Angular distributions in multiple scattering

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The theory of the random walk on a sphere is derived by elementary methods and by specializing the general theory of random walks on group (G) spaces. The result is applied to find the angular distribution in multiple elastic scattering after N steps. The case of electron scattering from a screened Coulomb potential is done in detail. Formulas which are exact in principle and asymptotic in practice are found, giving the distribution with many-place accuracy at all angles for $N \geq 20$. Methods for generating a random variable with this distribution are also given.

I. INTRODUCTION

Rutherford's famous analysis of α-particle scattering in gold foils distinguished two angular regimes[1]: a forward regime with a Gaussian distribution, dominated by many small-angle scatters, and a large-angle regime, dominated by single scattering. This approach to the angular distribution in multiple scattering of charged particles is widely used even now. It is described in textbooks.[2] It is difficult to avoid the impression that it is essentially correct.

Despite its intuitive plausibility, it is fair to ask how accurate this representation really is. Has it been thoroughly checked in physically interesting cases? The answer seems to be no. That check, for a realistic differential cross section, considered at all angles, is made in this paper for the first time. Previous partial results showing that the model of Fig. 1 is, in fact, not quantitatively accurate, are confirmed and the comparison is extended to the entire angular range.

The method employed is not entirely new. It was given by Goudsmit and Saunderson (GS),[3] who were, however, unable to evaluate certain coefficients in their formulas [the λ_l's of Eq. (7)]. Their results are rederived here in a more physical and transparent manner in Sec. II, and the calculation is completed in Sec. III. The result is that the usual model (Fig. 1) is significantly in error (see Fig. 2): A naive application of the central-limit theorem to the forward distribution is typically low by about 40% in the forward direction and may be high by a factor of 2 at somewhat larger angles where one might expect it still to apply. In addition, multiple-scattering corrections to the single-scattering distribution are typically significant even at the largest angles.

Curiously enough, the GS paper, which contains exact results of beautiful simplicity, has been ignored in favor of approaches which make small-angle approximations. If one makes too drastic an approximation one finds that the forward distribution is given by the central-limit theorem.[4] This is unequivocally wrong. In the late 1940s, more careful use of the small-angle approximation began to reveal the true shape of the forward distribution,[5,6] but the results were initially often described as if they approached the naive Gaussian, even when they clearly did not.[6,7] By 1963 it was appreciated that the forward distribution is non-Gaussian,[8] but the small-angle theories which revealed this fact are complicated and approximate, and have never fully displaced the model of Fig. 1 in actual use.

By contrast, the methods of this paper are so simple that exact angular distributions in multiple scattering can be routinely generated for use in accurate modeling of physical processes in detectors, biological materials, etc. Until now there has been no recognized good way to do this. The persistence of Rutherford's original model (Fig. 1), despite its (more or less known) shortcomings, was perhaps due to the absence of a simple, clear alternative free of dubious approximations.

In Sec. IV the method of Sec. II is related to the more general problem of a random walk on a group. The difficulty of extending this method from the angular distribu-

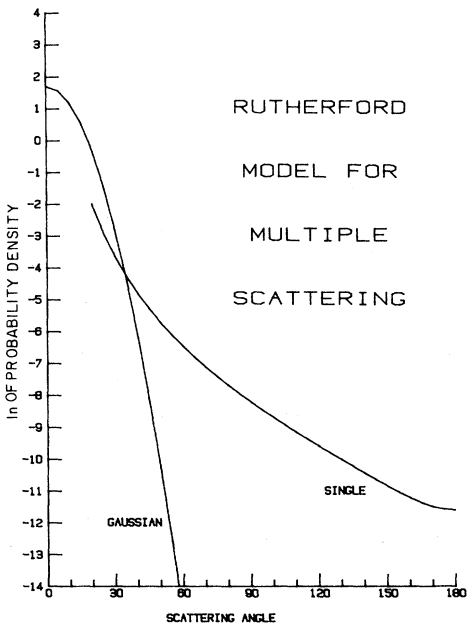

FIG. 1. Rutherford model for the angular distribution in multiple scattering, showing two angular regimes, with a Gaussian distribution forward and single scattering backward. Crossover region, where some kind of interpolation appears to be called for, is sometimes called "plural scattering."

