Aharonov-Bohm effect for trapped ions

Robert R. Lewis
Department of Physics, University of Michigan, Ann Arbor, Michigan 48109
(Received 3 June 1982)

The role of the Aharonov-Bohm effect is analyzed for the bound states of a charged particle in the superposition of two fields: A uniform magnetic field plus the vector potential of a solenoid. Exact solutions are given for a solenoid of zero size and the corrections due to finite size are discussed. It is shown that a series of new "betatron resonances" appear in the spectrum, dependent on the flux quantum number $f = e\Phi/hc$. The fundamental betatron resonance remains narrow and unshifted for solenoid radii large enough to broaden and shift the cyclotron resonance.

INTRODUCTION

It might seem that nothing is simpler than the cyclotron motion of a charged particle in a uniform magnetic field. A classical particle has circular orbits of all radii, with a frequency $\omega = qB/mc$ independent of the orbit radius and of the location of the orbit center. In quantum theory, the kinetic energy is quantized, giving a ladder of energy eigenvalues with equal spacing $\hbar\omega$ having the same degeneracy under displacement of the orbit center. This spectrum is modified by the inclusion of spin and relativity, but the problem seems to be a straightforward one without surprises or complications. These results for uniform fields are well known and have been used for many years for precision measurements of the masses and magnetic moments of ions.

While the theory of cyclotron motion in *uniform* fields is simple, the effect of *nonuniform* magnetic fields is more subtle, as we shall show here. At the classical level, magnetic inhomogeneities break the degeneracy under displacement of the orbit center, producing a broadening of the cyclotron resonance dependent on the magnetic field gradients. There is nothing particularly interesting in the classical theory of this line shape, and we have nothing to add to the usual discussion of it. We are concerned instead with the quantum theory of motion in a nonuniform field, and especially with the explicit appearance of the vector potential A, rather than just the magnetic field B, in the equations of motion. There is no way of separating the effects of these two fields for a uniform magnetic field, because the two fields are proportional and cannot be individually varied.

However, for nonuniform magnetic fields, the different roles of A and B must be considered because of the nonlocal nature of the relation between these two fields. We can obtain the local value of B from local properties of A, since $B = \vec{\nabla} \times \vec{A}$, but not vice versa. The local values of the vector potential depend on the global properties of the magnetic field, since the line integral of A is related by Stokes theorem to the flux of B through the entire area bounded by the path. This distinction is particularly clear for the case of a confined magnetic flux Φ in the interior of a physically inaccessible solenoid. The flux Φ produces a vector potential A but no magnetic field B outside the solenoid. If we superpose this solenoidal potential with a uniform magnetic field, we have a prototype for a nonuniform magnetic field which is different inside and outside the solenoid. This is the basic configuration of fields which we will consider; it is the configuration of a betatron accelerator. We will use the phrase "betatron motion" to distinguish this from the "cyclotron motion" in a uniform magnetic field. The basic question now is whether the betatron motion of an ion is determined only by the accessible magnetic field or whether it also depends on the presence of the vector potential and thus on the inaccessible flux Φ. Classical theory gives one answer (B alone), and quantum theory gives a different answer (B and Φ). We want to analyze the dependence on Φ and discuss the practical problem of how to observe this dependence.

Aharonov and Bohm[1] showed that quantum phenomena make it possible to detect the presence of the vector potential surrounding a solenoid, as long as one maintains phase coherence of a charged particle passing around the solenoid. Ehrenburg and Siday[2] had previously discussed the effect of the vector potential in electron interferometry, where the insertion of a tiny solenoid between the interfering paths was shown to give an additional phase difference

$$\left[\frac{e}{\hbar c}\right] \oint \vec{A} \cdot d\vec{s} = e\Phi/\hbar c$$

proportional to the flux through the solenoid. The resulting shift in the fringe system is called the Aharonov-Bohm (AB) effect. In terms of gauge theory,[3] such phase factors are necessary to maintain invariance of the theory under local gauge transformations:

$$A'_\mu(x)=A_\mu(x)+\partial_\mu\Lambda(x)\ ,\qquad \psi'(x)=e^{iq\Lambda(x)/\hbar c}\psi(x)\ . \tag{1}$$

The AB effect is an important test of the role of gauge fields in quantum field theory, as well as in quantum mechanics. It is also an effect of practical importance since it permits the measurement of very small magnetic fluxes[4] in the quantum unit

$$hc/e=4.136\times10^{-7}\ \mathrm{G\ cm}^2\ .$$

Several experiments have reported observation of a phase shift in electron interferometry, verifying the AB effect.[5] The most recent of these has determined the coefficient hc/e to 1.5% accuracy.[6] Despite this success, there is still an aura of controversy about the existence of this effect in electron interferometry and the analysis of these experiments.[7] It is clear that new experiments with greater precision could settle the controversy, and perhaps even contribute to the precision measurement of the fundamental constants hc/e.

