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Čerenkov radiation from periodic electron bunches

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Čerenkov radiation is calculated for electron beams which exceed the velocity of radiation in a nondispersive dielectric medium. The electron beam is assumed to be bunched as emitted from a traveling-wave accelerator, and the emission region is assumed to be finite. Predictions include (a) emission at harmonics of the bunch rate, (b) coherence of radiation at low frequencies, (c) smearing of the emission angle for finite emission regions, and (d) explicit evaluation of the power spectrum in terms of bunch dimensions. The results apply to microwave emission from fast electrons in air or other dielectrics.

I. INTRODUCTION

The radiation produced by γ rays incident on ordinary dielectric materials such as glass was first discovered by Čerenkov,[1] and was described in terms of a charged particle (electron) moving faster than light in the medium by Frank and Tamm.[2] A summary of work to 1958 is contained in the treatise by Jelly.[3] An important application is the Čerenkov particle detector which is familiar in any particle-physics laboratory, and an early and crucial application occurred in the discovery of the antiproton.[4]

Because the distribution of intensity of Čerenkov radiation is proportional to the frequency, the radiation at microwave frequencies would be low unless beams are intense and bunched so that coherent radiation by many electrons contributes. Danos[5] calculated radiation produced by a planar beam passing above a dielectric interface and a hollow cylindrical beam passing through a hole in a dielectric. Experimental and theoretical investigations at microwave frequencies were reviewed by Lashinsky.[6]

This investigation was motivated by a recently renewed interest in the study of stimulated Čerenkov radiation in which the electron may be in a medium consisting of a gas[7] or a hollow dielectric resonator.[8,9] Recent developments of electron accelerators for applications such as free-electron lasers (FEL) have aimed toward high-peak currents in bunches in contrast to nuclear- and particle-physics applications, where low-peak but high-average currents are desirable to avoid saturating detectors. The high-peak currents in the new accelerators should yield enhanced Čerenkov radiation, as is calculated in this paper.

II. CALCULATION OF THE POYNTING VECTOR

In the following derivation we consider the Čerenkov radiation produced in a dispersionless medium such as gases or other dielectrics by a series of pulses of electrons such as are produced by a traveling-wave electron accelerator (linac). The pulses or bunches are periodic, the total emission region is finite, and the bunches have a finite size.

In determining the radiated power, the procedure is to calculate the Poynting vector from fields which are, in turn, obtained from solutions of the wave equations for the potentials. Since the current and charge densities entering into the wave equations are expressed in Fourier form, the resulting fields and radiated power also have Fourier components. In the derivation, $\vec{r}$ is the coordinate at which the fields will be calculated, $\vec{r}\,'$ is the coordinate of an element of the charge which produces the fields, and $\hat{n}$ is a unit vector in the direction of $\vec{r}$. We assume that $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ have been expanded in a Fourier series, appropriate for the case where the source current is periodic. Then we have

$$\vec{E}(\vec{r},t)=\sum_{\omega=-\infty}^{\infty} e^{-i\omega t}\vec{\tilde{E}}(\vec{r},\omega)\ , \tag{1}$$

and a corresponding expansion for $\vec{B}$, where ω is a discrete variable, and $\vec{\tilde{E}}$ and $\vec{\tilde{B}}$ are Fourier-series coefficients. The Poynting vector $\vec{S}$ is given by

$$\vec{S}=\frac{1}{\mu}\vec{E}\times\vec{B}\ , \tag{2}$$

and it is easy to show that the average of $\vec{S}$ in a direction given by a normal vector $\hat{n}$ is

$$\frac{1}{T}\int_0^T \hat{n}\cdot\vec{S}\,dt=\frac{1}{\mu}\sum_{\omega=-\infty}^{\infty}\hat{n}\cdot\vec{\tilde{E}}(\vec{r},\omega)\times\vec{\tilde{B}}(\vec{r},-\omega)\ , \tag{3}$$

where T is an integer multiple of the period of the periodic current.

Letting $c=(\mu\epsilon)^{-1/2}$ be the velocity of light in the medium, the wave equations for $\vec{A},\phi$ and their solutions are

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A}(\vec{r},t)=\mu\vec{J}(\vec{r},t)\ ,$$
$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi(\vec{r},t)=1/\rho(\vec{r},t)/\epsilon\ , \tag{4}$$

$$\vec{A}(\vec{r},t)=\mu\int\int D(\vec{r}-\vec{r}',t-t')\vec{J}(\vec{r}',t')d^3r'dt' ,$$
$$\phi(\vec{r},t)=\frac{1}{\epsilon}\int\int D(\vec{r}-\vec{r}',t-t')\rho(\vec{r}',t')d^3r'dt' , \tag{5}$$

where the Green's function D is given by

$$D(\vec{r},t)=\frac{1}{4\pi r}\delta(t-r/c) . \tag{6}$$

The vector potential $\vec{A}(\vec{r},t)$ also can be developed in a Fourier-series expansion of a form similar to (1) with an expression for the Fourier-series coefficients given by

$$\begin{aligned}\vec{\tilde{A}}(\vec{r},\omega)&=\frac{1}{T}\int_0^T dt\,\vec{A}(\vec{r},t)e^{i\omega t}\\&=\mu\int d^3r'\vec{\tilde{J}}(\vec{r}',\omega)\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|/c} .\end{aligned} \tag{7}$$