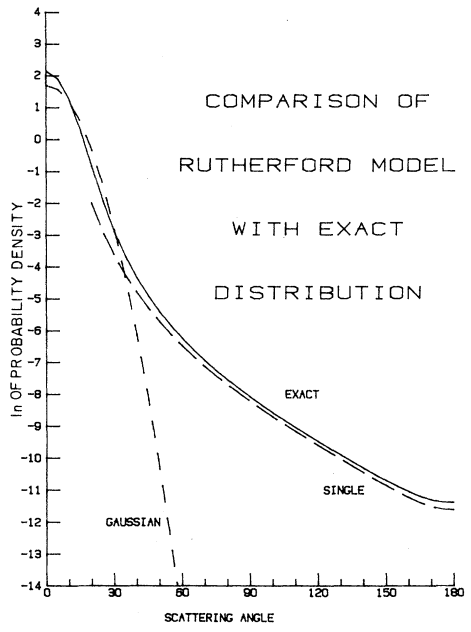

FIG. 2. Comparison of the Rutherford model with the exact angular distribution, found as described in the text, for a 1.9-MeV electron in germanium after 150 scatters. There is a significant discrepancy, of the order of 10–50%, at almost all angles.

tion to the spatial distribution of particles undergoing multiple scattering is pointed out.

Section V contains practical suggestions for modeling the exact distribution.

II. RANDOM WALK ON THE SPHERE

Consider the distribution of a particle on the sphere which makes random jumps. It is assumed that the law governing the jumps is isotropic, homogeneous, and Markovian, i.e., that there are no preferred directions or positions, and that each jump is independent of the others. (In applying this idea to multiple scattering, the direction of travel of a particle or wave will be assumed to make such a random walk. The distribution of this random direction will be found.)

Take coordinates (θ,ϕ) on the sphere in the usual way, fixed once and for all, and use unit vectors $\hat{p}$, $\hat{q}$, etc., as a shorthand for locations (θ_p,ϕ_p), (θ_q,ϕ_q), etc. At each step of the random walk one must use the conditional probability density $W_1(\hat{p},\hat{q})$ that a particle at $\hat{q}$ moves in the next step to $\hat{p}$. Clearly W_1 is positive with

$$\int W_1(\hat{p},\hat{q})d\hat{p}=1 . \tag{1}$$

Since it is assumed that the law which governs the single step depends only on the relative direction of $\hat{p}$ with respect to $\hat{q}$, and not on any third direction, one has

$$W_1(\hat{p},\hat{q})=f(\hat{p}\cdot\hat{q}) . \tag{2}$$

The Markovian assumption that each step is independent of the preceding one implies

$$W_N(\hat{p},\hat{q})=\int W_1(\hat{p},\hat{k})W_{N-1}(\hat{k},\hat{q})d\hat{k} , \tag{3}$$

where $W_N(\hat{p},\hat{q})$ is the conditional probability density that a particle initially at $\hat{q}$ arrives at $\hat{p}$ after exactly N steps.

The problem posed is to find a simple expression for $W_N(\hat{p},\hat{q})$ given $W_1(\hat{p},\hat{q})$. Expand $W_1(\hat{p},\hat{q})$ in Legendre polynomials and thence in spherical harmonics,

$$W_1(\hat{p},\hat{q})=f(\hat{p}\cdot\hat{q})=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\lambda_l P_l(\hat{p}\cdot\hat{q})$$

$$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\lambda_l Y_{lm}(\hat{p})Y_{lm}^*(\hat{q}) , \tag{4}$$

where

$$\lambda_l=2\pi\int_{-1}^{1}f(z)P_l(z)dz . \tag{5}$$

Then by induction in Eq. (3),

$$W_N(\hat{p},\hat{q})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\lambda_l^N Y_{lm}(\hat{p})Y_{lm}^*(\hat{q}) . \tag{6}$$

Equation (6) is the solution. In particular, taking $\hat{q}=\hat{z}$ (which only amounts to choosing the coordinate system in the most convenient way), one has

$$W_N(\hat{p},\hat{z})=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\lambda_l^N P_l(\cos\theta_p) . \tag{7}$$