The main idea presented in this paper is that the use of ions trapped in bound states enclosing a small solenoid should provide a source of new experiments with much greater precision than electron interferometry. A charged particle orbiting around a solenoid should experience a phase shift which grows linearly with time, and therefore an energy shift. Ion trapping techniques have demonstrated very great precision in the measurement of energy differences,[8] and therefore might be used for a precision measurement of the AB effect in bound states. The existence of an AB effect for bound states is not a new idea.[9] It is related to the quantization of flux in superconducting cylinders[10] and to periodicity of the relation between magnetic field and critical temperature of superconductors.[11] Indeed, it is related to the quantization of energy in cyclotron motion.[12] However, there has been no serious consideration of experiments to detect the AB effect for a single bound ion; that is my goal in this paper.

This has led to a study of various configurations used for trapping ions, the simplest of which is a "magnetic bottle," which has been extensively used at Michigan[13] and elsewhere. We will ignore the axial motion along the field lines, and simplify the radial motion by considering the field to be uniform. By choosing such a simple starting point, one can proceed analytically and obtain exact results, even for a finite radius of the solenoid inserted into the bottle. The inclusion of axial motion and the extension of these ideas to other trap configurations will be postponed to a later publication.

CYCLOTRON MOTION

A quick review of the transverse motion in a uniform magnetic field is necessary to record some well-known results[14] and to establish our notation. We begin with the nonrelativistic quantum theory in a gauge which preserves cylindrical symmetry about the magnetic field (z axis).

$$\vec{A}=\tfrac{1}{2}(\vec{B}\times\vec{r})\ . \tag{2}$$

The Schrödinger equation for a spinless particle with charge q and mass m

$$\frac{1}{2m}(\vec{p}-q\vec{A}/c)^2\psi=E\psi \tag{3}$$

can be separated in cylindrical coordinates

$$\psi(r,\theta)=G(r)e^{iM\theta}\ . \tag{4}$$

Choosing units based on $\hbar$, m, and ω, the radial equation becomes

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{M^2}{r^2}+M-\frac{r^2}{4}+2E\right]G(r)=0\ . \tag{5}$$

In this system, length is measured in units of

$$(\hbar/m\omega)^{1/2}=(\hbar c/qB)^{1/2}=2.6\ \mu\mathrm{m}/(qB)^{1/2}$$

with q in units of e and B in Gauss. The eigenvalues of this equation are found by standard methods to be

$$E(N,M)=(2N+1+|M|-M)/2\ , \tag{6a}$$

and the eigenfunctions are

$$\psi_{N,M}\propto r^{|M|}L_N^{|M|}(r^2/2)e^{-r^2/4}e^{iM\theta}\ , \tag{6b}$$

where the functions $L_N^{|M|}$ are the associated Laguerre polynomials. Here N and M count the number of radial and azimuthal nodes, respectively. The eigenvalues form a harmonic oscillator spectrum (ladder) with equally spaced levels of infinite degeneracy.

The same representation can be constructed in a different way,[15] using operator methods based on the generalized variables

$$\pi_x=p_x+y/2\ ,\quad x_0=x/2+p_y\ ,\qquad \pi_y=p_y-x/2\ ,\quad y_0=y/2-p_x\ . \tag{7}$$