Now, if we assume that the observer is far from the source so that $|\vec{r}|\gg|\vec{r}'|$ for regions where the integrand in (7) is important, we can let $|\vec{r}-\vec{r}'|=r-\hat{n}\cdot\vec{r}'$ in the exponential and $|\vec{r}-\vec{r}'|=r$ in the $|\vec{r}-\vec{r}'|^{-1}$ factor in (7), obtaining (where $\hat{n}=\vec{r}/r$)

$$\vec{\tilde{A}}(\vec{r},\omega)=\frac{\mu}{4\pi r}e^{i\omega r/c}\int d^3r'\vec{\tilde{J}}(\vec{r}',\omega)e^{-i(\omega/c)\hat{n}\cdot\vec{r}'} . \tag{8}$$

The Fourier-series coefficients of the fields are obtained from those for the vector potential (8) through the usual relations $\vec{B}=\vec{\nabla}\times\vec{A}$ and $\vec{E}=-\vec{\nabla}\phi-\partial\vec{A}/\partial t$. Under the conditions leading to (8), the field Fourier coefficients are[10]

$$\vec{\tilde{B}}(\vec{r},\omega)=i\frac{\omega}{c}\hat{n}\times\vec{\tilde{A}}(\vec{r},\omega) , \tag{9}$$

$$\vec{\tilde{E}}(\vec{r},\omega)=-c\hat{n}\times\vec{\tilde{B}}(\vec{r},\omega) . \tag{10}$$

The Poynting vector can now be found by using (9) and (10) in expansions like (1) and then substituting in (2). However, it is more convenient to deal with the frequency components of the radiated power by substituting (9) and (10) into the expression of the average radiated power (3) giving

$$\frac{1}{T}\int_0^T\hat{n}\cdot\vec{S}\,dt=\frac{1}{\mu}\sum_{\omega=-\infty}^{\infty}\frac{\omega^2}{c}|\hat{n}\times\vec{\tilde{A}}(\vec{r},\omega)|^2 . \tag{11}$$

III. FOURIER COMPONENTS OF THE CURRENT

The expression (7) for the Fourier components of the vector potential contains the Fourier components of the current density. Consequently, it is necessary to examine the form of the current and its Fourier development. Assume the current is in the z direction and periodic. If the electrons move with velocity v and we ignore for the moment the x and y variables, the charge or current functions should have the general form

$$f(z,t)=\sum_{k_z}e^{ik_zz}\sum_{\omega}e^{-i\omega t}\tilde{f}(k_z,\omega) . \tag{12}$$

Under the condition of rigid motion,

$$f(z,t)=f_0(z-vt) . \tag{13}$$

It is easy to show that

$$\tilde{f}(k_z,\omega)=\delta_{\omega,k_zv}\tilde{f}_0(k_z) , \tag{14}$$

where

$$\tilde{f}_0(k_z)=\frac{1}{Z}\int_0^Z e^{-ik_zz}f_0(z)dz . \tag{15}$$

Thus the restrictions of Eq. (13) reduce the two-dimensional Fourier series of Eq. (12) to essentially a one-dimensional series (14).

With (14) in mind, the current density associated with the electron beam from a linear accelerator should be periodic in both z,t, with a Fourier-series expansion, but the x and y dependence should be represented by a Fourier-integral form

$$J_z(\vec{r},t)=v\rho(r,t)=\frac{v}{(2\pi)^2}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\sum_{k_z=-\infty}^{+\infty}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\tilde{\rho}_0(\vec{k}) , \tag{16}$$

where the Fourier components of the charge density are

$$\tilde{\rho}_0(\vec{k})=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\frac{1}{Z}\int_0^Z dz\,e^{-i\vec{k}\cdot\vec{r}}\rho_0(\vec{r}) , \tag{17}$$

$\rho_0(\vec{r})$ is $\rho(\vec{r},t)$ evaluated at $t=0$, and $\vec{J}$ is assumed to be in the z direction. Note in Eq. (16) that k_z and ω are both discrete and from (14), $\omega=k_zv$.

IV. VECTOR POTENTIAL

The results of Sec. III can be applied to the evaluation of the vector potential and, in turn, to the fields. Let the infinite periodic pulse train be made finite, extending from $z=-Z'$ to $+Z'$, and let θ be the angle between $\hat{n}$ and $\vec{A}$. Then the cross product in (11) can be written

$$|\hat{n}\times\vec{\tilde{A}}(\vec{r},\omega)| = \sin\theta\frac{\mu}{4\pi r}e^{i\omega r/c}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'\int_{-Z'}^{Z'}dz' e^{-i\hat{n}\cdot\vec{r}'\omega/c} \times\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\sum_{k_z=-\infty}^{\infty}v\tilde{\rho}_0(\vec{k})\delta_{k_z v,\omega}e^{i\vec{k}\cdot\vec{r}'} . \tag{18}$$