In interpreting Eq. (7) it is useful to realize that, according to Eq. (5), the coefficient λ_l is

$$\lambda_l=\langle P_l\rangle_{W_1} , \tag{8}$$

i.e., just the average value of the Legendre polynomial P_l with respect to the distribution W_1. In particular,

$$\lambda_0=1 \tag{9}$$

and

$$|\lambda_l|<1 \text{ if } l\geq 1 \tag{10}$$

(except in uninteresting degenerate cases). Thus in the limit as $N\to\infty$, only the $l=0$ term survives in Eq. (7), and one has

$$\lim_{N\to\infty}W_N(\hat{p},\hat{q})=\frac{1}{4\pi} , \tag{11}$$

the uniform distribution on the sphere, as intuition would suggest.

If W_1 is sharply peaked forward, approximating a δ function, $\lambda_l\approx 1$ for a large number of terms. Finding these λ_l's accurately is the main computational difficulty in applying this method, but it is not particularly imposing. Scattering from a Thomas-Fermi atom, a problem considered by many previous investigators, yields to a simple trick (see Sec. III). A cross section parametrized by partial waves would also yield λ_l's in an obvious way.

III. MULTIPLE SCATTERING FROM THOMAS-FERMI ATOMS

To apply the ideas of Sec. II to multiple scattering, note that

$$W_1=\frac{1}{\sigma}\frac{d\sigma}{d\Omega}\ , \tag{12}$$

i.e., W_1 is just the differential cross section normalized to be a probability distribution. In particular, consider potential scattering of a Dirac electron from a screened Coulomb potential

$$V(r)=-\frac{Ze^2}{r}e^{-r/a}\ . \tag{13}$$

The result, in Born approximation, is the Mott cross section[9] modified by screening

$$W_1(x)\propto\frac{1-\beta^2x}{(x+A)^2}\ , \tag{14}$$

where

$$x=\sin^2(\theta/2)\ \text{ and }\ A=\left[\frac{\alpha Z^{1/3}}{2\beta\gamma}\right]^2, \tag{15}$$

the latter quantity corresponding to the screening length

$$a=\frac{e^2}{\alpha^2mc^2}Z^{-1/3} \tag{16}$$

suggested by Thomas-Fermi theory[10] (here α is the fine-structure constant, β and γ are the relativistic parameters of the electron, and Z is the atomic number of the target nucleus).

The average value of the Legendre function P_l, called λ_l in Eq. (7), can be found by a generating-function technique. The generating function for the average Legendre functions is just the average of their generating function, by linearity,

$$\langle[(1-h)^2+4xh]^{-1/2}\rangle_{W_1}=\sum_{n=0}^{\infty}h^n\lambda_n\ . \tag{17}$$

Define quantities $k_n(A)$ and $s_n(A)$ by

$$\int_0^1[(1-h)^2+4xh]^{-1/2}(x+A)^{-1}dx=\sum_{n=0}^{\infty}h^nk_n\ , \tag{18}$$

$$\int_0^1\left[1+\frac{4xh}{(1-h)^2}\right]^{-1/2}(x+A)^{-1}dx=\sum_{n=0}^{\infty}h^ns_n=F(h)\ . \tag{19}$$

It is easy to see that

$$k_n=\sum_{m=0}^{n}s_m\ , \tag{20}$$

$$\lambda_n=\left[\beta^2k_n+(1+\beta^2A)\frac{dk_n}{dA}\right]\times\left[\beta^2k_0+(1+\beta^2A)\frac{dk_0}{dA}\right]^{-1} \tag{21}$$

so that it is enough to find the s_n. Expand the first factor of the integrand in Eq. (19) by the binomial theorem and integrate. One has

$$F(h)=\sum_{n=0}\left\{\binom{-\frac{1}{2}}{n}z^n\left[(-A)^n\ln\left[1+\frac{1}{A}\right]+\sum_{m=0}^{n-1}\frac{(-A)^m}{n-m}\right]\right\}, \tag{22}$$

where $z(h)=4h/(1-h)^2$. Now collect like powers of A,

$$F(h)=\sum_{m=0}(-A)^m\left[\binom{-\frac{1}{2}}{m}z^m\ln\left[1+\frac{1}{A}\right]+F_m(h)\right], \tag{23}$$

where

$$F_m(h)=z^m\lim_{\epsilon\to0}\int_\epsilon^z[(1+\xi)^{-1/2}]_m\xi^{-m-1}d\xi \tag{24}$$

and the square brackets with subscript m mean the first m terms of the Maclaurin expansion are subtracted.