The variables π_x and π_y represent the particle velocity and x_0 and y_0 represent the coordinates of the orbit center. These variables have constant commutators, of which the only nonzero ones are

$$[\pi_y,\pi_x]=[x_0,y_0]=-i\ . \tag{8}$$

The Hamiltonian contains π_x and π_y alone and does not depend on the variables x_0 and y_0 of the orbit center. A complete set of states can be obtained by simultaneously diagonalizing the kinetic energy $(\pi_x^2+\pi_y^2)/2$ and some combination of the orbit center variables. For a uniform field, one can equally well choose either x_0 or y_0 as diagonal (Landau basis). Alternatively, one can choose to diagonalize the distance from the coordinate axis to the orbit center, $r_0^2=x_0^2+y_0^2$ (Johnson-Lippman basis). We will make the latter choice because it preserves the cylindrical symmetry and can still be used when we introduce the solenoid into the problem. The eigenvalues of the energy are

$$E_{n,l}=(n+\tfrac{1}{2})\ , \tag{9}$$

and the eigenvalues of the squared distance to the orbit center are

$$r_0^2=(2l+1)\ . \tag{10}$$

The normalized eigenfunctions can be generated by raising operators on the ground state:

$$\begin{aligned}\psi_{n,l}&=(2^{(n+l)}n!l!)^{-1/2}(\pi_y+i\pi_x)^n(x_0+iy_0)^l\psi_0\\&=(2^{(n-l)}n!/2\pi l!)^{1/2}r^{(l-n)}L_n^{(l-n)}\\&\quad\times(r^2/2)e^{-r^2/4}e^{i(l-n)\theta}\ ,\end{aligned} \tag{11}$$

where the ground-state wave function is

$$\psi_0=(2\pi)^{-1/2}e^{-r^2/4}\ . \tag{12}$$

This can be seen to be the same representation given by Eqs. (6a) and (6b) with the following connection between the two sets of quantum numbers:

$$M=l-n\ ,\quad N=\text{lesser of }(l,n)\ . \tag{13}$$

The use of the Johnson-Lippman variables simplifies some calculations and also clarifies the physical interpretation of these eigenstates. States with $l=0$ correspond classically to circular orbits centered on the (arbitrarily chosen) origin of the coordinates. As n increases, we obtain circular orbits with increasing radius and increasing kinetic energy, still centered on the origin. The mean-square radius of these states is easily found to be

$$\langle r^2\rangle=(2n+1)\ .$$

States with $l>0$ corresponds to a superposition of states with the same orbit radius, given by n, distributed in annular ring with mean-square distance from the origin

$$\langle r^2\rangle=2(l+n+1)\ .$$

States with $l\le n$ correspond classically to orbits which link the origin, since their orbit radii are larger than the distance of the orbit center from the origin. States with $l>n$ do not link the origin.

These energy eigenstates have another interesting property: The flux of the magnetic field through the orbit is also quantized.[12] The simplest way to see this is to calculate the mean-square radius of the semiclassical orbits centered on the origin

$$\langle n,0\,|\,r^2\,|\,n,0\rangle=(2n+1)\ . \tag{14}$$

We note that the flux through these discrete orbits is quantized in units of hc/e:

$$\Phi_n\equiv B\langle n,0\,|\,\pi r^2\,|\,n,0\rangle=(n+\tfrac{1}{2})hc/e\ . \tag{15}$$

This semiclassical derivation can be given a more precise meaning for any quantum state by calculating the flux through the mean area of the state, defined by its probability density. This quantity is also quantized:

$$\begin{aligned}\Phi_{n,l}&\equiv B\langle n,l\,|\,\pi r^2\,|\,n,l\rangle\\&=(n+l+1)hc/e\ .\end{aligned} \tag{16}$$

This result is the quantum analog of the well-known classical theorem that the magnetic moment of a charged particle in a static magnetic field is an adiabatic invariant. We can easily verify this invariance and find other adiabatic invariants by exhibiting the dependence of various expectation values on the magnetic field or on the cyclotron frequency. The kinetic energy is linear in ω and therefore is not an invariant; we can add energy to the particles by slowly increasing the field. The mean angular momentum about the origin can be shown to be independent of ω and therefore is an adiabatic invariant, as in classical mechanics. The derivation above shows that the flux is also independent of ω and is an adiabatic invariant. In fact, the flux and the angular momentum are independent invariants since they have different dependence on the quantum numbers n,l.