But

$$\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'\int_{-Z'}^{Z'}dz' e^{i\vec{r}'\cdot(\vec{k}-\hat{n}\omega/c)} =(2\pi)^2\delta(k_x-n_x\omega/c)\delta(k_y-n_y\omega/c)I(Z') , \tag{19}$$

where

$$I(Z')=\int_{-Z'}^{Z'}dz' e^{i(k_z-n_z\omega/c)z'}=\frac{2}{G}\sin(GZ') , \tag{20}$$

and $G=k_z-n_z\omega/c=\omega/v-n_z\omega/c$. Thus, the cross-product term is

$$|\hat{n}\times\vec{\tilde{A}}(\vec{r},\omega)| =\sin\theta\frac{\mu}{4\pi r}e^{i\omega r/c}v\tilde{\rho}_0\left[\frac{n_x\omega}{c},\frac{n_y\omega}{c},\frac{\omega}{v}\right]I(Z') . \tag{21}$$

Note that ω is a discrete variable, but from (19) the continuous variables k_x and k_y become evaluated at discrete points.

Returning to (17), a more symmetric form may be obtained by assuming that $\rho_0(\vec{r})$, which is periodic in z with period Z, is actually zero between the pulses. Denoting by $\rho_0'(\vec{r})$ the charge density of a single pulse, which is zero for $z<0$ and $z>Z$, the integral on z can be written

$$\int_0^Z dz\, e^{-ik_z z}\rho_0(\vec{r})=\int_0^Z dz\, e^{-ik_z z}\rho_0'(\vec{r}) =\int_{-\infty}^{\infty}dz\, e^{-ik_z z}\rho_0'(\vec{r}) . \tag{22}$$

Then (17), the Fourier coefficient of the charge density, becomes

$$\tilde{\rho}_0(\vec{k})=\frac{1}{Z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dx\,dy\,dz\, e^{-i\vec{k}\cdot\vec{r}}\rho_0'(\vec{r}) =\frac{1}{Z}\tilde{\rho}_0'(\vec{k}) , \tag{23}$$

where $\tilde{\rho}_0'(\vec{k})$ is the three-dimensional Fourier transform of the single pulse described by $\rho_0'(\vec{r})$. Substituting these expressions into (21) gives a final simple result for the cross-product form:

$$|\hat{n}\times\vec{\tilde{A}}(\vec{r},\omega)| =\sin\theta\frac{\mu}{4\pi r}e^{i\omega r/c}(v/Z)\tilde{\rho}_0'(\vec{k})I(Z') , \tag{24}$$

where

$$I(Z')=\frac{2}{G}\sin(GZ') ,$$
$$G=\omega/v-n_z\omega/c , \tag{25}$$
$$\vec{k}=(n_x\omega/c,n_y\omega/c,\omega/v) .$$

The components of the Čerenkov E and B fields may now be found by substituting (24) in (9) and (10).

V. RADIATED POWER

The frequency components of the average radiated power are obtained by substituting (24) into (11). The negative frequency terms equal the corresponding positive frequency terms, yielding a factor of 2 when the summation range is changed. Multiplying by r^2 converts to average power per unit solid angle, $dP/d\Omega$, yielding

$$\frac{dP}{d\Omega}=r^2\frac{1}{T}\int_0^T\hat{n}\cdot\vec{S}\,dt=r^2\frac{2}{\mu}\sum_{\omega=0}^{\infty}\frac{\omega^2}{c}|\hat{n}\times\vec{\tilde{A}}(\vec{r},\omega)|^2 =\sum_{\omega=0}^{\infty}W(\omega,\hat{n}) , \tag{26}$$

where $W(\omega,\hat{n})$ is defined as

$$W(\omega,\hat{n})=\frac{2\mu}{(4\pi)^2}\frac{\omega^2}{c}\sin^2\theta\frac{v^2}{Z^2}|\tilde{\rho}_0'(\vec{k})|^2 I^2(Z') . \tag{27}$$

$W(\omega,\hat{n})$ is the power per unit solid angle radiated at the frequency ω, which is a harmonic of the basic angular frequency ω_0 of the periodic pulse train.

To find P_ω, the total power radiated at the frequency ω, W is multiplied by $d\Omega$ and integrated over solid angle. Note that $n_z=\cos\theta$, and as θ varies, G changes according to (25), with $dG=-(\omega/c)dn_z$ so that

$$d\Omega=d\phi(c/\omega)dG . \tag{28}$$

Noting that the integral over ϕ yields 2π, we find the result for the total radiated power at the frequency ω for all angles

$$P_\omega=\frac{\mu}{4\pi}\frac{\omega^2}{c}\frac{v^2}{z^2}\int_{G'}^{G''}\sin^2\theta|\tilde{\rho}_0'(k)|^2 I^2(Z')\frac{c}{\omega}dG . \tag{29}$$