It is now straightforward, though tedious, to find $F(h)$ as a power series in h (and hence to find s_n) through any finite power of A in the expansion of Eq. (23). The result for s_n through terms involving A^3 is

$$s_n=a_n\ln(1+1/A)+b_n+c_nA\ln(1+1/A)+d_nA+e_nA^2\ln(1+1/A)+f_nA^2+g_nA^3\ln(1+1/A)+h_nA^3+O(n^7A^4\ln A)\ , \tag{25}$$

where

$$\begin{aligned}
&a_n=\delta_{n0},\quad b_n=(-2/n)\theta_{n1},\quad c_n=2n\theta_{n1}\ ,\\
&d_n=-[2(n-1)+4n(\tfrac{1}{2}+\tfrac{1}{3}+\cdots+1/n)]\theta_{n2},\quad e_n=(n+1)n(n-1)\theta_{n2}\ ,\\
&f_n=-\{4n(n-1)(n-2)/3-(n-1)(n-2)(n-3)/6+2(n+1)n(n-1)[\tfrac{1}{4}+\tfrac{1}{5}+\cdots+1/(n+1)]\}\theta_{n3}\ ,\\
&g_n=[(n+2)(n+1)n(n-1)(n-2)/6]\theta_{n3}\ ,\\
&h_n=-\{\tfrac{1}{4}(n+1)n(n-1)(n-2)(n-3)-\tfrac{1}{20}n(n-1)(n-2)(n-3)(n-4)+\tfrac{1}{180}(n-1)(n-2)(n-3)(n-4)(n-5)\\
&\qquad+\tfrac{1}{3}(n+2)(n+1)n(n-1)(n-2)[\tfrac{1}{6}+\tfrac{1}{7}+\cdots+1/(n+2)]\}\theta_{n4}\ ,\\
&\delta_{nm}=\begin{cases}1 & \text{if } n=m\\ 0 & \text{if } n\neq m\end{cases}\qquad\theta_{nm}=\begin{cases}1 & \text{if } n\geq m\\ 0 & \text{if } n<m\ .\end{cases}
\end{aligned} \tag{26}$$

A similar expression for λ_n follows from Eqs. (20) and (21). Using this expression for λ_n in Eq. (7) gives a formula for W_N which for many purposes is essentially exact, a remark we now justify.

The series obtained for λ_n from Eqs. (23) and (19)–(21) converges (in fact it is exact after $2n+1$ terms). More important for practical purposes, only the first few terms, typically just those given in Eq. (25), are necessary in applications, by the following argument.

The terms in Eq. (25) decrease rapidly if $n \ll A^{-1/2}$, so that one may expect Eq. (25) to be accurate up to some $n_{\max}$ which depends on A. An estimate of $n_{\max}$ is given by looking at the next term in the expansion for s_n in Eq. (25), namely,

$$s_n = \cdots + j_n A^4 \ln(1+1/A) , \tag{27}$$

where

$$j_n = (n+3)(n+2)(n+1)n(n-1)(n-2)(n-3)/72 .$$

Requiring this term to be less than, say, 0.001 gives

$$n_{\max} \approx [0.072 A^{-4}/\ln(1+1/A)]^{1/7} . \tag{28}$$

Now take N, the number of steps, to be so large that the contribution of the term containing $\lambda_{n_{\max}}^N$ is negligible, even at $\theta=\pi$, where the cancellation of terms is especially delicate. Using the estimate $W_N(\pi)=NW_1(\pi)$ and asking for three-figure accuracy there, one finds

$$N \gtrsim \ln\left[\frac{4\pi W_1 \times 10^{-3}}{2n_{\max}+1}\right][\ln(\lambda_{n_{\max}})]^{-1} . \tag{29}$$

