Report, we will show explicitly that for $N_e \gg 1$ the spontaneously emitted photons satisfy, to a very good approximation, thermal statistics. This report is self-contained only with respect to the physical results and underlying assumptions; for the formalism and notation we shall refer to I.

The amplitude specifying spontaneous emission of n photons while the electrons have final momenta $p_i(i)$, $i=1,\ldots,N_e$, is given by

$$A_{n,p_1,\ldots,p_{N_e}} = \left\langle 0,p_1(1),\ldots,p_{N_e}(N_e) \right. \times \left| \frac{a^n}{\sqrt{n!}} S(T/2,-T/2) \right| \left. \text{in} \right\rangle \ , \quad (1)$$

where $S(T/2,-T/2)$ is the time-evolution operator [paper I, Eq. (10)] and the initial state

$$|\text{in}\rangle = |0,\bar{p}(1),\ldots,\bar{p}(N_e)\rangle \equiv |0,(\bar{p})^{N_e}\rangle \quad (2)$$

describes the field vacuum and N_e electrons with identical momenta. The probability that n photons be emitted irrespective of the final momenta of the electrons is then

$$P_n(N_e) = \int dp_1 \cdots dp_{N_e} |A_{n,p_1,\ldots,p_{N_e}}|^2 \ . \quad (3)$$

In order to evaluate Eq. (1) we replace[4] the time-evolution operator S by its lowest-order approximation S_0 [paper I, Eq. (13)], make use of the commutation relation (18a) of paper I, and rewrite

$$S_0(T/2,-T/2) = \exp[i\theta(T/2,-T/2)]\exp\left(j(T) \sum_{i=1}^{N_e} A_i^+ \right) \times \exp\left(j(T) \sum_{i=1}^{N_e} A_i \right) \exp(-\tfrac{1}{2} z N_e B) \ . \quad (4)$$

The quantity $j(T)$ and the operator B are defined in Eqs. (17b) and (8) of paper I, respectively. We have also introduced the abbreviation

$$[j(T)]^2 \equiv z \ . \quad (5)$$

The integration over the final momenta of the electrons in Eq. (3) can now be done via closure, and we obtain

$$P_n(N_e,z) = \frac{z^n}{n!} \left(-\frac{\partial}{\partial z} \right)^n F(N_e,z) \ , \quad (6)$$

where

$$F(N_e,z) = \langle e^{-zN_eB} \rangle \ , \quad (7)$$

and the expectation value is with respect to the initial state (2). Proper normalization, i.e.,

$$\sum_{n=0}^{\infty} P_n(N_e) = 1 \ ,$$

is evident from Eqs. (6) and (7), because $F(N_e,0)=1$. Either from the general expression

$$N_e^n \langle B^n \rangle = \sum_{\nu_1=0}^{\infty} \cdots \sum_{\nu_{N_e}=0}^{\infty}{}_{(\sum_i \nu_i = n)} \left(\frac{n!}{\nu_i! \cdots \nu_{N_e}!} \right)^2 \ , \quad (8)$$

or by direct inspection of the definition of B, it is easily shown that, for $N_e \gg 1$,

$$\langle B^n \rangle = n! \left[1 + O\left(\frac{n}{N_e} \right) \right] \ . \quad (9)$$

We then find

$$F(N_e,z) = (1+N_e z)^{-1} \quad (10)$$

and, consequently,

$$P_n(N_e,z) = (1+N_e z)^{-1} \left(\frac{N_e z}{1+N_e z} \right)^n \ . \quad (11)$$

This expression is the thermal statistics mentioned in the introduction. The above derivation is justified whenever the mean number of emitted photons per electron and per mode is very small compared with unity, i.e., $z \ll 1$ [cf. paper I, Eq. (27)]. In fact, Eq. (11) is valid under more general conditions as we shall demonstrate below.

