Quantum statistics of parametric oscillation

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We present a quantum-statistical analysis of the steady-state light fields in driven parametric oscillation in a cavity. Using the solution of a Fokker-Planck equation for the generalized P representation of the signal and idler modes, we calculate the mean photon numbers, second-order correlation functions, and intermode correlation function as functions of the driving field. The generalized P distribution describes the signal and idler modes' statistics over the whole range of driving-field strengths except in the region far below the oscillation threshold. The second-order correlation functions are found to violate the Cauchy-Schwartz inequality, a violation allowed because the P distribution is complex. Squeezing is also found in a linear combination of the signal and idler fields.

I. INTRODUCTION

Over the last two decades there has been great interest in the field of nonlinear optics, for both the practical applications and the theoretical aspects of the nonlinear effects possible. Among the more popular processes studied are second harmonic and subharmonic generation, and the corresponding nondegenerate processes, frequency upconversion and parametric oscillation.[1-8]

Here we shall investigate the quantum statistics of the steady-state signal fields in an intracavity parametric oscillator. One of the problems in the study of the quantum statistics is the divergence of the quasiprobability functions normally used. We avoid this problem by using the generalized probability functions introduced by Drummond and Gardiner.[9]

II. MODEL AND EQUATIONS OF MOTION

We assume the parametric oscillation will occur in a suitable medium inside an optical cavity tuned to allow three modes of the light field of frequencies ω_1, ω_2, and ω_3 with $\omega_1=\omega_2+\omega_3$. Mode 1 is pumped by an external laser at frequency ω_1. Modes 2 and 3 are the signal and idler modes.

The interaction Hamiltonian for this process is

$$H_{\rm int}=i\hbar K(a_1a_2^\dagger a_3^\dagger - a_1^\dagger a_2 a_3)\ , \tag{2.1}$$

where a_i and $a_i^\dagger$ are the annihilation and creation operators for mode i. The coupling constant K is proportional to the second-order susceptibility of the medium and to $(e^{\Delta\vec{k}\cdot\vec{L}}-1)/(\Delta\vec{k}\cdot\vec{L})$, where L is the path length in the medium and $\Delta\vec{k}=\vec{k}_1-(\vec{k}_2+\vec{k}_3)$ is the wave-number mismatch between the modes.

The pumping of mode 1 is described by the following interaction picture Hamiltonian:

$$H_{\rm pump}=i\hbar\epsilon_1(a_1^\dagger - a_1)\ , \tag{2.2}$$

where ϵ_1 is proportional to the amplitude of the driving laser field. Irreversible damping of the modes is represented by the Hamiltonian

$$H_{\rm irrev}=\sum_{i=1}^{3}(a_i\Gamma_i^\dagger + a_i^\dagger\Gamma_i)\ , \tag{2.3}$$

where the Γ_i and $\Gamma_i^\dagger$ are heat-bath operators.

Using standard techniques to eliminate the heat baths,[10] we find the following interaction picture equation of motion for the reduced density operator of the three modes:

$$\frac{\partial\rho}{\partial t}=\frac{1}{i\hbar}[H_{\rm int}+H_{\rm pump},\rho]+\sum_{i=1}^{3}\gamma_i(2a_i\rho a_i^\dagger - a_i^\dagger a_i\rho - \rho a_i^\dagger a_i)+\sum_{i=1}^{3}2n_i^{\rm th}\gamma_i(a_i\rho a_i^\dagger - \rho a_i a_i^\dagger - a_i^\dagger a_i\rho + a_i^\dagger\rho a_i)\ , \tag{2.4}$$

where the γ_i are the mode damping rates, and the n_i^{th} are the mean numbers of thermal photons of frequency ω_i in the heat baths.

