Renormalization theory of the interacting Bose fluid

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We derive an approximate closed form for the infinitesimal generator of the renormalization group for the interacting Bose fluid. The Bose-condensed phase is treated by the method of Bogoliubov, and a simple scaling law is found for the condensate density. It is shown that the quantum-mechanical features are irrelevant close to the λ line. This is demonstrated by calculating the critical behavior in $4-\epsilon$ dimensions to order ϵ.

I. INTRODUCTION

The renormalization-group approach to phase transitions has proved successful in accounting for the universal nature of critical phenomena in classical systems. The system we study in this paper, the interacting Bose fluid, is inherently quantum mechanical, and it is therefore somewhat surprising that its critical exponents are given by those of a corresponding classical system.[1–13] Thus it appears that universality transcends the distinction between classical and quantum-mechanical systems.

The fact that such a corresponding classical system exists is a consequence of the collective behavior of a large number of Bose particles occupying the same quantum state. It should be emphasized that the corresponding classical system is not a system of classical particles, but a classical two-component field theory. Hence, the quantum-mechanical origins of the λ transition are manifest in the nature of the corresponding classical system but not in the numerical values of the critical exponents. A clear discussion of this point can be found in the paper of Lee.[11] By contrast, a system of interacting fermions cannot exhibit this type of collective behavior, and one therefore expects the critical properties of a Fermi system to be quite different.

In a series of papers[1–4] Singh applied diagrammatic methods to derive a set of renormalization equations which reproduce the standard results of the ϵ expansion.[14] The transition to the classical two-component field theory is understood in this context by observing that the boson mass scales to infinity after many renormalizations. A similar approach was followed by Stella and Toigo,[5] and by De Cesare and Busiello.[7]

Lee has constructed an alternative but equivalent theory to Singh's in which the basic boson field operators a are scaled according to $a\rightarrow\sqrt{\zeta}b$, where the scaling factor ζ is larger than unity. This leads to a commutation relation for the b operators $[b,b^\dagger]=\zeta^{-1}$. Hence, after many renormalizations this commutator becomes vanishingly small and the theory goes over to a classical field theory.

The contents of this paper are organized as follows. In Sec. II the functional integral for the grand canonical partition function is introduced, and the formal renormalization equations are derived. This work is the natural extension to quantum systems of the method of Houghton and Wegner.[15] In Sec. III we apply the "separation-of-scales" approximation to obtain a closed form for the renormalization equations. In addition, an expression for the contribution to the canonical pressure arising from the renormalization process is found.

In Sec. IV we consider the question of Bose condensation. It is shown that the renormalization of the condensate wave function decouples from the renormalization of the potential, and leads to a simple scaling law for the condensate density. The remaining equation for the potential function is identical to that obtained in the normal phase.

The fixed point equation is considered in Sec. V, and specific results are obtained for $D=4-\epsilon$ to $O(\epsilon)$. As observed by several authors[1–13] the fixed point and critical exponents for this system correspond to those of a two-component classical field theory. The linear dynamics around the fixed point is solved completely to $O(\epsilon)$ with the result that the scaling fields are simple polynomials in $|\phi|^2$ and the spectrum of the linearized transformation is discrete with a single negative or "relevant" eigenvalue. In Sec. VI we present our conclusions and discuss possible applications of this work.

II. THE INFINITESIMAL RENORMALIZATION TRANSFORMATION

The grand canonical partition function $Z(\beta,\mu)$ for the interacting Bose fluid may be represented as a functional integral over the complex scalar field $\psi(\vec{x},\tau)$ (for derivations see the reviews in Refs. 9, 16, and 17 and the papers quoted there):

$$Z(\beta,\mu)=N(\beta,\mu)\int D[\psi]e^{-S[\psi;\beta,\mu]}\ , \tag{2.1}$$

where in our model we take

$$S[\psi;\beta,\mu]=\int d^Dx\int_0^\beta d\tau\left[\psi^*\frac{\partial\psi}{\partial\tau}+\frac{\hbar^2}{2m}|\vec{\nabla}\psi|^2-\mu|\psi|^2+\frac{g}{2}|\psi|^4\right]. \tag{2.2}$$

The factor $N(\beta,\mu)$ is a normalization constant

$$\frac{1}{\Omega}\ln N(\beta,\mu) = -\frac{V_D(\Lambda)}{(2\pi)^D}\frac{\beta\mu}{2} - \int_0^\Lambda d^D k \ln(1-e^{-\beta\epsilon_k^{(0)}}) . \tag{2.3}$$

In the above equations β and μ are, as usual, $(k_B T)^{-1}$ and the chemical potential, respectively. In addition, m is the boson mass, g is the effective coupling constant, Λ is the momentum cutoff, Ω is the volume, $V_D(\Lambda)$ is the volume of a D sphere of radius Λ, and $\epsilon_k^{(0)}$ is the free-particle energy $\epsilon_k^{(0)} = \hbar^2 k^2/2m$.