First, we want to reemphasize the physical model underlying our formalism: (1) Since we start from the Bambini-Renieri Hamiltonian we are restricted to one fixed (though arbitrary) mode of the electromagnetic field. Results, such as the spectrum and photon statistics of spontaneous emission, can be summed over various modes according to the resolution properties of the detector. However, there are no mode correlation effects. (2) The electron beam is described by a definite number N_e of electrons initially in identical momentum eigenstates. This implies a cw description of both the electrons and the electromagnetic field. The possible consequences of a more general formulation are discussed in Appendix C of I. (3) The direct electron-electron interaction is neglected.

We now turn to a more precise evaluation of the function $F(N_e,z)$ defined in Eq. (7). Recalling the definition [paper I, Eq. (8)] of the operator B, introducing $\lambda_i = \exp(ikz_i)$ and noting that, in view of $\langle \bar{p} | \bar{p} + \hbar k \rangle = 0$,

$$\langle \lambda_i^n \rangle = \delta_{n,0} \quad , \tag{12}$$

we can write

$$F(N_e,z) = \oint \frac{d\lambda_1 \cdots d\lambda_{N_e}}{(2\pi i)^{N_e} \lambda_1 \cdots \lambda_{N_e}} \exp\left[-z \sum_{i=1}^{N_e} \lambda_i \sum_{j=1}^{N_e} \lambda_j^{-1} \right] . \tag{13}$$

When the integration over λ_{N_e} is performed, we obtain a formula which relates $F(N_e,z)$ to $F(N_e-1,z)$:

$$\begin{aligned} F(N_e,z) &= \oint \frac{d\lambda_1 \cdots d\lambda_{N_e-1}}{(2\pi i)^{N_e-1} \lambda_1 \cdots \lambda_{N_e-1}} \sum_{\nu=0}^{\infty} \frac{1}{(\nu!)^2} \left[z^2 \sum_{i=1}^{N_e-1} \lambda_i \sum_{j=1}^{N_e-1} \lambda_j \right]^{\nu} \exp\left[-z - z \sum_{i=1}^{N_e-1} \lambda_i \sum_{j=1}^{N_e-1} \lambda_j^{-1} \right] \\ &= e^{-z} I_0 \left[2z \left[-\frac{\partial}{\partial z} \right]^{1/2} \right] F(N_e-1,z) \quad . \end{aligned} \tag{14}$$

Here I_0 denotes a modified Bessel function and the operators $\partial/\partial z$ in a power-series expansion of I_0 only act on the variable z in $F(N_e-1,z)$. Note that Eq. (14) is exact.

For $N_e \gg 1$, Eq. (14) turns into a condition to be imposed on $F(N_e,z)$. We can now check whether or not the previous result (10) derived for $z \ll 1$ satisfies Eq. (14). We realize that

$$e^{-z} I_0 \left[2z \left[-\frac{\partial}{\partial z} \right]^{1/2} \right] \frac{1}{1+N_e z} = \frac{1}{1+N_e z} \exp\left[-\frac{z}{1+N_e z} \right] .$$

Since the last exponential equals $1 - O(1/N_e)$ irrespective of the value of z as long as $N_e z \gg 1$, we take it for granted that the validity of Eq. (10) is not restricted by the condition that $z \ll 1$.

For small N_e, Eq. (14) can be used to generate the exact function $F(N_e,z)$. For example, we obtain for $N_e = 2$

$$F(2,z) = e^{-2z} I_0(2z) \quad , \tag{15}$$

since $F(1,z) = \exp(-z)$. It is instructive to compare the probabilities derived to lowest order in z from Eq. (15):

$$P_n(2,z) = 2z,\ 3z^2,\ 10z^3/3,\ 35z^4/12, \ldots \tag{16}$$

for $n = 1,2,3,4$, respectively, with the corresponding Poisson distribution[5] which would result for one electron radiating on the average twice as many photons,

$$P_n(1,2z) = 2z,\ 2z^2,\ 4z^3/3,\ 2z^4/3, \ldots \ . \tag{17}$$

Hence, already for just two electrons the photon number distribution differs significantly from a Poisson distribution. Comparison of Eqs. (16) and (17) makes clear that radiation from one electron stimulates radiation from the other so that the probability of several photons being emitted increases. Note, however, that the mean number of emitted photons, if evaluated exactly, would be $2z$ in either case.