This operator equation may be converted into a c-number Fokker-Planck equation using the generalized form of the Glauber P representation developed by Drummond and Gardiner[9]:

$$\rho=\int d\mu(\{\alpha\},\{\alpha^\dagger\})P(\{\alpha\},\{\alpha^\dagger\})\,|\{\alpha\}\rangle\langle\{\alpha^\dagger\}|\ , \tag{2.5}$$

where $|\{\alpha\}\rangle\equiv|\alpha_1,\alpha_2,\alpha_3\rangle$ is a three-mode coherent state. $d\mu(\{\alpha\},\{\alpha^\dagger\})$ is the integration measure chosen so that P is a well-behaved function on the region over which integrations are done. The stochastic variables α_i and $\alpha_i^\dagger$ correspond to the operators a_i and $a_i^\dagger$; however, α_i and $\alpha_i^\dagger$ are no longer complex conjugates as in the original formulation by Glauber. That is, $\alpha_i^\dagger\neq\alpha_i^*$. Using the usual operator algebra rules[10] we derive from Eq. (2.4) the following Fokker-Planck equation for P:

$$\begin{aligned}\frac{\partial P}{\partial t}=\Bigg[&-\frac{\partial}{\partial\alpha_1}(\epsilon_1-\gamma_1\alpha_1-K\alpha_2\alpha_3)-\frac{\partial}{\partial\alpha_1^\dagger}(\epsilon_1-\gamma_1\alpha_1^\dagger-K\alpha_2^\dagger\alpha_3^\dagger)-\frac{\partial}{\partial\alpha_2}(-\gamma_2\alpha_2+K\alpha_1\alpha_3^\dagger)\\&-\frac{\partial}{\partial\alpha_2^\dagger}(-\gamma_2\alpha_2^\dagger+K\alpha_1^\dagger\alpha_3)-\frac{\partial}{\partial\alpha_3}(-\gamma_3\alpha_3+K\alpha_1\alpha_2^\dagger)-\frac{\partial}{\partial\alpha_3^\dagger}(-\gamma_3\alpha_3^\dagger+K\alpha_1^\dagger\alpha_2)+\frac{\partial^2}{\partial\alpha_2\partial\alpha_3}K\alpha_1\\&+\frac{\partial^2}{\partial\alpha_2^\dagger\partial\alpha_3^\dagger}K\alpha_1^\dagger+\sum_{i=1}^{3}\gamma_i n_i^{\text{th}}\frac{\partial^2}{\partial\alpha_i^\dagger\partial\alpha_i}\Bigg]P\ .\end{aligned} \tag{2.6}$$

The Fokker-Planck equation derived by Graham[2,3] using the Glauber P representation is formally identical to Eq. (2.6).

The Langevin equations corresponding to this equation (using Itô rules) are

$$\begin{aligned}&\frac{\partial}{\partial t}\begin{bmatrix}\alpha_1\\ \alpha_1^\dagger\end{bmatrix}=\begin{bmatrix}\epsilon_1-\gamma_1\alpha_1-K\alpha_2\alpha_3\\ \epsilon_1-\gamma_1\alpha_1^\dagger-K\alpha_2^\dagger\alpha_3^\dagger\end{bmatrix}+\begin{bmatrix}0 & \frac{1}{2}\gamma_1 n_1^{\text{th}}\\ \frac{1}{2}\gamma_1 n_1^{\text{th}} & 0\end{bmatrix}^{1/2}\begin{bmatrix}\eta_1(t)\\ \eta_1^\dagger(t)\end{bmatrix},\\&\frac{\partial}{\partial t}\begin{bmatrix}\alpha_2\\ \alpha_2^\dagger\\ \alpha_3\\ \alpha_3^\dagger\end{bmatrix}=\begin{bmatrix}-\gamma_2\alpha_2+K\alpha_1\alpha_3^\dagger\\ -\gamma_2\alpha_2^\dagger+K\alpha_1^\dagger\alpha_3\\ -\gamma_3\alpha_3+K\alpha_1\alpha_2^\dagger\\ -\gamma_3\alpha_3^\dagger+K\alpha_1^\dagger\alpha_2\end{bmatrix}+\begin{bmatrix}0 & \frac{1}{2}\gamma_2 n_2^{\text{th}} & \frac{1}{2}K\alpha_1 & 0\\ \frac{1}{2}\gamma_2 n_2^{\text{th}} & 0 & 0 & \frac{1}{2}K\alpha_1^\dagger\\ \frac{1}{2}K\alpha_1 & 0 & 0 & \frac{1}{2}\gamma_3 n_3^{\text{th}}\\ 0 & \frac{1}{2}K\alpha_1^\dagger & \frac{1}{2}\gamma_3 n_3^{\text{th}} & 0\end{bmatrix}^{1/2}\begin{bmatrix}\eta_2(t)\\ \eta_2^\dagger(t)\\ \eta_3(t)\\ \eta_3^\dagger(t)\end{bmatrix},\end{aligned} \tag{2.7}$$