VI. ČERENKOV ANGLE

The remaining integral over G may now be examined. The $\sin^2\theta$ and $\tilde{\rho}_0$ factors may often be slowly varying compared to the $I^2(Z')$ factor, the latter being shown in Fig. 1. For large Z', the peak in $I^2(Z')$ becomes narrow, and if the integrand may be neglected outside the physical range $G'<G<G''$,

$$\int_{G'}^{G''}I^2(Z')dG=\int_{-\infty}^{\infty}4(Z')^2\left[\frac{\sin(GZ')}{GZ'}\right]^2 dG =4\pi Z' . \tag{30}$$

Then, evaluating the $\sin\theta$ factor and $\tilde{\rho}_0'(k)$ at the point corresponding to $G=0$ (which is $\cos\theta=n_z=c/v$) shows that θ at the peak of $I(Z')$ is the usual Čerenkov angle θ_C. We thus obtain for large Z'

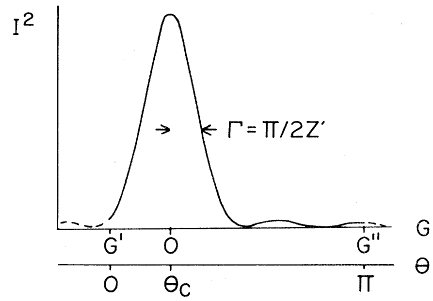

FIG. 1. Qualitative behavior of the function $I^2(Z')$. Both the function G, from Eq. (25) in the text, and the emission angle are displayed as independent variables. G' and G'' are upper and lower limits.

$$P_\omega = \frac{\mu}{4\pi}\omega v^2 \sin^2\theta_C \,|\tilde{\rho}_0(\vec{k})|^2 4\pi Z'/Z \ . \tag{31}$$

Now let $2Z'/Z$ = ratio of the interaction length to pulse spacing $=N$, the number of pulses. Also $Z = v2\pi/\omega_0$ or $2\pi/Z = \omega_0/v$ so that (in the large Z' limit)

$$P_\omega = \frac{\mu}{4\pi}\omega\omega_0 v \sin^2\theta_C \,|\tilde{\rho}_0'(\vec{k})|^2 N \ . \tag{32}$$

To compare with usual formulations, (32) is divided by Nv to obtain the energy loss per unit path length per pulse

$$\frac{d\mathcal{E}}{dx} = \frac{\mu}{4\pi}\omega\omega_0 \sin^2\theta_C \,|\tilde{\rho}_0'(\vec{k})|^2 \ . \tag{33}$$

If the pulse is, in fact, a point charge, the Fourier transform $\tilde{\rho}_0'(\vec{k})$ reduces to q, the total charge per pulse, and (33) is very similar to the usual Čerenkov energy-loss formula where for a single charge q, the radiation is continuous and the factor $\omega\omega_0$ in (33) is replaced by $\omega\, d\omega$. In the present case the pulse train is periodic at angular frequency ω_0 and the radiation is emitted at the harmonic frequencies denoted by ω.

VII. DISCUSSION OF RESULTS

Equation (29) and the approximate evaluation expressed as (32) form the main results. Some consequences will now be noted.

A. Effect of pulse size

The spatial distribution of the charge in the pulse appears in $\tilde{\rho}_0'(\vec{k})$, which is the Fourier transform of the charge distribution. The peak of $I^2(Z')$ in Fig. 1 occurs at $G=0$ or $n_z = c/v$. Thus, at the peak, $\omega/v = n_z\omega/c$ so that $\vec{k}$, the argument of $\tilde{\rho}_0'(\vec{k})$, is evaluated at [from (25)]

$$\vec{k} = \hat{n}\omega/c \ . \tag{34}$$

We may also define a charge form factor $F(\vec{k})$

$$\tilde{\rho}_0'(\vec{k}) = qF(\vec{k}) \ . \tag{35}$$

The form factor $F(\vec{k})$ is identically one for a point charge,

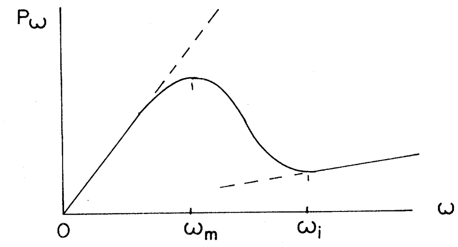

FIG. 2. Schematic behavior of power emitted as a function of angular frequency.

and for a finite distribution $F(\vec{k})=1$ for $k=0$.

Furthermore, $F(\vec{k})$ must fall off as a function of $\vec{k}$ near the origin if all the charge has the same sign. If the pulse were spherically symmetric, $F(\vec{k})$ would behave as elastic electron scattering form factors defined for nuclear charge distributions.[11] In that case, the mean-square radius $\langle r^2\rangle$ of the charge distribution is given by the behavior of $F(\vec{k})$ near the origin

$$F(\vec{k}) \to 1 - \langle r^2\rangle k^2/6 \tag{36}$$

(spherical pulse).