So long as N satisfies this inequality (which is quite insensitive to the detailed assumptions in its derivation), W_N is accurate at all angles. In fact, the inequality turns out to be

$$N \gtrsim 20 \tag{30}$$

showing little explicit dependence on electron energy up to 3 MeV, which covers the range of most natural β emitters. Of course, for higher energy W_1 is sharper (A is smaller) so that more terms in Eq. (7) are needed to represent W_N. But just because A is smaller, the expansion of Eq. (25) does provide those terms. (W_{20} for a 3-MeV electron in germanium requires nearly 500 terms. For larger N, far fewer are needed.)

The procedure outlined above could be extended to $N<20$ by going farther in the expansion of Eq. (25). For example, with the next term [Eq. (27)] included, W_{15} appears to be accurate.

The above discussion does not give rigorous estimates of error, but experience has shown it to be an accurate rule of thumb. The formula for W_N is asymptotic in N. The error in W_N appears to be $O(\epsilon^N)$ where $\epsilon^{20} \lesssim 0.02 W_1(\pi) \ll 1$.

A typical result for W_N is shown in Fig. 2 with the Gaussian distribution of the central-limit theorem and the single-scattering tail NW_1 for comparison. The results of work using small-angle approximations is confirmed in the forward direction, and comparisons at large angles are possible for the first time. Multiple-scattering corrections to the single-scattering distribution are quite noticeable.

In summary, Eq. (7) with λ_l given by Eqs. (20), (21), and (25) is an explicit solution to the multiple-scattering problem for $N \gtrsim 20$ with many-place accuracy at all angles.

IV. RANDOM WALKS ON GROUPS

One gets a deeper insight into the formulas of Sec. II by considering them in a more general setting, that of a random walk on a group.[11] In particular one sees that the reason the forward distribution in multiple scattering is non-Gaussian is that it arises as a random walk in a non-Euclidean space, whereas the central-limit theorem is associated in an essential way with random walks in Euclidean spaces and more particularly with the Euclidean translation group.

A random walk on a group G can be identified with a random sequence of group operations in which G operates on itself by right multiplication. The group operations have a linear representation ρ on the space $\mathcal{F}$ of real-valued functions on the group in the usual way[12]

$$\rho(g)f(h) = f(hg^{-1}) . \tag{31}$$

In what follows, the notation will imply that G is a finite group, but nothing is changed if G is a compact Lie group and summation over G is interpreted as integration with respect to Haar measure.[12]

Let p_1 be a probability measure on G such that $p_1(g)$ is the probability of a translation in a single step by the action of g on G. (We assume that the random walk can be characterized in this way, i.e., that the probability of the step $h \to hg$ is independent of h.) Similarly, let $p_N(g)$ be the probability that in N steps one has translated by the action of g. Then

$$p_N(g) = \sum_{h \in G} p_1(gh^{-1}) p_{N-1}(h) . \tag{32}$$

Define an operator[13] on functions using the representation ρ,

$$W[p_1] = \sum_{g \in G} p_1(g)\rho(g) . \tag{33}$$

Then it is easy to prove by induction that

$$(W[p_1])^N = W[p_N] . \tag{34}$$

For suppose $(W[p_1])^N = \sum_{h \in G} p_N(h)\rho(h)$. Then

$$(W[p_1])^{N+1} = \sum_{g \in G}\sum_{h \in G} p_1(g)\rho(g)p_N(h)\rho(h) = \sum_{g \in G}\sum_{h \in G} p_1(g)p_N(h)\rho(gh)$$
$$= \sum_{j \in G}\sum_{h \in G} p_1(jh^{-1})p_N(h)\rho(j) = \sum_{j \in G} p_{N+1}(j)\rho(j) = W[p_{N+1}] . \tag{35}$$

$W[p_1]$ is essentially the stochastic matrix for the random walk.