where the $\eta_i(t)$ and $\eta_i^\dagger(t)$ are delta-correlated stochastic forces with zero mean:

$$\begin{aligned}&\langle\eta_i(t)\rangle=\langle\eta_i^\dagger(t)\rangle=0\ ,\\&\langle\eta_i(t)\eta_j^\dagger(t')\rangle=\delta_{ij}\delta(t-t')\ .\end{aligned} \tag{2.8}$$

These Langevin equations will be useful later when we adiabatically eliminate mode 1.

III. STEADY-STATE SOLUTION OF THE FOKKER-PLANCK EQUATION IN THE ADIABATIC LIMIT

First, let us review the semiclassical behavior of this system. The semiclassical equations of motion for the field amplitudes α_i follow from Eqs. (2.7) when the stochastic forces $\eta_i(t)$ and $\eta_i^\dagger(t)$ and the stochastic nature of the α_i are ignored[1]:

$$\begin{aligned}&\dot\alpha_1=\epsilon_1-\gamma_1\alpha_1-K\alpha_2\alpha_3\ ,\\&\dot\alpha_2=-\gamma_2\alpha_2+K\alpha_1\alpha_3^*\ ,\\&\dot\alpha_3=-\gamma_3\alpha_3+K\alpha_1\alpha_2^*\end{aligned} \tag{3.1}$$

together with the complex-conjugate equations. The steady-state behavior of the signal mode with increasing driving ϵ_1 shows an abrupt transition behavior

$$|\alpha_2|^2=\begin{cases}0, & \epsilon_1<\epsilon_1^{\text{thres}}\\ \dfrac{1}{K}\left[\dfrac{\gamma_3}{\gamma_2}\right]^{1/2}(\epsilon_1-\epsilon_1^{\text{thres}}), & \epsilon_1\geq\epsilon_1^{\text{thres}}\end{cases} \tag{3.2}$$

where the threshold driving field

$$\epsilon_1^{\text{thres}}=\frac{\gamma_1(\gamma_2\gamma_3)^{1/2}}{K}\ .$$

We now return to the stochastic equations, Eq. (2.7). We shall assume that mode 1 is so heavily damped ($\gamma_1 \gg \gamma_2, \gamma_3$) that it may be adiabatically eliminated. That is, we assume $\dot{\alpha}_1 = \dot{\alpha}_1^\dagger = 0$ and we use the resulting expressions for α_1 and $\alpha_1^\dagger$ in the equations for $\dot{\alpha}_2$ and $\dot{\alpha}_3$. This step simplifies the mathematics without altering the physics too much. (cf. Ref. 3). As a further simplification, we note that at normal to low temperatures the thermal means $n_i^{\text{th}} = (e^{\hbar\omega/kT} - 1)^{-1}$ are much less than 1. This means the terms $\frac{1}{2}\gamma_i n_i^{\text{th}}$ in the diffusion matrices in Eqs. (2.7) are, in general, much smaller than the terms $\frac{1}{2}K\alpha_1$ and $\frac{1}{2}K\alpha_1^\dagger$ which are of the order $\frac{1}{2}\gamma_2$ according to Eqs. (3.1) and (3.2). Hence we shall ignore the *thermal* noise terms $\frac{1}{2}\gamma_i n_i^{\text{th}}$ in the diffusion matrices, retaining only the *quantum* noise terms $\frac{1}{2}K\alpha_1$ and $\frac{1}{2}K\alpha_1^\dagger$. The resulting set of Langevin equations for modes 2 and 3 is then