For simplicity we shall define dimensionless variables, taking Λ^{-1} as the unit of length, $\hbar^2\Lambda^2/m$ as the unit of energy, and defining a dimensionless temperature parameter $\theta = \tau/\beta$, and a new field variable $\phi = \Lambda^{-(D-2)/2}(\hbar^2\beta/m)^{1/2}\psi$. In terms of these dimensionless quantities, the action is

$$S = \int d^D x \int_0^1 d\theta \left[\frac{1}{\beta}\phi^* \frac{\partial\phi}{\partial\phi} + \tfrac{1}{2}|\vec{\nabla}\phi|^2 - \mu|\phi|^2 + \frac{g}{2}|\phi|^4 \right] . \tag{2.4}$$

Following Houghton and Wegner,[15] we separate the field into two components $\phi^{(S)}$ and $\phi^{(L)}$, where $\phi^{(S)}$ has Fourier components in a thin shell near the cutoff (which is now unity) $\phi_{\vec{k}}^{(S)}$ for $(1-\eta \le |\vec{k}| 1)$ and $\phi_{\vec{k}}^{(L)}$ has Fourier components lying in the interior of the sphere $\phi_{\vec{k}}^{(L)}$ for $(0 \le |\vec{k}| < 1-\eta)$.

The first step in the renormalization program is to "integrate out" the rapidly varying fields $\phi^{(S)}$. This yields a new form for the interaction potential of the slowly varying fields $\phi^{(L)}$, and a new momentum cutoff $1-\eta$. The potential may be expanded in a functional Taylor series about $\phi^{(L)}$,

$$U[\phi^{(L)}+\phi^{(S)}] = U[\phi^{(L)}] - \frac{\delta U}{\delta\phi_\alpha}[\phi^{(L)}]\phi_\alpha^{(S)} + \frac{1}{2}\frac{\delta^2 U}{\delta\phi_\alpha\delta\phi_\beta}[\phi^{(L)}]\phi_\alpha^{(S)}\phi_\beta^{(S)} + \cdots . \tag{2.5}$$

In (2.5) a two-component vector of fields is introduced

$$\phi_\alpha = \begin{cases} \phi, & \alpha=1 \\ \phi^*, & \alpha=2 \end{cases} \tag{2.6}$$

as well as an extended summation convention in which repeated indices are summed and the associated coordinates (which we do not exhibit explicitly) are integrated over. Thus, for example,

$$A_\alpha B_\alpha \equiv \int_0^1 d\theta \int d^D x [A(\vec{x},\theta)B(\vec{x},\theta) + A^*(\vec{x},\theta)B^*(\vec{x},\theta)] . \tag{2.7}$$

Our original functional integral is now

$$Z(\beta,\mu) = N(\beta,\mu)\int D[\phi^{(L)}]e^{-S[\phi^{(L)}]} \times \int D[\phi^{(S)}]e^{-S_0[\phi^{(S)}]}e^{-U'[\phi^{(L)};\phi^{(S)}]}, \tag{2.8}$$

where we have defined $U'[\phi^{(L)};\phi^{(S)}] = U[\phi^{(L)}+\phi^{(S)}] - U[\phi^{(L)}]$ and S_0 is the quadratic form

$$S_0 = \int d^D x \int_0^1 d\theta \left[\frac{1}{\beta}\phi^* \frac{\partial\phi}{\partial\theta} + \tfrac{1}{2}|\vec{\nabla}\phi|^2 \right] . \tag{2.9}$$

Hence the problem of evaluating the integral over the fields $\phi^{(S)}$ requires that we calculate $\langle e^{-U'[\phi^{(L)};\phi^{(S)}]}\rangle_0$, where the average is taken with respect to the Gaussian weight functional $e^{-S_0[\phi^{(S)}]}$. Such an average is given by $\langle e^{-U'[\phi^{(L)};\phi^{(S)}]}\rangle = e^{\mathscr{C}}$, where $\mathscr{C}$ is the sum of all closed connected diagrams[18] formed from the Green function, or propagator,

$$G_{\alpha\beta} = \langle \phi_\alpha \phi_\beta \rangle_0 \tag{2.10}$$

and the vertex functions $\delta^n U/\delta\phi^n$ which arise in the Taylor expansion (2.5).