B. Smearing of the Čerenkov angle

For a finite region over which emission is allowed, namely, if $2Z'$ is finite, the function $I^2(Z')$, appearing in the integral in (29), will have a finite width. Since the peak height is $4Z'^2$ and the area is $4\pi Z'$, (30), we can assign an effective width 2Γ = area/height $= \pi/Z'$, or

$$\Gamma = \pi/2Z' \ . \tag{37}$$

Thus the radiation is emitted mainly near $G=0$ (which corresponds to $\theta=\theta_C$) but in a range $\Delta G = \pm\Gamma$. But from (25), $\Delta G = (\omega/c)\Delta n_z = (\omega/c)\Delta(\cos\theta)$ so that there is a range in $\cos\theta$ over which emission occurs

$$\Delta(\cos\theta) = \frac{c}{\omega}\frac{\pi}{2Z'} \ . \tag{38}$$

Note that the finite angular width of the Čerenkov cone angle in (38) has the factor $1/\omega$, indicating that the higher harmonics are emitted in a sharper cone.

C. Behavior at high frequencies related to pulse parameters

To be specific, let the charge distribution for a single pulse be given by Gaussian functions

$$\rho_0'(\vec{r}) = A\exp(-x^2/a^2 - y^2/a^2 - z^2/b^2) \ . \tag{39}$$

Then $F(\vec{k})$ may be found:

$$F(\vec{k}) = \exp(-k_x^2 a^2/4 - k_y^2 a^2/4 - k_z^2 b^2/4) \ . \tag{40}$$

Beam pulse parameters could then be determined by measuring the Čerenkov radiation. For example, fast electrons from an accelerator in air will emit with a θ_C of several degrees in which case k_x and k_y in (40) can be neglected, giving

$$F(\vec{k})=\exp(-k_z^2b^2/4)=\exp[-\omega^2b^2/(4v^2)] \ . \qquad (41)$$

The expected behavior of P_ω as a function of ω is shown qualitatively in Fig. 2 as a linear rise at low frequencies followed by a falloff at higher frequencies, the peak occurring at

$$\omega_m=v/b \ . \qquad (42)$$

Furthermore, a different behavior would be expected at very high frequencies. The formulation from the beginning represents coherent radiation from all charges, not only from one pulse, but coherence from pulse to pulse. $F(\vec{k})$ then describes interference of radiation emitted from different parts of the pulse; but note that expressions (29) and (32) will still be proportional to $q^2=n^2e^2$ where n is the number of electrons in a pulse. Thus the n^2 dependence of P_ω indicates coherence. But above some high frequency ω_i such that $\omega_i/c=2\pi/l$, where l is the mean spacing of electrons in the cloud, the radiation should switch over to incoherent radiation from each charge and P_ω should be proportional to n. The incoherent radiation should then rise again as a function of ω.

VIII. CONCLUDING REMARKS

The general results presented here describe the Čerenkov radiation produced by fast electrons produced by a linear accelerator (linac). For an S-band linac operating at about 3 GHz (10-cm radiation), the electron bunches are separated by 10 cm and would be about 1-cm long at 1% energy resolution. Microwave Čerenkov radiation is expected and has been seen in measurements at the Naval Postgraduate School linac.

Two types of measurements were made. In measurements of series A, an X-band antenna mounted near the beam path, oriented to intercept the Čerenkov cone, was connected to a spectrum analyzer. Harmonics 3—7 of the 2.85-GHz bunch frequency were seen, but power levels could not be measured quantitatively. Harmonics 1 and 2 were below the waveguide cutoff. In the series B measurements, the electron beam emerged from the end window of the accelerator and passed through a flat metal sheet 90 cm downstream oriented at an angle ϕ from the normal to the beam. The metal sheet defined a finite length of gas radiator and reflected the Čerenkov cone of radiation toward the accelerator but rotated by an angle 2ϕ from the beam line. A microwave X-band antenna and crystal detector with response from 7 to above 12 GHz could be moved across the (reflected) Čerenkov cone as a probe.

As mentioned earlier, the series A measurements showed the radiation is produced at the bunch repetition rate and its harmonics. Series B measurements performed with several antennas always indicated a broadened Čerenkov cone with strong radiation occurring at angles up to 10°, well beyond the predicted Čerenkov angle of 1.3°. Since a broadband detector was used, it was impossible to verify the prediction [see Eq. (38)] that the broadening of the cone should depend on the harmonic number. However, it should be noted that incoherent radiation by a beam of 1 μA at $\theta_C=1.3°$ for a 1-m path in air would be about 10^{-14} W at microwave frequencies, so that observation of any signal by either method A or B clearly demonstrated coherent radiation by the electron bunches.

Many of the concepts were clearly noted by Jelly in his treatise (Jelly[3], Sec. 3.4 especially). The form factor was noted, but a general expression was not given. In fact, the form factor quoted by Jelly represents the special case of a uniform line charge of length L' with a projected length $L=L'\cos\theta_C$ in the direction of the radiation. The coherence of the radiation from the bunch was noted, but no broadening of the cone nor harmonic structure were developed.

Casey, Yeh, and Kaprielian[12] considered an apparently related problem in Čerenkov radiation in which a single electron passes through a dielectric medium where a spatially periodic term is added to the dielectric constant. The result is radiation occurring even for electrons which do not exceed the velocity of light in the medium and at angles other than the Čerenkov cone angle. The non-Čerenkov part of the radiation is attributed to transition radiation.