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha_2\\ \alpha_2^\dagger\\ \alpha_3\\ \alpha_3^\dagger\end{bmatrix} = \begin{bmatrix}-\gamma_2\alpha_2 + K\alpha_1\alpha_3^\dagger\\ -\gamma_2\alpha_2^\dagger + K\alpha_1^\dagger\alpha_3\\ -\gamma_3\alpha_3 + K\alpha_1\alpha_2^\dagger\\ -\gamma_3\alpha_3^\dagger + K\alpha_1^\dagger\alpha_2\end{bmatrix} + \begin{bmatrix}0 & 0 & \frac{1}{2}K\alpha_1 & 0\\ 0 & 0 & 0 & \frac{1}{2}K\alpha_1^\dagger\\ \frac{1}{2}K\alpha_1 & 0 & 0 & 0\\ 0 & \frac{1}{2}K\alpha_1^\dagger & 0 & 0\end{bmatrix}^{1/2}\begin{bmatrix}\eta_2(t)\\ \eta_2^\dagger(t)\\ \eta_3(t)\\ \eta_3^\dagger(t)\end{bmatrix} \tag{3.3}$$

with $\alpha_1 = 1/\gamma_1(\epsilon_1 - K\alpha_2\alpha_3)$ and $\alpha_1^\dagger = 1/\gamma_1(\epsilon_1 - K\alpha_2^\dagger\alpha_3^\dagger)$.

The Fokker-Planck equation corresponding to this set of equations is

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial\alpha_2}(-\gamma_2\alpha_2 + K\alpha_1\alpha_3^\dagger) - \frac{\partial}{\partial\alpha_2^\dagger}(-\gamma_2\alpha_2^\dagger + K\alpha_1^\dagger\alpha_3) - \frac{\partial}{\partial\alpha_3}(-\gamma_3\alpha_3 + K\alpha_1\alpha_2^\dagger) - \frac{\partial}{\partial\alpha_3^+}(-\gamma_3\alpha_3^\dagger + K\alpha_1^\dagger\alpha_2) + \frac{\partial^2}{\partial\alpha_2\partial\alpha_3}K\alpha_1 + \frac{\partial^2}{\partial\alpha_2^\dagger\alpha_3^\dagger}K\alpha_1^\dagger \right] P\ , \tag{3.4}$$

where $\alpha_1 = 1/\gamma_1(\epsilon_1 - K\alpha_2\alpha_3)$, $\alpha_1^\dagger = 1/\gamma_1(\epsilon_1 - K\alpha_2^\dagger\alpha_3^\dagger)$, and $\{\alpha\}$ is the set $(\alpha_2, \alpha_2^\dagger, \alpha_3, \alpha_3^\dagger)$. We are interested in the steady-state solution of this equation, from which we can calculate the steady-state statistical properties of modes 2 and 3.

Equation (3.4) has a potential solution[4] in the steady state provided $\gamma_2 = \gamma_3$, so we shall make this not unreasonable assumption (cf. Ref. 11). The steady-state solution is then $P(\{\alpha\}) = N\exp[\phi(\{\alpha\})]$, where N is a suitable normalization factor and the potential $\phi(\{\alpha\})$ is

$$\phi(\{\alpha\}) = 2\alpha_2^\dagger\alpha_2 + 2\alpha_3^\dagger\alpha_3 + \left[\frac{2\gamma_1\gamma_2}{K^2} - 1\right]\ln(K\alpha_2\alpha_3 - \epsilon_1) + \left[\frac{2\gamma_1\gamma_2}{K^2} - 1\right]\ln(K\alpha_2^\dagger\alpha_3^\dagger - \epsilon_1)\ . \tag{3.5}$$

We note that the terms $2\alpha_2^\dagger\alpha_2$ and $2\alpha_3^\dagger\alpha_3$ would cause $P(\{\alpha\})$ to diverge at large α_2 or α_3 if the integration domain was defined in the normal way with $\alpha_i^\dagger = \alpha_i^*$.

IV. STATISTICS OF THE STEADY-STATE LIGHT FIELDS

The solution (3.5) may now be used to calculate the statistics of the modes 2 and 3 in the steady state. Here we shall evaluate normally ordered averages of the type $\langle a_2^{\dagger m} a_2^m a_3^{\dagger n} a_3^n \rangle$ which are intensity moments or intensity correlations. The non-normalized moments are

$$I_{mn} = \int (\alpha_2^\dagger\alpha_2)^m (\alpha_3^\dagger\alpha_3)^n P(\{\alpha\}) d\mu(\{\alpha\}) \tag{4.1}$$

using the non-normalized function $P(\{\alpha\}) = \exp[\phi(\{\alpha\})]$. The averages are obtained by dividing the appropriate I_{mn} by I_{00}.