The sum of the closed connected diagrams is also often referred to as the "cumulant expansion."[9] A connected diagram is one in which all parts of the diagram are connected by at least one propagator line. The simplest such diagrams may be constructed from the first two terms of the series (2.5). These are the "string diagrams," Fig. 1(a), and "ring diagrams," Fig. 1(b). In Fig. 1 the solid line represents a Green function G, the open circle a factor $\delta U/\delta\phi$, and the shaded circle $\delta^2 U/\delta\phi^2$. In addition to these diagrams, one can imagine more complicated diagrams arising from vertex functions with more than two incoming propagator lines. However, since each such incoming line introduces a new momentum summation, and each summation is over the momentum shell of width η at the cutoff, these more complicated diagrams contribute to $O(\eta^2)$ and shall be neglected. Thus we need retain only the linear and quadratic terms in the Taylor expansion (2.5). This leaves a Gaussian functional integral to perform, which we write as

$$e^{\mathscr{C}} = \frac{\int D[\phi^{(S)}]\exp\left[-\tfrac{1}{2}\phi_\alpha^{(S)}(G^{-1})_{\alpha\beta}\phi_\beta^{(S)} - \frac{\delta U}{\delta\phi_\alpha}[\phi^{(L)}]\phi_\alpha^{(S)} - \frac{1}{2}\frac{\delta^2 U}{\delta\phi_\alpha\delta\phi_\beta}[\phi^{(L)}]\phi_\alpha^{(S)}\phi_\beta^{(S)} \right]}{\int D[\phi^{(S)}]\exp[-\tfrac{1}{2}\phi_\alpha^{(S)}(G^{-1})_{\alpha\beta}\phi_\beta^{(S)}]} . \tag{2.11}$$

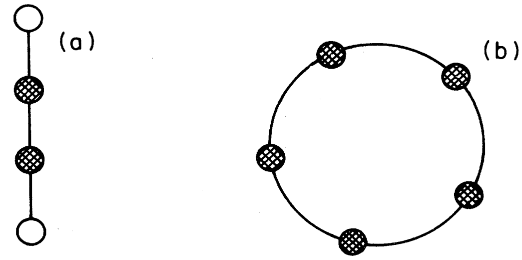

FIG. 1. Simplest closed connected diagrams in the cumulant expansion.

The linear term in the integral may be eliminated by "completing the square," and the remaining purely Gaussian functional integral is easily evaluated to give

$$\mathscr{C}=-\frac{1}{2}\frac{\delta U}{\delta\phi_\alpha}\left[(G^{-1})_{\alpha\beta}+\frac{\delta^2 U}{\delta\phi_\alpha\delta\phi_\beta}\right]^{-1}\frac{\delta U}{\delta\phi_\beta}-\tfrac{1}{2}\operatorname{tr}\ln\left[1-G_{\alpha\beta}\frac{\delta^2 U}{\delta\phi_\beta\delta\phi_\alpha}\right]. \tag{2.12}$$

Note that (2.11) has been written as the ratio of two functional integrals. This is a convenient device as we are about to rescale the fields, and the constants which arise from this change of variable will cancel in numerator and denominator.

The renormalization process is completed by restoring the momentum cutoff, which now lies at $1-\eta$, to its original value of unity. The functional integral is then rewritten in terms of the rescaled variables

$$\begin{aligned}\vec{k}&=(1-\eta)\vec{k}^{(R)}\,,\\ \vec{x}&=(1-\eta)^{-1}\vec{x}^{(R)}\,,\\ \phi&=(1-\eta)^{(D-2)/2}\phi^{(R)}\,,\\ \beta&=(1-\eta)^{-2}\beta^{(R)}\,.\end{aligned} \tag{2.13}$$

The renormalized interaction is given, to $O(\eta)$, by the expression

$$U^{(R)}-U=\eta\left[DU-\frac{D-2}{2}\frac{\delta U}{\delta\phi_\alpha}\phi_\alpha\right]-\frac{1}{2}\frac{\delta U}{\delta\phi_\alpha}\left[(G^{-1})_{\alpha\beta}+\frac{\delta^2 U}{\delta\phi_\alpha\delta\phi_\beta}\right]^{-1}\frac{\delta U}{\delta\phi_\beta}+\tfrac{1}{2}\operatorname{tr}\ln\left[1+G_{\alpha\beta}\frac{\delta^2 U}{\delta\phi_\beta\delta\phi_\alpha}\right]. \tag{2.14}$$

The first two terms arise from rescaling of the coordinates and fields, respectively, the third term is the sum of string diagrams, and the last term is the sum of ring diagrams.

III. SEPARATION OF SCALES AND AN APPROXIMATE CLOSED FORM FOR THE RENORMALIZATION EQUATION

The result (2.14) is purely formal in the sense that one has no hope of carrying out the indicated operations for a general functional $U[\phi]$. To proceed we make the following approximation, which will be called the separation of scales. If $f[\phi^{(L)}(x)]$ is an implicit function of $x\equiv(\vec{x},\theta)$ through $\phi^{(L)}$, then we approximate

$$f[\phi^{(L)}(x)]G(x-y)\cong G(x-y)f[\phi^{(L)}(y)]\,. \tag{3.1}$$

The assumption here is that G, which contains momentum components in a thin shell near the cutoff, varies rapidly in comparison to $\phi^{(L)}(x)$ and simple functions of $\phi^{(L)}$.