Our discussion so far has concerned only a spinless, nonrelativistic charged particle. The inclusion of spin and relativity can easily be carried out; we will only give the results for reference. The eigenvalues of the Klein-Gordon equation for a spinless charged particle in a uniform field are

$$E_{n,l}=\pm[m^2c^4+\hbar\omega mc^2(2n+1)]^{1/2}\ . \tag{17}$$

The eigenvalues of the Pauli equation for a spin-$\frac{1}{2}$ particle with arbitrary magnetic moment are

$$E_{n,l,m_s}=(n+\tfrac{1}{2})\hbar\omega-gm_s\hbar\omega/2\ , \tag{18}$$

and the eigenvalues of the Dirac equation for a spin-$\frac{1}{2}$ particle with $g=2$ are

$$E_{n,l,m_s}=\pm[m^2c^4+\hbar\omega mc^2(2n-2m_s+1)]^{1/2} . \tag{19}$$

In each case the eigenfunctions can easily be constructed using the Schrödinger eigenfunctions. We turn next to a description of the "betatron motion" of a charged particle in the same uniform field with a singular magnetic field (solenoid of zero size) added at the origin.

BETATRON MOTION

The addition of a solenoid at the origin alters the translational invariance of the equations of motion by introducing a preferred position for the orbit center. We should expect this to break the degeneracy in l. The cylindrical symmetry of the problem is not altered if we choose a suitable gauge for the potential of the solenoid:

$$\vec{A}=(\vec{\Phi}\times\vec{r})/2\pi r^2 , \tag{20}$$

where the direction of $\vec{\Phi}$ is along the magnetic field in the solenoid. We can solve the eigenvalue problem as before with a straightforward separation of variables in cylindrical coordinates. The azimuthal dependence must still be $\exp(iM\theta)$ with integer M to maintain single valuedness. This implies that the angular momentum about the field axis has nonintegral eigenvalues $M-f$. The added vector potential of the solenoid simply replaces M by $M-f$ everywhere in the radial equation

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(M-f)^2}{r^2}+(M-f)-\frac{r^2}{4}+2E\right]G(r)=0 . \tag{21}$$

This has the immediate consequence that the replacement of f by $f+j$, where j is an integer, will leave the eigenvalue problem unchanged except for a relabeling of the states. We will refer to this as the periodicity property of the eigenvalues: The set of eigenvalues is periodic in f with period unity or in Φ with period hc/e. Consequently, we can restrict the range of f to lie between zero and unity. Since $M-f$ appears in the Schrödinger equation as a parameter, we can obtain the new eigenvalues and eigenfunctions from the old by simply continuing the parameter M from integer to noninteger values. The new eigenvalues are therefore

$$E_{N,M}=(2N+1+|M-f|-(M-f))/2 , \tag{22}$$

and the new eigenfunctions become

$$\psi_{N,M}\propto r^{|M-f|}L_N^{|M-f|}(r^2/2)e^{-r^2/4}e^{iM\theta} . \tag{23}$$

We immediately see that the eigenvalues form two distinct ladders, depending on whether M is greater or lesser than f. In the former case, the ladder is the same as for cyclotron motion; in the latter case, the ladder has the same spacing as before, but is shifted upward by a constant energy $f\hbar\omega$. This is certainly not the only effect of the solenoid, since the radial functions also depend on f, but this energy shift is the principal consequence of adding the solenoid, and justifies our earlier comment about the AB effect for bound states. Before attempting to discuss the physical interpretation and the experimental consequences of this energy shift, we would like to reanalyze the problem using operator methods. There is a simple extension of the Johnson-Lippman method, and it substantially aids both the calculation and the interpretation of the results.

The added potential of the solenoid changes the definition of the velocity and orbit center variables:

$$\begin{aligned}\pi_x&=p_x+y/2+fy/r^2 ,\\ \pi_y&=p_y-x/2-fx/r^2 ,\\ x_0&=x/2+p_y-fx/r^2 ,\\ y_0&=y/2-p_x-fy/r^2 .\end{aligned} \tag{24}$$

It also changes their commutation rules, adding terms which are singular at the origin. The nonzero commutators are now

$$\begin{aligned}[\pi_x,\pi_y]&=i+2\pi if\delta(\vec{r}) ,\\ [x_0,y_0]&=-i+2\pi if\delta(\vec{r}) ,\\ [\pi_x,x_0]&=[\pi_y,y_0]=+2\pi if\delta(\vec{r}) .\end{aligned} \tag{25}$$