This integral is evaluated in several steps. First, the α_2 and $\alpha_3^\dagger$ integrals are done using the change of variables $u = (K/\epsilon_1)\alpha_2\alpha_3$ and $v = (K/\epsilon_1)\alpha_2^\dagger\alpha_3^\dagger$. The u and v integrals are then the integral forms of degenerate hypergeometric functions; each integration path is a Pochammer contour.[12] The $\alpha_2^\dagger$ and α_3 integrals are done using the change of variables $\omega = (K/\epsilon_1)\alpha_2^\dagger\alpha_3$ and $z = \alpha_2^\dagger/\alpha_3$. The ω integration path is any closed path around the origin, and gives a constant independent of m or n. At this stage the non-normalized moments have the following form:

$$I_{mn}=\left[\frac{\epsilon_1}{K}\right]^{m+n}\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+q+2)\Gamma(n+q+2)}$$
$$\times\int z^{m-n-1}M\left[m+1,m+q+2;\frac{2\epsilon_1}{K}z\right]$$
$$\times M\left[n+1,n+q+2;\frac{2\epsilon_1}{K}\frac{1}{z}\right]dz\ , \qquad (4.2)$$

where the Γ are gamma functions, the M are degenerate hypergeometric functions, and $q=2\gamma_1\gamma_2/K^2-1$. The single remaining integration can be done straightforwardly using the residue theorem, taking a closed integration contour around $z=0$. Thus, for example,

$$I_{00}=\frac{1}{[\Gamma(q+2)]^2}\sum_{k=0}^{\infty}\frac{1}{[(q+2)_k]^2}\left[\frac{2\epsilon_1}{K}\right]^{2k}\ ,$$
$$I_{10}=I_{01}=\frac{2(\epsilon_1/K)^2}{[\Gamma(q+3)]^2}\sum_{k=0}^{\infty}\frac{(k+1)}{[(q+3)_k]^2}\left[\frac{2\epsilon_1}{K}\right]^{2k}\ ,$$
$$I_{11}=\frac{(\epsilon_1/K)^2}{[\Gamma(q+3)]^2}\sum_{k=0}^{\infty}\left[\frac{(k+1)}{(q+3)_k}\right]^2\left[\frac{2\epsilon_1}{K}\right]^{2k}\ , \qquad (4.3)$$
$$I_{20}=I_{02}=\frac{8(\epsilon_1/K)^4}{[\Gamma(q+4)]^2}\sum_{k=0}^{\infty}\frac{(3)_k}{k![(q+4)_k]^2}\left[\frac{2\epsilon_1}{K}\right]^{2k}\ ,$$

where $(x)_k=x(x+1)(x+2)\cdots(x+k-1)$ and $(x)_0=1$. The fact that $I_{10}=I_{01}$, $I_2=I_{02}$, and so forth, is due to the symmetry imposed by taking $\gamma_2=\gamma_3$.

Figure 1 shows the mean photon number $\langle a_2^\dagger a_2\rangle$, the second-order correlation function

$$g^{(2)}0=\langle a_2^{\dagger 2}a_2^2\rangle/\langle a_2^\dagger a_2\rangle^2\ ,$$

and the intermode correlation

$$\langle a_2^\dagger a_2 a_3^\dagger a_3\rangle/(\langle a_2^\dagger a_2\rangle\langle a_3^\dagger a_3\rangle)$$

as functions of the driving field ϵ_1. The mean photon number follows the semiclassical result [Eq. (3.2) with $\gamma_2=\gamma_3$] closely, except there is a small nonzero intensity below threshold due to amplification of spontaneous emission by the noise term. The modes are highly correlated for low driving fields, but rapidly become uncorrelated for higher ϵ_1. The second-order correlation function $g^{(2)}(0)$ for a single mode starts from the value $2[(q+2)/(q+3)]^2$ [see Eq. (4.3) with $\epsilon_1=0$], then drops quite sharply through the threshold region, reaching an asymptotic value of 1. The asymptotic (large ϵ_1) forms can be straightfowardly calculated from Eq. (4.2) using the asymptotic forms for the M functions[13] and evaluating the integrals by the method of steepest descents:

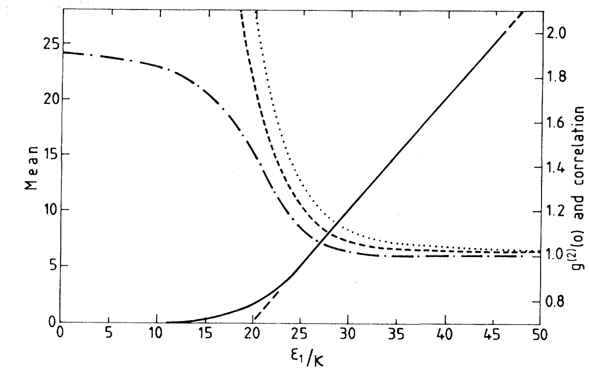

FIG. 1. Semiclassical photon number (— — —), mean photon number (——), second-order correlation function (—·—), intermode correlation (— — —), and the quantity $g^{(2)}(0)+1/\langle a^\dagger a\rangle$ ($\cdots$) as functions of ϵ_1/K; $\epsilon_1^{\text{thres}}=20$.

$$\langle a_2^\dagger a_2\rangle\sim\frac{\epsilon_1}{K}-\frac{\gamma_1\gamma_2}{K^2}-\frac{1}{8}\ ,$$
$$g^{(2)}(0)\sim 1-\frac{1}{4(\epsilon_1/K)}\ , \qquad (4.4)$$
$$\frac{\langle a_2^\dagger a_2 a_3^\dagger a_3\rangle}{\langle a_2^\dagger a_2\rangle\langle a_3^\dagger a_3\rangle}\sim 1+\frac{1}{4(\epsilon_1/K)}\ .$$

Note that the mean is less than the deterministic value by $\frac{1}{8}$. However, this difference is far too small to be observable.

More general moments, of the form

$$\langle(a_2^\dagger)^k(a_2)^l(a_3^\dagger)^m(a_3)^n\rangle$$
$$=\int\alpha_2^{*k}\alpha_2^l\alpha_3^{*m}\alpha_3^n P(\{\alpha\})d\mu(\{\alpha\}) \qquad (4.5)$$

may be calculated using a straightforward generalization of the above method. In this case, the w integration leading to the form corresponding to Eq. (4.2) is zero unless $(k-l)=(m-n)$; otherwise the moments take forms similar to those in Eq. (4.3). Thus

$$\langle a_2\rangle=\langle a_3\rangle=0\ ,$$
$$\langle(a_2)^2\rangle=\langle(a_3)^2\rangle=0\ , \qquad (4.6)$$
$$\langle a_2a_3\rangle=\frac{(\epsilon_1/K)}{\Gamma(q+2)\Gamma(q+3)}$$
$$\times\sum_{k=0}^{\infty}\frac{(k+1)}{(q+2)_k(q+3)_k}\left[\frac{2\epsilon_1}{K}\right]^{2k}\ .$$

These particular moments have been given for later use in Sec. V.

V. SUMMARY

We have considered a model for parametric oscillation in a cavity driven by an external laser. A quantum-statistical analysis showed that as well as thermal noise, noise arising from quantum effects is also present (see also Refs. 2 and 3). For normal to low temperatures, this quantum noise dominates the thermal noise except for very much below the oscillation threshold.

Here we derived a Fokker-Planck equation for the (generalized) P representation describing the pump, signal, and idler modes. The steady-state solution of this equation was found in the limit when the pump mode can be adiabatically eliminated, the signal and idler modes have equal loss rates, and the thermal noise is negligible compared with the quantum noise. This gave a single expression describing the statistics of the signal and idler modes over the whole range of driving-field strengths except in a very small region far below threshold where thermal noise and quantum noise become comparable.

Using this distribution function we derived expressions for the mean photon numbers, the second-order correlation functions and the intermode correlation as functions of the driving field. Far enough above threshold the modes become uncorrelated, and their second-order correlation functions tend to 1, the value for a coherent state.