It is difficult to formally assess the validity of this approximation, but one may argue heuristically that very near the critical point the behavior of the system is dominated by very long-wavelength fluctuations, which supports (3.1) so far as the coordinate dependence is concerned. We must, however, also consider the θ dependence of G and $\phi^{(L)}$. The quantum variable θ enters essentially through the purely imaginary term $(1/\beta)\int_0^1 d\theta\,\phi^*(\partial\phi/\partial\theta)$ in the thermodynamic action. If this term were absent the variable θ would be superfluous and one could rewrite the functional integral in terms of $\vec{x}$-dependent fields only. In fact, the structure of our renormalization transformation has just this effect. By (2.13) we see that the parameter β rescales according to $\beta=(1-\eta)^{-2}\beta^{(R)}$. This shows that after many renormalization β tends to zero and the "quantum term" becomes rapidly varying for θ-dependent fields. Thus as $\beta\to 0$ only θ-independent fields contribute to the functional integral.

In addition it is known that "momentum-independent" approximations such as (3.1) are valid to first order in $\epsilon=4-D$. Therefore, if by such an approximation one can derive a closed form for the renormalization equation, one may regard it as a model with which to test the ϵ expansion or other techniques which may be intractable when applied to the exact renormalization equation.

Thus we regard the separation-of-scales approximation in two ways. First, it gives the renormalized potential correct to first order in ϵ, and this is discussed more fully in Sec. V. Second, it yields a compact, closed form for the renormalization equation which is of intrinsic interest.

If one applies (3.1) to the sum of string and ring diagrams one finds that the string diagrams vanish and the sum of ring diagrams becomes

$$-\tfrac{1}{2}\operatorname{tr}\ln\left[1+G_{\alpha\gamma}\frac{\delta^2 U}{\delta\phi_\gamma\delta\phi_\beta}\right]\cong\tfrac{1}{2}\int dx\frac{1}{(2\pi)^D}\int d^Dk\sum_{n=-\infty}^{\infty}\operatorname{tr}\ln\left[1+\frac{\partial^2 u(x)}{\partial\phi_\alpha\partial\phi_\beta}G_{\beta\gamma}(\vec{k},n)\right]. \tag{3.2}$$

In (3.2) the symbol tr indicates the trace of the 2×2 matrix, and we have assumed that U is a local function of $\phi(x)$, $U=\int dx\, u[\phi(x)]$. In fact, for a matrix M, $\mathrm{tr}\ln M=\ln\det M$, which gives for (3.2)

$$\mathrm{tr}\ln\left[1+\frac{\partial^2 u}{\partial\phi_\alpha\partial\phi_\beta}G_{\beta\gamma}\right]=\ln\{1+u_{12}(x)[G(\vec{k},n)+G^*(\vec{k},n)]+[u_{12}^2(x)-|u_{11}|^2(x)]\,|G(\vec{k},n)|^2\}\ , \tag{3.3}$$

where $G(\vec{k},n)=(\frac{1}{2}-2\pi in/\beta)^{-1}$. Observing that $G+G^*=|G|^2$ we have

$$\mathrm{tr}\ln\left[1+\frac{\partial^2 u}{\partial\phi_\alpha\partial\phi_\beta}G_{\beta\gamma}\right]=\ln[1+(u_{12}+u_{12}^2-|u_{11}|^2)\,|G|^2]\ . \tag{3.4}$$

In (3.2) the integral over the $\vec{k}$ vectors is trivial, yielding simply a factor which is the volume of a unit spherical shell of thickness η in D dimensions, $\eta\sigma_D$ (σ_D is the area of a unit D sphere). The remaining sum over the Matsubara frequencies is of the form $\sum_{n=-\infty}^{\infty}\ln\{1+[A^2/B^2+(2\pi n)^2]\}$. This sum may be represented in the usual way by a contour integral in the complex z plane

$$\sum_{n=-\infty}^{\infty}\ln\left[1+\frac{A^2}{B^2+(2\pi n)^2}\right]=\frac{1}{2\pi i}\oint_\gamma dz\frac{1}{1-e^{-z}}\ln\left[1+\frac{A^2}{B^2-z^2}\right]\ . \tag{3.5}$$

The contour γ is shown in Fig. 2. We may deform γ to γ' which encircles the two branch cuts of the logarithm. This yields