The presence of these extra singular terms in the commutators might appear to interfere with the use of annihilation and creation operators, but it does not in fact do so. The reason is that on states which are sufficiently regular at the origin, the singular terms simply vanish, and the usual methods of raising and lowering states proceed without difficulty. We must, however, divide the eigenfunctions into two classes and introduce two separate "vacuum" states. The two classes correspond to the states on the two different ladders and are characterized by different indicial behavior at the origin. We shall distinguish the states which link the origin with a subscript $(-)$ as a reminder that they have $(M-f)$ negative. These states have indicial behavior r^f times a polynomial. The states which do not link the origin we shall denote with a subscript $(+)$, since $(M-f)$ is positive. They have indicial behavior r^{1-f} times a polynomial. Care must be taken in the construction of the eigenstates never to cross the boundary $M=f$ separating these two classes of states. It is convenient to introduce the raising and lowering operators

$$a^{\dagger}=\pi_y+i\pi_x\ ,\quad b^{\dagger}=x_0+iy_0\ ,$$
$$a=\pi_y-i\pi_x\ ,\quad b=x_0-iy_0\ . \tag{26}$$

We find that $a^{\dagger}$ and b reduce M by 1, while $b^{\dagger}$ and a increase M by 1. Therefore in constructing the $(+)$ ladder out of its vacuum state, we must order the operators so that $b^{\dagger}$ acts first and $a^{\dagger}$ acts next; for the $(-)$ ladder the ordering must be different, with $a^{\dagger}$ first and $b^{\dagger}$ next. With this simple precaution, the usual methods of constructing the oscillator eigenstates can be carried out without difficulty. The ground state of the $(+)$ ladder is characterized by $a\psi_+=0$ and is found to be

$$\psi_+=[2^{(2-f)}\pi\Gamma(2-f)]^{-1/2}r^{(1-f)}e^{-r^2/4}e^{i\theta}\ . \tag{27}$$

The lowest state in the $(-)$ ladder is defined by $b\psi_-=0$ and is given by

$$\psi_-=[2^{(1+f)}\pi\Gamma(1+f)]^{-1/2}r^f e^{-r^2/4}\ . \tag{28}$$

The other states in the $(+)$ ladder can now be constructed by raising operators acting on ψ_+:

$$\begin{aligned}\psi_{n,l}^{(+)}&=[2^{(n+l-1)}n!\Gamma(l+1-f)/\Gamma(2-f)]^{-1/2}(a^{\dagger})^n(b^{\dagger})^{l-1}\psi_+\\&=(2^n n!)^{1/2}[2^{(l+1-f)}\pi\Gamma(l+1-f)]^{-1/2}r^{(l-n-f)}L_n^{(l-n-f)}(r^2/2)e^{-r^2/4}e^{i(l-n)\theta}\ .\end{aligned} \tag{29}$$

The states in the $(-)$ ladder can be constructed from ψ_-:

$$\begin{aligned}\psi_{n,l}^{(-)}&=[2^{(n+l)}l!\Gamma(n+f+1)/\Gamma(f+1)]^{-1/2}(b^{\dagger})^l(a^{\dagger})^n\psi_-\\&=(-)^{l+n}(2^l l!)^{1/2}[2^{(n+l+f)}\pi\Gamma(n+l+f)]^{-1/2}r^{(n-l+f)}L_l^{(n-l+f)}(r^2/2)e^{-r^2/4}e^{i(l-n)\theta}\ .\end{aligned} \tag{30}$$

These are of course the same states one would discover by separation of variables, but the operator method can not be used to calculate the normalization integrals and other expectation values. Again we caution the reader that care must be exercised in ordering the operators acting on these states to avoid generating functions with the wrong indicial behavior. For example, the two different ground states cannot be connected by raising operators; if we operate on ψ_- with $b^{\dagger}$, we do not get the regular eigenfunction ψ_+. Instead, we get a linear combination of both the regular and the irregular solutions of the radial equation. It is straightforward but laborious derivation to check that the eigenstates constructed in Eqs. (29) and (30) are in fact energy eigenstates. One finds

$$H\psi_{n,l}^{(+)}=(n+\tfrac{1}{2})\psi_{n,l}^{(+)}\ ,$$
$$H\psi_{n,l}^{(-)}=(n+f+\tfrac{1}{2})\psi_{n,l}^{(-)}\ . \tag{31}$$