It is interesting to ask whether or not the light fields here exhibit any nonclassical effects such as photon antibunching [$g^{(2)}(0)$ values less than 1]. The $\epsilon_1 \to 0$ limit of $g^{(2)}(0)$ is $2[(q+2)/(q+3)]^2$ which is less than 1 for $q \epsilon (-1, \sqrt{2}-1)$ so antibunching is predicted in the region near $\epsilon_1 = 0$. However, such q values seems incompatible with the adiabatic elimination requirement that γ_1 be large. Further, in the region near $\epsilon_1 = 0$ thermal noise cannot be ignored, and this will tend to increase $g^{(2)}(0)$ towards the thermal field value of 2. According to the asymptotic expressions, Eq. (4.4), for large driving fields $g^{(2)}(0)$ is slightly less than 1. However, this difference is very small and cannot be resolved on the scale used in Fig. 1.

However, there is one nonclassical effect apparent. If the P distribution were a standard real, non-negative distribution, the intermode correlation and second-order functions would have to satisfy the following Cauchy-Schwartz inequality[14]:

$$\left[\frac{\langle a_2^\dagger a_2 a_3^\dagger a_3 \rangle}{\langle a_2^\dagger a_2 \rangle \langle a_3^\dagger a_3 \rangle} \right]^2 \le g_2^{(2)}(0) g_3^{(2)}(0) . \tag{5.1}$$

Zubairy has recently shown that this inequality is violated in the two-photon laser.[15] In our case, because of the symmetry between modes 2 and 3, $g_2^{(2)}(0) = g_3^{(2)}(0)$ and Eq. (5.1) reduces to

$$\frac{\langle a_2^\dagger a_2 a_3^\dagger a_3 \rangle}{\langle a_2^\dagger a_2 \rangle \langle a_3^\dagger a_3 \rangle} \le g_2^{(2)}(0) . \tag{5.2}$$

Figure 1 shows that $g_2^{(2)}(0)$ is always *less* than or equal to the intermode correlation, with equality in the limit of large driving field ϵ_1. The nonreal nature of the P distribution thus allows the relation (5.1) to be violated. This violation exists even above threshold, so for this system the effect should be relatively accessible to experimental investigation. The experiments of Burnham and Weinberg on parametric fluorescence[16] have shown that the intermode correlation is in fact very high, although they did not measure the corresponding $g^{(2)}(0)$ values for comparison.

Although the nonpositive-real behavior of the P distribution allows the "strong" inequality (5.1) to be violated, there is a weaker inequality which cannot be violated. Since $a^\dagger a$ is a Hermitian operator, we must have $\langle (a_2^\dagger a_2 + \lambda a_3^\dagger a_3)^2 \rangle \ge 0$ for all real λ. The left-hand side (LHS) of this inequality is a quadratic in λ, and the inequality means this quadratic has no real roots. Thus its discriminant must be less than or equal to zero, which gives the following inequality:

$$\langle a_2^\dagger a_2 a_3^\dagger a_3 \rangle^2 \le \langle (a_2^\dagger a_2)^2 \rangle \langle (a_3^\dagger a_3)^2 \rangle . \tag{5.3}$$

In the present case, because of the symmetry between the two modes, this reduces to

$$\frac{\langle a_2^\dagger a_2 a_3^\dagger a_3 \rangle}{\langle a_2^\dagger a_2 \rangle \langle a_3^\dagger a_3 \rangle} \le \frac{\langle (a_2^\dagger a_2)^2 \rangle}{\langle a_2^\dagger a_2 \rangle^2} = g^{(2)}(0) + \frac{1}{\langle a_2^\dagger a_2 \rangle} , \tag{5.4}$$

which is a weaker inequality than (5.1). Figure 1 shows that this inequality is, as expected, always obeyed. In fact, from this graph we see that

$$g^{(2)}0 \le \frac{\langle a_2^\dagger a_2 a_3^\dagger a_3 \rangle}{\langle a_2^\dagger a_2 \rangle \langle a_3^\dagger a_3 \rangle} \le g^{(2)}(0) + \frac{1}{\langle a_2^\dagger a_2 \rangle} . \tag{5.5}$$

According to Fig. 1, the LHS of (5.4) is very close to the right-hand side (RHS), especially above threshold. That is, for the parametric oscillator, the intermode correlation is so high that it approaches the limit allowed by the inequality (5.4). This high correlation is not unexpected here, since each photon in one mode is produced as the partner of a photon produced in the other mode.