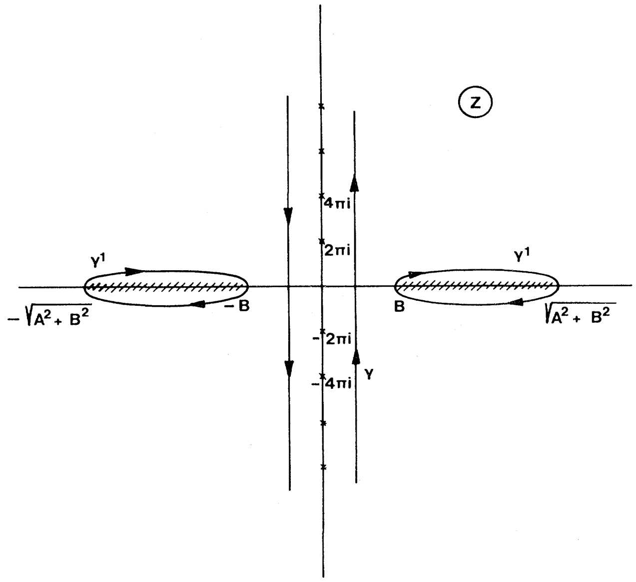

FIG. 2. Contours in the complex z plane used to evaluate the sum (3.5).

$$\sum_{n=-\infty}^{\infty}\ln\left[1+\frac{A^2}{B^2+(2\pi n)^2}\right]=\int_B^{(A^2+B^2)^{1/2}}dz\frac{1}{1-e^{-z}}-\int_{-(A^2+B^2)^{1/2}}^{-B}dz\frac{1}{1-e^{-z}}=2\ln\frac{\sinh[\frac{1}{2}(A^2+B^2)^{1/2}]}{\sinh\frac{B}{2}}\ . \tag{3.6}$$

Returning to (3.2) we have for the sum of ring diagrams

$$-\tfrac{1}{2}\,\mathrm{tr}\ln\left[1+G\frac{\delta^2 U}{\delta\phi^2}\right]\simeq-\eta\frac{\sigma_D}{(2\pi)^D}\int dx\,\ln\frac{\sinh\{(\beta/4)[1+4(u_{12}+u_{12}^2-|u_{11}|^2)]^{1/2}\}}{\sinh(\beta/4)}\ . \tag{3.7}$$

In the important special case that u is a function of $|\phi|^2$ only, the renormalized potential is

$$u^{(R)}=u+\eta\left[Du-(D-2)|\phi|^2u'+\frac{\sigma_D}{(2\pi)^D}\ln\frac{\sinh\{(\beta/4)[(1+2u')(1+2u'+4|\phi|^2u'')]^{1/2}\}}{\sinh\{(\beta/4)[1+2u'(\phi=0)]\}}\right]\ . \tag{3.8}$$

The prime in (3.8) and following equations indicates differentiation with respect to the variable $|\phi|^2$. In (3.8) the part of the renormalized interaction which is independent of ϕ has been subtracted out so that if $u(\phi=0)=0$ then $u^{(R)}(\phi=0)=0$ as well. These constant terms may be taken out of the functional integral and their sum yields a part of the canonical pressure.

We may now take the limit $\eta\to 0$. The scale variable $t_n=(1-\eta)^n$ becomes a continuous parameter t in the limit $\eta\to 0$ and (3.8) becomes

$$-t\frac{\partial u}{\partial t}=Du-(D-2)|\phi|^2u'+\frac{\sigma_D}{(2\pi)^D}\ln\frac{\sinh\{(\beta t^2/4)[(1+2u')(1+2u'+4|\phi|^2u'')]^{1/2}\}}{\sinh\{(\beta t^2/4)[1+2u'(\phi=0,t)]\}}\ . \tag{3.9}$$

Collecting all the "constant" terms mentioned above, and the normalization constant (2.3), we find for the canonical pressure p,

$$p=-\frac{\sigma_D}{(2\pi)^D}\left[\frac{\mu}{2D}+\frac{1}{\beta}\int_0^1 dt\,t^{D-1}\ln(1-e^{-\beta t^2/2})+\frac{1}{\beta}\int_0^1 dt\,t^{D-1}\ln\frac{\sinh\{(\beta t^2/4)[1+2u'(0,t)]\}}{\sinh(\beta t^2/4)}\right]\ . \tag{3.10}$$

Equation (3.9) is a closed-form approximation to the renormalization equation with the following fortuitous features. If U is initially a local function of the field ϕ, then it remains local under the action of the renormalization transformation. In addition, if U is locally gauge invariant, as is appropriate for the normal phase of the Bose system, then this property is also preserved under renormalization.