In the present paper the transition radiation associated with the gas-cell boundaries is included, and radiation appears outside the Čerenkov cone. If the electron velocity were lower so that v/c were close to but less than unity, the peak in I would be pushed to the left in Fig. 1 such that $\cos\theta_C=v/c$ would be larger than 1. But the tails of the diffraction function I would extend into the physical range $1\le\cos\theta\le-1$, and this would be called transition radiation and be ascribed to the passage of the electrons through the boundaries of the gas cell. Now return to the case $v/c>1$, with the situation as shown in Fig. 1. The radiation is then a combination of Čerenkov and transition radiation. The formalism of Ref. 12 does admit a decomposition into the two types of radiation but is inherently much more cumbersome.

As a final remark, one might extend the analysis further in the region near ω_i. Consider electron bunches emitted from a traveling-wave linac which could be 1 cm long spaced 10 cm apart. Let these bunches enter the wiggler magnet of a free-electron laser (FEL). Then, if gain occurs, the 1-cm bunches would be subdivided into bunches of a finer scale, with the spatial scale appropriate to the output wavelength of the FEL.[13] If the (partially) bunched beam from the FEL were passed into a gas Čerenkov cell, then the observed radiation should be reinforced because of partial coherence at the FEL bunch frequency and harmonics. This would lead to bumps in the spectrum in the region near ω_i.

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APPENDIX A: DERIVATION OF ČERENKOV RADIATION FOR A SINGLE PULSE OF CHARGE

Let the pulse be described by

$$\rho'(\vec{r},t)=\rho_0'(\vec{r}-\vec{v}t)\ . \tag{A1}$$

Both k_z and ω are continuous variables in this case; $\vec{v}$ is again along the z axis. If we expand in terms of a four-dimensional Fourier integral,

$$\rho'(\vec{r},t)=[1/(2\pi)^4]\int\int\int\int e^{i(\omega t-\vec{k}\cdot\vec{r})}\tilde{\rho}'(\vec{k},\omega)\times d^3k\,d\omega\ . \tag{A2}$$

It may be shown that the condition (A1) gives

$$\tilde{\rho}'(\vec{k},\omega)=2\pi\delta(\omega-k_z v)\tilde{\rho}_0'(\vec{k})\ , \tag{A3}$$

where $\tilde{\rho}_0'(\vec{k})$ is the three-dimensional spatial transform of ρ' evaluated at $t=0$. All the fields have Fourier-integral rather than Fourier-series expansions, and the energy radiated per unit solid angle becomes

$$r^2\int_{-\infty}^{\infty}dt\,\hat{n}\cdot\vec{S}=\frac{1}{2\pi}\frac{1}{(4\pi)^2}\frac{\mu}{c}\int_{-\infty}^{\infty}\omega^2 d\omega\left|\int d^3r'\int dt' e^{i(ct'-\hat{n}\cdot\vec{r}')\omega/c}\hat{n}\times\vec{J}(\vec{r}',t')\right|^2=\int_0^{\infty}W(\omega,\hat{n})d\omega\ . \tag{A4}$$

The integrand is a symmetric function of ω so that

$$W(\omega,\hat{n})=\frac{1}{16\pi^3}\frac{\mu}{c}\omega^2\int\int\int\int d^3r'dt'e^{i\omega(t'-\hat{n}\cdot\vec{r}'/c)}\hat{n}\times\vec{J}(\vec{r}',t')=\frac{1}{16\pi^3}\frac{\mu}{c}\omega^2(\hat{n}\times\vec{v})^2M^2\ , \tag{A5}$$

where

$$M=\int\int\int\int d^3r'dt'e^{i(\omega t'-\hat{n}\cdot\vec{r}'/c)}\rho'(\vec{r}',t')\ . \tag{A6}$$

Now we may write $\rho'(\vec{r}',t')$ in a Fourier-integral representation

$$\rho'(\vec{r}',t')=\frac{1}{(2\pi)^4}\int\int\int\int d^3k'd\omega'\tilde{\rho}'(\vec{k}',\omega')e^{-i(\omega't'-\vec{k}'\cdot\vec{r}')}\ . \tag{A7}$$

Inserting Eq. (A3) into Eq. (A7) and the result into Eq. (A6), the integral over d^3k' involves only exponentials and yields $(2\pi)^3\delta^3(\vec{k}'-\omega\hat{n}/c)$, so that Eq. (A6) becomes

$$M=\int dt'\int\int\int\int d^3k'd\omega'e^{i(\omega-\omega')t'}\delta^3(\vec{k}'-\hat{n}\omega/c)\delta(\omega'-k_z'v)\tilde{\rho}_0'(\vec{k}')\ .$$

Now the integral over ω' may be done. Because of the δ function, ω' is evaluated at $k_z'v$:

$$M=\int dt'\int\int\int d^3k'e^{i(\omega-k_z'v)t'}\delta^3(\vec{k}'-\omega\hat{n}/c)\times\tilde{\rho}_0'(\vec{k}')\ .$$