There is one other nonclassical effect attracting much attention at present: "squeezing." In this effect, one of the quadrature fluctuations $\langle(\Delta X)^2\rangle = \langle(X-\langle X\rangle)^2\rangle$ or $\langle(\Delta Y)^2\rangle = \langle(Y-\langle Y\rangle)^2\rangle$ in the complex field amplitude operator $a = X+iY$ becomes less than $\frac{1}{4}$. A state where $\langle(\Delta X)^2\rangle = \frac{1}{4} = \langle(\Delta Y)^2\rangle$ is a minimum uncertainty state, in which the product $\langle(\Delta X)^2\rangle\langle(\Delta Y)^2\rangle$ takes the minimum value allowed by the Heisenberg uncertainty principle. A state in which either $\langle(\Delta X)^2\rangle$ or $\langle(\Delta Y)^2\rangle$ is less than $\frac{1}{4}$ is usually called a "squeezed" state.[17,18] The potential application of such states in gravity wave detection and optical communications has been discussed in the literature.[18,19]

Calculation of the appropriate quantities for the parametric oscillator here shows that some squeezing does occur. However, this squeezing is not in the field modes a_2 or a_3 themselves; rather, in the "coupled mode" $d = (1/\sqrt{2})(a_2+a_3)$. If we let $d = X+iY$, then, since $[d^\dagger, d] = 1$, we have

$$\begin{aligned}\langle(\Delta X)^2\rangle &= \tfrac{1}{4}+\tfrac{1}{4}(2\langle d^\dagger d\rangle + \langle d^{\dagger 2}\rangle + \langle d^2\rangle - 2\langle d^\dagger\rangle\langle d\rangle - \langle d^\dagger\rangle^2 - \langle d\rangle^2)\,,\\ \langle(\Delta Y)^2\rangle &= \tfrac{1}{4}+\tfrac{1}{4}(2\langle d^\dagger d\rangle - \langle d^{\dagger 2}\rangle - \langle d^2\rangle - 2\langle d^\dagger\rangle\langle d\rangle + \langle d^\dagger\rangle^2 + \langle d\rangle^2)\,.\end{aligned} \tag{5.6}$$

Using the results (4.6) and the symmetry $\langle a_2^\dagger a_2\rangle = \langle a_3^\dagger a_3\rangle$, these can be written for the parametric oscillator as

$$\begin{aligned}\langle(\Delta X)^2\rangle &= \tfrac{1}{4}+\tfrac{1}{2}(\langle a_2^\dagger a_2\rangle + \langle a_2 a_3\rangle)\,,\\ \langle(\Delta Y)^2\rangle &= \tfrac{1}{4}+\tfrac{1}{2}(\langle a_2^\dagger a_2\rangle - \langle a_2 a_3\rangle)\,.\end{aligned} \tag{5.7}$$

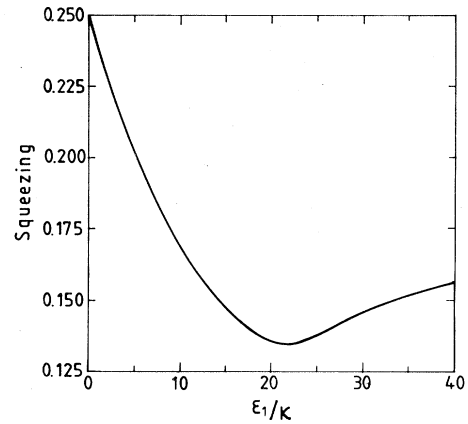

FIG. 2. Squeezing in the coupled mode $(1/\sqrt{2})(a_2+a_3)$ as a function of ϵ_1/K; $\epsilon_1^{\rm thres} = 20$.

Figure 2 shows the behavior of $\langle(\Delta Y)^2\rangle$ as a function of the driving field. Maximum squeezing [minimum value of $\langle(\Delta Y)^2\rangle$] occurs just above threshold, where the fluctuations in Y are reduced to about 0.125. This behavior is similar to the squeezing found in the degenerate parametric oscillator,[20] although in that case, the squeezing occurs directly in the signal mode. As pointed out in Ref. 20 this amount of squeezing is not enough for practical applications in optical communication systems or gravity wave detectors.

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