In the limit of infinitely many renormalizations $t \to 0$ and, assuming that u remains finite, the renormalization equations take the simpler form

$$-t\frac{\partial u}{\partial t}=Du-(D-2)|\phi|^2u'+\frac{1}{2}\frac{\sigma_D}{(2\pi)^D}\ln\frac{(1+2u')(1+2u'+4|\phi|^2u'')}{[1+2u'(0,t)]^2} . \tag{3.11}$$

Note that the right-hand side is independent of t; this is essential for the existence of a fixed point function u^*. Such a fixed point, if it exists, satisfies the ordinary second-order nonlinear differential equation

$$Du^*-(D-2)|\phi|^2u^{*\prime}+\frac{1}{2}\frac{\sigma_D}{(2\pi)^D}\ln\frac{(1+2u^{*\prime})(1+2u^{*\prime}+4|\phi|^2u^{*\prime\prime})}{[1+2u^{*\prime}(0)]^2}=0 . \tag{3.12}$$

In Sec. IV we shall solve (3.12) and the linearized version of (3.11) to first order in $\epsilon=4-D$.

IV. THE CONDENSED PHASE

In the low-temperature phase of the interacting Bose fluid the order parameter $\langle\phi\rangle$ is different from zero, and the equilibrium state is no longer gauge invariant. Rather than introducing a "source term" which couples linearly to the fields in analogy with an external magnetic field for a spin system, we introduce symmetry breaking by modifying the form of the potential according to the Bogoliubov prescription[19]

$$u(\phi;\phi_0)=f(|\phi+\phi_0|^2)-f(|\phi_0|^2) . \tag{4.1}$$

In (4.1) the Bose field ϕ has been replaced by $\phi=\phi_1+\phi_0$ where $\phi_0\equiv\langle\phi\rangle$ and ϕ_1 may be thought of as the fluctuation of the field about its ensemble average ϕ_0. In (4.1) and all subsequent equations the subscript is dropped from ϕ_1 since ϕ_1 is simply the random field over which we integrate.

Since (4.1) is not gauge invariant one must use the more general form of our renormalization equation

$$-t\frac{\partial u}{\partial t}=Du-\frac{D-2}{2}\phi_\alpha u_\alpha+\frac{\sigma_D}{(2\pi)^D}\ln\frac{\sinh\{(\beta t^2/4)[1+4(u_{12}+u_{12}^2-|u_{11}|^2)]^{1/2}\}}{\sinh((\beta t^2/4)\{1+4[u_{12}(0,t)+u_{12}^2(0,t)-|u_{11}(0,t)|^2]\}^{1/2})} . \tag{4.2}$$

The "condensate wave function" ϕ_0 must scale according to (2.13), which gives

$$\phi_0(t)=\sqrt{n_0}\,t^{-(D-2)/2} . \tag{4.3}$$

Indeed, if one inserts the Bogoliubov form of the potential into the renormalization equation, consistency requires that ϕ_0 satisfy

$$-t\frac{\partial\phi_0}{\partial t}=\left[\frac{D-2}{2}\right]\phi_0 \tag{4.4}$$

and the unknown function f is given by

$$-t\frac{\partial f}{\partial t}=Df-(D-2)|\phi|^2f'+\frac{\sigma_D}{(2\pi)^D}\ln\frac{\sinh\{(\beta t^2/4)[(1+2f')(1+2f'+4|\phi|^2f'')]^{1/2}\}}{\sinh\{(\beta t^2/4)[1+2f'(0)]\}} . \tag{4.5}$$

The above equation for f, from which we construct by (4.1) the renormalized potential in the broken symmetry phase, is identical to the renormalization equation for the potential in the gauge-invariant, or normal phase. Thus the solutions to the simpler gauge-invariant form of the renormalization equation for initial parameters on either side of the phase boundary are sufficient to determine the renormalized potential everywhere in the phase plane.

The term in the canonical pressure which arises from renormalization is, however, different in the broken symmetry phase, and is given by

$$P_s=-\frac{\sigma_D}{(2\pi)^D}\frac{1}{\beta}\int_0^1 dt\, t^{D-1}\frac{\sinh((\beta t^2/4)\{[1+2f'(|\phi_0|^2,t)][1+2f'(|\phi_0|^2,t)+4|\phi_0|^2f''(|\phi_0|^2,t)]\}^{1/2})}{\sinh(\beta t^2/4)} . \tag{4.6}$$

The condensate density n_0 is fixed by requiring that the canonical pressure by maximal (cf. Ref. 18),

$$\frac{\partial p}{\partial n_0}(\beta,\mu,n_0)=0 . \tag{4.7}$$

By (4.3) one sees that n_0 is a "relevant" variable, and at the fixed points of the renormalization transformation $n_0^*=0$. Thus we may conclude that the phase transition we are studying is characterized by the onset of Bose condensation.