We can now examine the adiabatic invariance of various quantities as we slowly increase the flux through the solenoid. For example, the kinetic energy of the $(-)$ states has a term linear in f and is therefore not an adiabatic invariant. We can increase the energy of particles in these states by slowly increasing the flux through the solenoid; this is clearly the quantum version of the betatron acceleration mechanism. On the contrary, the energy of the $(+)$ states is an adiabatic invariant. We can evaluate the mean-square radius of the states and find that there is a dependence on f for both classes:

$$\langle n,l(\pm)\,|\,r^2\,|\,n,l,(\pm)\rangle=2(n+l+1\mp f)\ . \tag{32}$$

The radius of the nonlinking $(+)$ states is decreased by an increase in the flux; the $(-)$ states have an increased radius. This illustrates the fact that in addition to modifying the energy of states, the vector potential also changes the radial functions. There are two different consequences of this dependence of the radius on f. One involves the behavior of the angular momentum about the solenoid:

$$L_z\equiv x\pi_y-y\pi_x=-i\frac{\partial}{\partial\theta}-f-r^2/2\ . \tag{33}$$

We readily discover that the mean angular momentum of the particles in the $(+)$ states is independent of f and therefore adiabatically invariant. The angular momentum of the $(-)$ states depends on f:

$$\langle n,l,+\,|\,L_z\,|\,n,l,+\rangle=-(2n+1)\ ,$$
$$\langle n,l,-\,|\,L_z\,|\,n,l,-\rangle=-(2n+1+2f)\ . \tag{34}$$

We also find that the mean flux through the $(+)$ states is an adiabatic invariant, but not through the $(-)$ states. By the flux, we mean the total flux of the (accessible) uniform magnetic field plus the flux of the inaccessible field in the solenoid. There is an exact compensation between the flux added to the solenoid and the flux through the increased or decreased area of the state. For the $(+)$ states, these changes cancel and the total flux remains unchanged. For the $(-)$ states these effects have the same sign and add together:

$$\begin{aligned} \Phi_{n,l}^{(+)} &\equiv fhc/e + B\langle n,l,+\,|\,\pi r^2\,|\,n,l,+\rangle \\ &= (n+l+1)hc/e\ , \\ \Phi_{n,l}^{(-)} &\equiv fhc/e + B\langle n,l,-\,|\,\pi r^2\,|\,n,l,-\rangle \\ &= (n+l+1+2f)hc/e\ . \end{aligned} \tag{35}$$

Again we notice that these adiabatic invariants are independent since they have different dependence on the quantum numbers n and l.

DISCUSSION

One conclusion that should be pointed out is that some of the effects of the solenoid are classical in nature, as well as quantum mechanical. For example, the energy shift $f\hbar\omega$ is in fact independent of $\hbar$ and therefore a classically defined quantity:

$$\Delta E = f\hbar\omega = q^2 B\Phi/2\pi mc^2\ . \tag{36}$$

This can easily be seen to be the remanent effect of the electric fields which accompanied the build up of the flux in the solenoid. The adiabatic increase in the flux produces an electric field which accelerates the particles linking the solenoid. Similarly, the added term in the angular momentum of particles linking the solenoid can be traced to the torque exerted on the particles by the electric fields during the build up of the flux:

$$\Delta L_z = f\hbar = q\Phi/2\pi c\ . \tag{37}$$

Another conclusion is that the overall shift in the energies of the $(-)$ states can in principle be measured. In the nonrelativistic limit, the energy is undefined to within an additive constant and so the uniform shift of the entire ladder of $(-)$ states cannot be measured. However, in the relativistic limit, there is an absolute scale of energy set by the rest energy. The constant energy shift in Eq. (36), like the zero-point energy, will affect the cyclotron frequency of a particle at high velocity. The change in the cyclotron frequency as f is increased from 0 to $\frac{1}{2}$ is of order $\hbar\omega/mc^2$, which is at the parts per 10^9 level for an electron in a strong magnetic field. While this is correct in principle, the effects are very small in practice and do not represent a useful method of testing the AB effect.