Now do the integrals over k_x', k_y', and k_z'. Noting that k_z' appears in the exponential but k_x' and k_y' do not,

$$M=\int dt'\,e^{i\omega t'}e^{-i\omega t'n_z v/c}\tilde{\rho}_0'(\omega n_x/c,\omega n_y/c,\omega n_z/c)\ .$$

This may be written as

$$M=\int dt'e^{i\omega t'H}\tilde{\rho}_0'(\omega\hat{n}/c)\ , \tag{A8}$$

where

$$H=1-n_z v/c\ . \tag{A9}$$

If we let the time interval be finite, from $-T$ to $+T$, the integral is easily done:

$$M=\frac{2}{\omega}\sin(\omega HT)\tilde{\rho}_0'(\hat{n}\omega/c)\ , \tag{A10}$$

$$M^2=4T^2\frac{\sin^2(\omega HT)}{(\omega HT)^2}\,|\tilde{\rho}_0'(\hat{n}\omega/c)|^2\ . \tag{A11}$$

This result, Eq. (A11), may be inserted into (A5) for ω. The factor $\hat{n}\times\vec{v}$ is just $\sin\theta$ where θ is the angle between the radiation and the beam axis

$$W(\omega,\hat{n})=\frac{1}{16\pi^3}\frac{\mu}{c}\omega^2\sin^2\theta\,4T^2\frac{\sin^2(\omega HT)}{(\omega HT)^2}\,|\tilde{\rho}_0'(\hat{n}\omega/c)|^2\ . \tag{A12}$$

W is the energy radiated per unit solid angle per unit angular frequency ω. To proceed to the total energy, multiply by $d\Omega$ (solid angle) and integrate. But $n_z=\cos\theta$ so that $d\Omega$ may be related to dH

$$d\Omega=d(\cos\theta)d\phi=-\frac{c}{v}dH\,d\phi\ . \tag{A13}$$

The functions in Eq. (A12) do not contain ϕ so that integration over ϕ yields 2π. Thus

$$\int W(\omega,\hat{n})d\Omega=\frac{1}{2\pi^2}\frac{\mu}{v}\omega^2T^2\int\sin^2\theta\,|\tilde{\rho}_0'|^2\frac{\sin^2(\omega HT)}{(\omega HT)^2}dH. \tag{A14}$$

The $\sin^2(\omega HT)/(\omega HT)^2$ factor in the integral is peaked at $H=0$, which by Eq. (A9) is at $n_z=\cos\theta=c/v$, or the usual Čerenkov angle θ_C. This function is more strongly peaked about $H=0$ for large values of T, and, in fact, for large T we may evaluate $\sin^2\theta$ and $\tilde{\rho}_0'$ at the point corresponding to $H=0$. Then the integral

$$\int_{-\infty}^{\infty}dx\,\sin^2(ax)/(ax)^2=\pi/a\ ,$$

may be used to evaluate Eq. (A14), yielding

$$\int W\,d\Omega=\frac{\mu}{4\pi}\frac{\omega}{v}2T\sin^2\theta_C\,|\tilde{\rho}_0'(\hat{n}\omega/c)|^2\,. \tag{A15}$$

The emission was assumed to occur in a time interval from $-T$ to $+T$; accordingly, dividing by $2T$ yields at a rate of emission, and multiplying by v converts to emission per unit path length. Thus we obtain, for the large-T limit

$$\frac{d^2\mathscr{E}}{dx\,d\omega}d\omega=\frac{\mu}{4\pi}\omega\,d\omega\sin^2\theta_C\,|\tilde{\rho}_0'(\hat{n}\omega/c)|^2\,, \tag{A16}$$

where $d^2\mathscr{E}/d\omega\,dx$ is the energy emitted per unit path length per unit angular frequency range ω.

The corresponding expression for T not large is

$$\frac{d^2\mathscr{E}}{dx\,d\omega}d\omega=\frac{\mu}{4\pi}\omega\,d\omega\left[\frac{\omega T}{\pi}\right]\int_{H'}^{H''}\sin^2\theta\,|\tilde{\rho}_0'(n\omega/c)|^2\times\frac{\sin^2(\omega HT)}{(\omega HT)^2}dH\,, \tag{A17}$$

where H'' and H' are the value of H corresponding to $\theta=0$ and π, respectively.

Equations (A16) and (A17) then describe the energy radiated per unit path length and per unit angular frequency range. For the nonperiodic (single) pulse the radiation has a continuous frequency spectrum. For a point charge q, $\tilde{\rho}_0'(\vec{k})$ is identically q and the usual Čerenkov formula is obtained. Equation (A16) is quoted by Jelly, but only with the form factor corresponding to a uniform line charge of length L'.