Very close to the critical point one may approximate in (4.5) $f'(n_0,t) \cong f'(0,t)+n_0 f''(0,t)$ and $n_0 f''(n_0 t) \cong n_0 f''(0,t)$. The pressure now contains the two relevant scaling fields $r(t)=f'(0,t)-f'^*(0)$ and $n_0(t)$. Very near the fixed point, $r(t)$ is assumed to have the behavior $r(t)=r_0 t^{-1/\nu}$. Note that we do not associate the critical exponent ν with the temperature variable, but with a variable which is essentially the chemical potential. This point has been discussed by Busiello and De Cesare,[13] the essential conclusion being that the usual two-parameter universal scaling laws hold provided one expresses all thermodynamic quantities in terms of r rather than $|T-T_c|$.

The singular part p_s of the canonical pressure is the contribution to the integral (4.5) from the neighborhood of the origin. In this neighborhood the integrand can be written as a function h of the two relevant scaling fields $r_0 t^{-1/\nu}$ and $n_0 t^{-(D-2)}$

$$p_s(r,n_0) \sim \int_0^\delta dt\, t^{D-1} h(r_0 t^{-1/\nu}, n_0 t^{-(D-2)}) \ . \tag{4.8}$$

If we now rescale r to λr, we see that

$$p_s(\lambda r, n_0) = \lambda^{\nu D} p_s(r, \lambda^{(D-2)\nu} n_0) \ . \tag{4.9}$$

The condition (4.7) that p be maximal with respect to n_0 immediately yields

$$n_0(r) = c r^{2\beta} \ , \tag{4.10}$$

where c is a constant and β, the critical exponent for the order parameter, is given by

$$\beta = \frac{D-2}{2}\nu \ . \tag{4.11}$$

By the scaling relation $2\beta - \eta\nu = (D-2)\nu$ we see that, as is generally the case with "momentum-independent" approximations,[20,21] the critical exponent η vanishes.

V. SOLUTION NEAR FOUR DIMENSIONS

To illustrate our results thus far we shall solve the fixed point equation (3.12) to $O(\epsilon)$ where, as usual, $\epsilon = 4 - D$. This is possible because, as is well known, for $D=4$ the Wilson-Fisher fixed point and the trivial Gaussian fixed point coincide. Near four dimensions we expect the fixed point potential to be of $O(\epsilon)$, and so we expand the logarithm to second order in powers of the potential. To simplify the equations somewhat write

$$u(|\phi|^2) = \frac{\sigma_D}{(2\pi)^D} f\left[\frac{(2\pi)^D}{\sigma_D}|\phi|^2\right], \qquad y = \frac{(2\pi)^D}{\sigma_D}|\phi|^2 \ . \tag{5.1}$$

The renormalization equation in the limit $t \ll 1$ (after many renormalizations) becomes

$$-t\frac{\partial f}{\partial t} = Df - (D-2)y\frac{\partial f}{\partial y} + \tfrac{1}{2}\ln \frac{\left[1+2\dfrac{\partial f}{\partial y}\right]\left[1+2\dfrac{\partial f}{\partial y}+4y\dfrac{\partial^2 f}{\partial y^2}\right]}{\left[1+2\dfrac{\partial f(0,t)}{\partial y}\right]^2} \ . \tag{5.2}$$

Expanding the logarithm to second order, one finds

$$-t\frac{\partial f}{\partial t} = Df - (D-2)y\frac{\partial f}{\partial y} + 2\left[\frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2}\left[\frac{\partial f}{\partial y}\right]^2 - 2y\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial y^2} - 2y^2\left[\frac{\partial^2 f}{\partial y^2}\right]^2 - \frac{\partial f}{\partial y}(0) + \left[\frac{\partial f(0)}{\partial y}\right]^2\right] \ . \tag{5.3}$$

Assuming that the fixed point function is analytic near $y=0$ one can expand it in powers of y. One finds that the coefficients of the first two terms are of $O(\epsilon)$ while the coefficients of all powers of y greater than two are at least of $O(\epsilon^3)$. Thus we write

$$f(y,t) = r(t)y + \tfrac{1}{2}s(t)y^2 \ . \tag{5.4}$$

Applying (5.3) to (5.4) one finds the following equations for r and s:

$$-t\frac{\partial r}{\partial t} = 2(1-4s)r + 4s \tag{5.5}$$

and

$$-t\frac{\partial s}{\partial t} = \epsilon s - 20 s^2 \ . \tag{5.6}$$

These equations have the nontrivial fixed point $s^* = \epsilon/20$ and $r^* = -\epsilon/10$. One may solve (5.6) directly by the substitution $s = 1/W$,

$$s(t) = \frac{s^*}{1+(s^*/s_0 - 1)t^\epsilon} \ . \tag{5.7}$$

Thus s always approaches its value at the fixed point in the limit $t \to 0$ and is "irrelevant." Replacing s by s^* in (5.5) we find for r

$$r(t) = r^* + (r_0 - r^*)t^{-2(1-\epsilon/5)} \ . \tag{5.8}$$

By our argument of Sec. IV the exponent in (5.8) can be identified with ν^{-1}, which gives to $O(\epsilon)$

$$\nu = \tfrac{1}{2} + \frac{\epsilon}{10} \ . \tag{5.9}$$

Using (4.10) one finds for the order-parameter exponent

$$\beta = \tfrac{1}{2} - \tfrac{3}{20}\epsilon \ . \tag{5.10}$$

These are just the usual values of β and ν for a classical two-component field theory.[14]