An important simplification occurs if $p_1(g)$ is a class function, i.e., if

$$p_1(g)=p_1(hgh^{-1}),\quad \forall h\in G\ . \tag{36}$$

In this case, $W[p_1]$ commutes with $\rho(g)$ for any $g\in G$, as one sees from

$$W[p_1]\rho(g)=\sum_{h\in G}p_1(h)\rho(h)\rho(g)=\sum_{h\in G}p_1(h)\rho(hg)=\sum_{j\in G}p_1(gjg^{-1})\rho(gj)$$

$$=\sum_{j\in G}p_1(j)\rho(gj)=\rho(g)W[p_1]\ . \tag{37}$$

Thus, by Schur's lemma, $W[p_1]$ is just a multiple of the identity operator I on each irreducible representation contained in ρ. If we take as a basis in $\mathcal{F}$ a basis for the irreducible representations contained in ρ, then $W[p_1]$ is diagonal, so that raising it to the Nth power (i.e., computing the effect of a random walk of N steps) is trivial.

A space M on which G acts transitively can be identified with G/H, where H is the isotropy group of a point of M.[14] If H is a symmetry of p_1, the random walk analysis on G passes to the quotient M. Thus the random walk on the sphere can be regarded as arising from a random walk on SO(3) with p_1 a class function. It passes to the quotient SO(3)/SO(2), which is just the two-sphere. Equation (6) displays W_1 in diagonal form, as it was guaranteed to be, because the spherical harmonics are a basis for the irreducible representations of SO(3) on S^2. [The spherical harmonics themselves arise by the quotient procedure; they are just the symmetric-top wave functions which are invariant under the action of the SO(2) in the quotient.]

It is disappointing that the same observation cannot be used to solve the multiple-scattering problem in its entirety, i.e., to find the spatial distribution of particles together with their angular distribution. This amounts to solving the random walk on the affine group of translations and rotations in three-space. Curiously enough, the probability measure that corresponds to physical scattering processes is not a class function on this group. Thus the stochastic matrix W_1 cannot be diagonalized using group theory alone: The correct basis functions depend on the details of p_1 and cannot be found once and for all, as in the simpler angular problem.

V. MODELING WITH W_N

In modeling multiple scattering one wishes to generate a random variable θ with the distribution W_N. This is not immediately easy, even though W_N is a known function.[15,16]

Call f a *rapidly generable* (RG) function on a domain D if it is a probability distribution, i.e., $f\geq 0$ on D,

$$\int_D f=1 \tag{38}$$

and if there is a rapid procedure for generating a random variable with distribution f. (This definition is flexible: What it means depends on context, but it seems to be a useful notion.)

To be concrete, let D be the interval [0,1]. Then by most standards the uniform distribution $f=1$ is RG. Also, if for any probability distribution f we define

$$g(x)\equiv\int_0^x f(x')dx'\ , \tag{39}$$

then the random variable $x=g^{-1}(y)$, where y is uniformly distributed on [0,1], has distribution f. Thus f is RG if g^{-1} is rapidly computable.

If $f_1,\ldots,f_n$ are RG, and non-negative numbers $a_1,\ldots,a_n$ satisfy $a_1+\cdots+a_n=1$, then $f=a_1f_1+\cdots+a_nf_n$ is RG, that is, appropriate linear combinations of RG functions are RG.

In this language, the problem is to approximate W_N by an RG function. In general, it is not clear that a solution exists at all, since the RG functions may not be dense in the space of the W_N's in the appropriate norm.

The following observations constitute a rough-and-ready solution to the problem of finding an RG approximation to W_N on the sphere. This is not a solution in the sense of giving an RG sequence $\{f_n\}$ which converges to W_N. Rather we find functions which are within a few percent at all angles.

Define a family of functions

$$F(\alpha;x)=\frac{e^{-4\alpha x}}{1-e^{-4\alpha}}\frac{\alpha}{\pi}\ . \tag{40}$$

For any α, F is a probability distribution in x on [0,1] with respect to the measure $4\pi\,dx$, and it is RG. [We are really thinking of $x=\sin^2(\theta/2)$ and F as a distribution on the sphere. The form is chosen so that F looks as nearly as possible like a Gaussian in θ with width α^{-1}.] It turns out that linear combinations of such functions can approximate W_N well in the forward direction.