APPENDIX B: DERIVATION OF EQ. (7)

Equation (7) is derived for the case in which $\vec{J}(\vec{r},t)$ is expanded in the Fourier series. Let the Fourier coefficients for $\vec{A}$ be given by

$$\vec{\tilde{A}}(\vec{r},\omega)=\frac{1}{\tau}\int_0^{\tau}dt\,\vec{A}(\vec{r},t)e^{i\omega t}\,. \tag{B1}$$

Assume that the Green's-function solution for $\vec{A}(\vec{r},t)$ is given

$$\vec{A}(\vec{r},t)=\mu\int d^3r'\int dt'\vec{J}(\vec{r}',t')D(\vec{r}-\vec{r}',t-t')\,, \tag{B2}$$

where

$$D(\vec{r},t)=\frac{1}{4\pi r}\delta(t-r/c)\,. \tag{B3}$$

Let the current density be expanded in a Fourier series

$$\vec{J}(\vec{r}',t')=\sum_{\omega'}e^{-i\omega't'}\vec{\tilde{J}}(\vec{r}',\omega')\,. \tag{B4}$$

Then insert (B2), (B3), and (B4) into (B1) to obtain

$$\vec{\tilde{A}}(\vec{r},\omega)=\frac{\mu}{\tau}\int_0^{\tau}dt\,e^{i\omega t}\int d^3r'\int dt'\frac{1}{4\pi}\left|\frac{1}{\vec{r}-\vec{r}'}\right|\delta(t-t'-|\vec{r}-\vec{r}'|/c)\sum_{\omega'}e^{-i\omega't'}\vec{\tilde{J}}(\vec{r}',\omega')\,. \tag{B5}$$

Do the integral over t', and note that t' appears in the δ function and in $e^{-i\omega't'}$. The result is t' is evaluated at $t'=t-|\vec{r}-\vec{r}'|/c$:

$$\vec{\tilde{A}}(\vec{r},\omega)=\frac{\mu}{\tau}\int_0^{\tau}dt\,e^{i\omega t}\int d^3r'\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|}\sum_{\omega'}e^{-i\omega't}e^{i\omega'|\vec{r}-\vec{r}'|/c}\vec{\tilde{J}}(\vec{r}',\omega')\,. \tag{B6}$$

Do the integral on t. Note that

$$\frac{1}{\tau}\int_0^{\tau}dt\,e^{i(\omega-\omega')t}=\delta_{\omega',\omega}\,. \tag{B7}$$

Then do the sum on ω'

$$\vec{\tilde{A}}(\vec{r},\omega)=\frac{\mu}{4\pi}\int\int\int d^3r'\frac{1}{|\vec{r}-\vec{r}'|}\vec{\tilde{J}}(\vec{r}',\omega)e^{i\omega|\vec{r}-\vec{r}'|/c}\,. \tag{B8}$$

This proves the desired result, (B8) is Eq. (7) as used in the main text.

APPENDIX C: TEMPORAL STRUCTURE OF THE ELECTRON PULSE FROM A TRAVELING-WAVE ACCELERATOR

Assume that the energy of a single electron emerging from a linac with phase ψ relative to the traveling-wave field is

$$E=E_0\cos\psi\,. \tag{C1}$$

This relation is shown in Fig. 3, along with some dots representing electrons near the maximum energy E_0, with

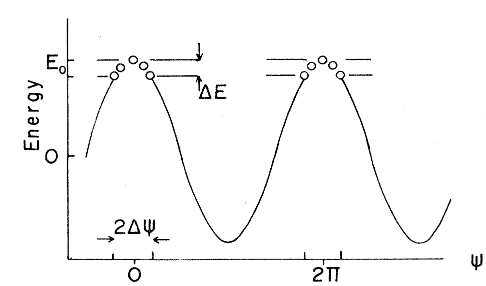

FIG. 3. Structure of charge pulse from a traveling-wave accelerator. ψ is the phase angle of an electron relative to the peak of the traveling-wave accelerating field. Electrons in the range $\pm\Delta\psi$ are passed by magnetic deflection system.

phases clustered about $\psi=0$ and 2π. Two bunches, separated by a phase difference of 2π, are separated by a time $T_1=1/f_0$ where f_0 is the accelerator frequency which is $f_0=2.85\times10^9$ Hz for a typical S-band accelerator of the Stanford type.

If a deflection system with an energy resolution slit passes only energies E from E_0 to E_0-E, the corresponding range of phase $\Delta\psi$ is

$$\Delta E=E-E_0=E_0(1-\cos\Delta\psi)\ . \tag{C2}$$

For small $\Delta\psi$, this reduces to

$$\frac{\Delta E}{E_0}=\frac{(\Delta\psi)^2}{2}\ . \tag{C3}$$

The temporal pulse length T_2 is

$$T_2=2\,\Delta\psi\, T_1/2\ ,$$

or

$$T_2=T_1(2\,\Delta\psi/2\pi)\ . \tag{C4}$$

If (C3) is used to evaluate $\Delta\psi$ in terms of the fractional energy resolution $\Delta E/E_0$,

$$T_2=T_1\left[\frac{2\,\Delta E}{E_0}\right]^{1/2}\frac{1}{\pi}\ . \tag{C5}$$

For 1% energy resolution, T_2/T_1 is about $\frac{1}{20}$. The electrons thus emerge in short bunches, and the charge and current, when expressed in a Fourier expansion, should have very strong harmonic content up to and above the 20th harmonic.

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