The best way to reveal the AB effect in this system is through the appearance of the new "betatron" resonance frequencies owing to radiative transitions between the two ladders. These resonances will merge with the cyclotron resonance for $f=0$ and $f=1$ but are distinct for $0<f<1$. Consider the resonances generated by transverse electric fields of low frequency. If the wavelength is much longer than the size of the cyclotron orbits, then the dipole approximation is valid. For transitions within either class $(+)$ or $(-)$, the normal dipole selection rules apply:

$$\Delta n = \pm 1\ , \quad \Delta l = \pm 1\ . \tag{38}$$

These rules follow from the fact that the dipole operator is a linear combination of the raising and lowering operators (a's and b's), which can change either quantum number by unity. Thus for transitions along either ladder, we obtain the usual cyclotron resonance at ω. At high power levels, there will be sidebands of these resonances at integral multiples of ω, owing to multiphoton transitions. Transitions between the two ladders generate a new set of frequencies, based on $f\omega$ and $(1-f)\omega$. These transitions exactly satisfy the rule $\Delta M = \pm 1$, which follows from the azimuthal variation of the dipole operator. There is no rule limiting the change in the individual quantum numbers n,l however. There are dipole matrix elements connecting states with arbitrarily large Δn and Δl. In the mathematics, this can be traced to the extra singular terms in the commutators, which have finally had some physical effect. Of course these extra terms must eventually affect the matrix elements of the a's and b's. It is to be expected that they influence the states with $n \cong l$ since this corresponds with orbits which graze the solenoid. We have evaluated the dipole matrix elements for a transition between any $(-)$ state with $M=0$, to any other $(+)$ state with $M=1$:

$$\begin{aligned} D_{n,n'} &\equiv \langle n,n+1,+\,|\,x+iy\,|\,n',n',-\rangle \\ &= (-)\frac{\sin\pi f}{\pi}[2\Gamma(n+2-f)\Gamma(n'+1+f)/n!n'!]^{1/2}[(n-n'+1-f)(n-n'-f)]^{-1}\ . \end{aligned} \tag{39}$$

The method used is rather tedious but straightforward: One can use the Rodrigues formula for the Laguerre polynomials plus repeated integration by parts. The principal result is the appearance of two new resonances at $f\omega$ and $(1-f)\omega$, corresponding to transitions with $n=n'$ and $n=n'+1$. These resonances are accompanied by a series of sidebands separated by ω, which still result from one-photon dipole transitions. Unlike the sidebands on the cyclotron resonance, they have a strength which is linear in the rf power, rather than being quadratic in the power, etc. Their strength does decrease with increasing $n-n'$ however. One can derive from Eq. (39) an approximate result valid for large n,n' and for $f=\frac{1}{2}$:

$$D_{n,n'} \cong \frac{-1}{\pi}(2n)^{1/2}[(n-n')^2-\tfrac{1}{4}]^{-1} . \qquad (40)$$

For this special case, the betatron resonances seen in absorption would consist of lines at $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots)\omega$, equally spaced between the cyclotron resonance and its overtones. The relative strengths of the transitions $n(+)$ to $n'(-)$ would be $(1, \frac{1}{9}, \frac{1}{225}, \ldots)$ and for the transitions $n'(-)$ to $n(+)$ would be $(\frac{1}{9}, \frac{1}{225}, \frac{1}{1225}, \ldots)$. Thus if both spectra were equally weighted, the strength of the $(\frac{3}{2}\omega$ line would be about 10% of the fundamental at $(\frac{1}{2})\omega$; the line at $(\frac{5}{2})\omega$ would be about 0.5% of the fundamental. This characteristic pattern should enhance the distinguishability of the betatron resonances in the spectrum.

ROLE OF FINITE SOLENOID RADIUS

Up to this point in the discussion, we have considered only a solenoid of zero size. The reasons for this are primarily mathematical, not physical: we can then evaluate everything exactly. Such a solvable model is of course valuable, but must not obscure the truth of the matter. It is only with great difficulty that one can actually make a solenoid whose dimensions are smaller than the characteristic size of the quantum levels. In this section, we would like to open the discussion of how the betatron resonances might actually be observed and of the role of the finite size of the solenoid. Of course the last word is for the experimenter; we can only open the discussion. To begin, we note that it is possible to treat the quantum theory of a solenoid of finite size exactly. If we assume the solenoid to be contained within an impenetrable cylinder of radius a, then we must solve the original radial equation with the additional boundary condition that the solution vanish at a. The desired solution is therefore that linear combination of the regular and irregular solutions which remains regular at infinity. This solution is well known and is generally called $U(b,c,r^2/2)$, where $2b=|\mu|-\mu+1-2E$ and $c=|\mu|+1$ and where μ is the combination $(M-f)$. The eigenvalue condition is therefore the transcendental equation[16]

$$U(\tfrac{1}{2}|\mu|-\tfrac{1}{2}\mu+\tfrac{1}{2}-E;|\mu|+1;a^2/2)=0 . \qquad (41)$$

We have not tried to follow this approach, however, since it would clearly involve lengthy numerical calculations.