In lowest order of ϵ the complete linear dynamics around the fixed point can be calculated. In general, if we write $f(y,t) = f^* + g(y,t)$, and keep terms linear in g, we have

$$-t\frac{\partial g}{\partial t}=(Dg-(D-2)y\frac{\partial g}{\partial y}+\left[\frac{1}{1+2\frac{\partial f^*}{\partial y}}+\frac{1}{1+2\frac{\partial f^2}{\partial y}+4y\frac{\partial^2 f^*}{\partial y^2}}\right]\frac{\partial g}{\partial y}+\frac{2y}{1+2\frac{\partial f^*}{\partial y}+4y\frac{\partial^2 f^*}{\partial y^2}}\frac{\partial^2 g}{\partial y^2}-\frac{\frac{\partial g(0,t)}{\partial y}}{1+2\frac{\partial f^*(0)}{\partial y}}\;. \tag{5.11}$$

Using the fixed point solution, and keeping only terms of $O(\epsilon)$, one finds that g satisfies

$$-t\frac{\partial g}{\partial t}=Dg-(D-2)y\frac{\partial g}{\partial y}+\frac{2\frac{\partial g}{\partial y}}{1+2r^*}-\frac{8s^*}{1+2r^*}y\frac{\partial g}{\partial y}+\frac{2}{1+2r^*}y\frac{\partial^2 g}{\partial y^2}-\frac{12s^*}{1+2r^*}y^2\frac{\partial^2 g}{\partial y^2}-\frac{2\frac{\partial g(0,t)}{\partial y^2}}{1+2r^*}\;. \tag{5.12}$$

The linear scaling fields $g_\lambda(y,t)$ are eigenfunctions of the linear operator on the right-hand side of the last equation, and have the simple t dependence $g_\lambda(t,y)=t^\lambda g_\lambda(y)$. If one assumes that $g_\lambda(y)$ has a power-series expansion

$$g_\lambda(y)=\sum_{n=1}^{\infty}\frac{a_n}{n!}y^n \tag{5.13}$$

one then finds the recursion formula for the a's;

$$a_{n+1}=-\frac{1+2r^*}{2}\frac{1}{n+1}\times\left[D+\lambda-n(D-2)-\frac{8s^*}{1+2r^*}n-\frac{12s^*}{1+2r^*}n(n-1)\right]a_n\;. \tag{5.14}$$

The coefficients grow for large n and the series diverges unless for some N, $a_{N+1}=0$. This gives for the eigenvalue

$$\lambda_N=N(D-2)-D+\frac{4s^*}{1+2r^*}N(3N-1) \tag{5.15}$$

and the eigenfunctions will be simple polynomials of order N. The first two are

$$\lambda_1=-2\left[1-\frac{\epsilon}{5}\right],\quad g_1(y)=y\;, \tag{5.16}$$

and

$$\lambda_2=\epsilon,\quad g_2(y)=y-\tfrac{1}{2}\left[1+\frac{\epsilon}{10}\right]y^2 \tag{5.17}$$

which agrees with the exact solutions (5.7) and (5.8). For higher-order polynomials the eigenvalues are larger and positive, and therefore, as expected, irrelevant.

VI. CONCLUSIONS

In this paper the infinitesimal renormalization transformation has been derived from the functional integral for the quantum Bose fluid in a way which closely parallels the work of Houghton and Wegner for classical systems. By applying the separation-of-scales or momentum-independent approximation one is able to obtain a closed form for the renormalization equation. In the limit of many renormalizations the quantum aspects of the system disappear, and the renormalization transformation takes a form very similar to that found by Nicoll, Chang, and Stanley for classical systems in the momentum-independent approximation. The validity of the separation of scales approximation and the irrelevance of quantum mechanics to the critical exponents is verified by working out the solution to first order in $\epsilon=4-D$ dimensions.

The canonical pressure is represented as an integral of a function of the renormalized potential over the renormalization trajectory. In this way, one can view the renormalization procedure as a means of calculating the functional integral.

The closed-form approximation of the infinitesimal renormalization transformation is also of interest as a means of examining in analytic detail the properties of renormalization trajectories. Using this as a model, one may address questions such as the convergence of the ϵ expansion or the effect of higher-order fixed points on the renormalization trajectories. It is in this direction that the authors feel the present work will prove most useful.

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