For large angles ($x\geq x_T\simeq\sin^2\theta_{\rm rms}$), the form

$$G(A,B,C,x_T;x)=\frac{B+Cx}{(A+x)^2}\Theta(x-x_T)\ , \tag{41}$$

suggested by the single-scattering distribution, works well.

In order that the two forms join well at x_T, it is advantageous to define "folded F's" on $[0,x_T]$,

$$\hat{F}(\alpha,x_T;x)=[F(\alpha;x)+(1-x_T)F(\alpha,x')/x_T]$$

$$\times\Theta(x_T-x)\ , \tag{42}$$

TABLE I. Form of the RG distribution approximating W_N in germanium for $100\leq N\leq 200$. Functional forms F, $\hat{F}$, and G are defined in the text.

Electron energy (MeV)	Distribution
$E\leq 0.52$	Uniform
$0.52<E\leq 0.6$	$F(\alpha)$
$0.6<E\leq 0.9$	$a\hat{F}(\alpha)+(1-a)G$
$0.9<E\leq 4.5$	$a_1\hat{F}(\alpha_1)+a_2\hat{F}(\alpha_2)+a_3G$

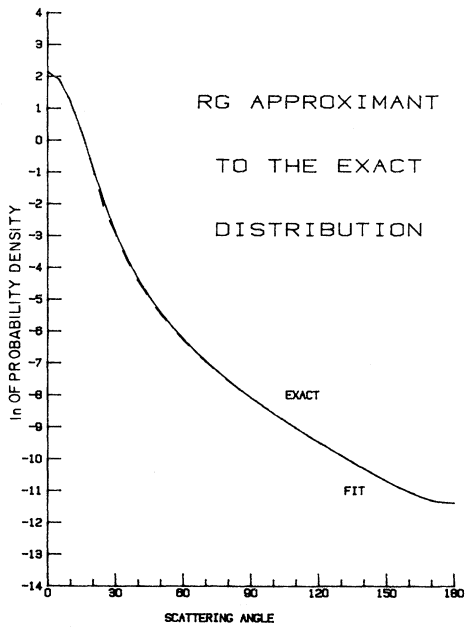

FIG. 3. Exact distribution of Fig. 2 is well approximated by the form given in Table I with parameters $a_1=0.572$, $a_2=0.393$, $a_3=0.034$, $\alpha_1=18.1$, $\alpha_2=40.0$, $A=-0.0190$, $B=0.003\,81$, $C=-0.003\,49$, and $x_T=0.005\,86$.

where $x'(x)=1-x(x_T^{-1}-1)$. $\hat{F}$ is RG because it corresponds to the following procedures: (1) generate a variable ξ' with distribution F, (2) if $\xi'\leq x_T$ set $\xi=\xi'$, and (3) if $\xi'>x_T$, set $\xi=x_T(1-\xi')/(1-x_T)$. ξ is a random variable with distribution $\hat{F}$.

The function $G(A,B,C,x_T;x)$ is RG because if y is uniformly distributed on [0,1], the iterative procedure

$$x_0=x_T\ , \tag{43}$$

$$x_n=\xi_1+(A+x_{n-1})(A+x_T)C\ln\left[\frac{A+x_{n-1}}{A+x_T}\right]\Big/\xi_2\ ,$$

where

$$\xi_2=y(A+x_T)-(B-AC)\ ,$$

$$\xi_1=[(AC-B)x_T-yA(A+x_T)]/\xi_2$$

converges to a random variable x whose distribution is G.

It is straightforward to find positive constants a_1, $a_2,\ldots,a_{n+1}$, and $\alpha_1,\ldots,\alpha_n$ such that $a_1+\cdots+a_{n+1}=1$ and

$$W_N\approx a_1\hat{F}_1+\cdots+a_n\hat{F}_n+a_{n+1}G\ . \tag{44}$$

It does not seem to be possible to use the same form for all parameter values, however. Modeling electrons in germanium, with $100\leq N\leq 200$ required the rather inelegant choices of Table I at various energies.

A typical W_n, with its RG approximant (nearly indistinguishable from it) is shown in Fig. 3.

In summary, the observations of this section facilitate fast, accurate modeling of angular distributions in multiple scattering.

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