The Poisson distribution has the unique property that if two (or more) events are independent and satisfy Poisson statistics so does their sum, viz.,

$$\sum_{m=0}^{n} P_m(1,z) P_{n-m}(1,\bar{z}) = P_n(1,z+\bar{z}) \quad . \tag{18}$$

This property is not shared by the thermal distribution (11). Any photodetector invariably has only a finite resolving power with respect to frequency. Hence, for $N_e = 1$ it would still detect a Poissonian distribution according to Eq. (18), since in our description there are no mode correlation effects so that the probabilities for the emission of photons with different frequencies are independent. On the other hand, for $N_e \gg 1$, the thermal distribution (11), since it refers to a single mode, can never be directly observed.

We shall finally write down the noise-to-signal ratios for spontaneous and stimulated emission. If we restrict ourselves to the leading terms we find, from Eqs. (32) and (35) of paper I, for $N = 0$,

$$\frac{[(\hbar\omega)^2 \Delta(N^2)]^{1/2}}{\hbar\omega\, \Delta N} = \left[\frac{1+(N_e-1)z}{N_e z} \right]^{1/2} . \tag{19}$$

This ratio tends to unity for $N_e z \gg 1$, as it should for a thermal distribution. On the other hand, for $N_e z \ll 1$ and, in particular, for $N_e = 1$, it is proportional to $z^{-1/2} \sim \hbar^{1/2}$, indicating that in this case the fluctuations in the photon number are due to the quantum-mechanical vacuum fluctuations. For stimulated emission ($N \gg N_e z$) we have

$$\frac{[(\hbar\omega)^2 \Delta(N^2)]^{1/2}}{\hbar\omega\, \Delta N} = \left[\frac{\hbar k^2}{2m} \left[\frac{(2N+1)N_e}{z} \right]^{1/2} z' \right]^{-1} , \tag{20}$$

where $z' = \partial z / \partial \beta$ is the derivative with respect to the detuning. Essentially the same results hold true, if the initial state of the laser field is not an eigenstate of the photon number, but a coherent state $|\alpha\rangle$ with $|\alpha|^2 = N$. The ratio (20) is independent of $\hbar$, since $N \sim \hbar^{-1}$ and $z \sim \hbar^{-1}$, and goes to zero for $N \to \infty$ or $N_e \to \infty$.

The crucial parameter in the above discussion is $N_e z$.

Transformed to the laboratory frame it is given by

$$N_e z = \frac{1}{2\gamma^2}\frac{e^2}{\hbar c}\left(\frac{ea_w}{mc^2}\right)^2 L^2\lambda\left(\frac{\sin(\Delta\omega\, T/2)}{\Delta\omega\, T/2}\right)^2\frac{N_e}{V} \quad . \qquad (21)$$

Here γ is related to the electron energy by $E = mc^2\gamma$, $e^2/\hbar c = \frac{1}{137}$, a_w is the amplitude of the circularly polarized wiggler field so that $(ea_w/mc^2)^2$ agrees with the frequently used parameter K^2, $L = \beta_0 T$ is the wiggler length, λ is the wavelength of the mode in question, $\Delta\omega$ is the detuning, and V is a normalization volume so that N_e/V is the actual electron density. Typical values of $N_e z$ range between $\sim 10^6$ for the Stanford experiment[6] and $\sim 10^3$ for the ACO (Orsay, France) experiment,[7] so that in any event $N_e z >> 1$ as required for thermal photon statistics.

Cooperative effects very similar to those discussed here for the FEL have been dealt with in the case of the ordinary laser in Ref. 8.

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[4]In doing this we drop all gain-related effects on photon statistics. For $N_e = 1$ it has been shown in Ref. 3 that a pure Poissonian results, the gain-related effect being photon bunching or antibunching, which is quantitatively minute. For $N_e >> 1$ this approximation is expected to be less accurate, since, e.g., amplified spontaneous emission is lost.

[5]Here and in the following we use $P_n(1,z) = z^n/n!\exp(-z)$ to denote a Poisson distribution. This is in accordance with Eq. (6) since $F(1,z) = \exp(-z)$.

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