Instead we have searched for approximation methods which give the general features of the finite size effects. The best approximation method is undoubtedly the asymptotic approximation (WKB), since it describes the approach to classical theory. The eigenvalue condition now becomes a single quadrature

$$\pi(N+\tfrac{1}{2})=\int_{r_1}^{r_2} dr(2E+\mu-\mu^2/r^2-r^2/4)^{1/2} , \qquad (42)$$

where r_1 and r_2 are the turning points. It is reassuring to note that this integral can be evaluated analytically and for a solenoid of zero size leads to the exact eigenvalues given in Eq. (22). If the radius a is less than the inner turning point r_1, then there is no change in the eigenvalues in this approximation. The turning points can easily be located for the grazing states of interest, which have μ of order unity and N large. This already introduces a critical size for the radius

$$a<\mu(2N+1)^{-1/2} . \qquad (43)$$

This condition on the radius is overly restrictive. It corresponds to a solenoid much smaller than the quantum scale of length. We must consider a radius larger than the inner turning point with some resulting shift in the eigenvalues. If we treat N as a large parameter and expand in powers of $1/N$, then we can evaluate the integral analytically and still carry out the inversion to find E

$$E_{N,M} \cong N+\tfrac{1}{2}+\tfrac{1}{2}|\mu|-\tfrac{1}{2}\mu + \frac{a}{2\pi}(2N+1)^{1/2} . \qquad (44)$$

This shows a nonuniform stretching of the ladders of eigenvalues with a gradual increase in the separation for larger N. The states of high N are raised more by the presence of the solenoid, since the inner turning point r_1 decreases with increasing N. The result is a broadening of the cyclotron resonance

$$\delta(E_{N+1}-E_N)=\frac{a}{2\pi}(2N+1)^{-3/2}\delta N . \qquad (45)$$

This is a type of homogeneous line broadening, since it is due to a spread in the distribution of orbit radii, not to magnetic field gradients. However, the more important conclusion from Eq. (44) is that the betatron resonance is neither shifted nor broadened: the two ladders are still displaced by an amount $f\hbar\omega$, independent of a in this approximation.

An even stronger theorem can be established for the special case of $f=\frac{1}{2}$. It is evident by inspection of Eq. (42) that N depends only on the combinations $2E+\mu$ and μ^2. It follows that on inversion, the eigenvalues will have the form

$$E_{N,M}=F(N,\mu^2)-\mu/2 \ , \tag{46}$$

where F is an unknown function of N and μ^2. Our results in Eqs. (22) and (44) exemplify this, but the same property holds for arbitrary size a. Indeed, this property is true for the exact eigenvalues of the Schrödinger equation, as is evident by inspection of Eqs. (21) and (41). The interesting consequence of this concerns the betatron resonances for $f=\frac{1}{2}$, which involve the energy difference of states with $\mu=\pm\frac{1}{2}$. We find that the fundamental betatron resonance frequency for $f=\frac{1}{2}$ is given by

$$E(N,\mu=-\tfrac{1}{2})-E(N,\mu=+\tfrac{1}{2})=\omega/2 \ . \tag{47}$$

The two ladders undergo the *same* stretching and are simply displaced by the energy $\hbar\omega/2$. We conclude that the main effect of the finite size of the solenoid is to broaden the cyclotron resonance, without broadening or shifting the betatron resonance, for $f=\frac{1}{2}$. The sidebands at $(\frac{3}{2}, \frac{5}{2},\ldots)\omega$ will be broadened, since ΔE depends on N. We see from Eq. (45) that the broadening does not become serious until $a\simeq\pi(2N+1)^{1/2}$, which is much less restrictive than Eq. (43). It is much more difficult to assess the effect of finite size on the strength of the betatron resonance. The line strength is probably reduced when a increases. This is one question perhaps better decided by measurement than by calculation.

We conclude that the betatron resonances can be expected to appear as sharp features in the microwave absorption spectrum, even for solenoid radii fairly large on the quantum scale of length. The betatron resonance could be observed through expansion of the orbit; either by repeated transitions at the betatron frequencies or by a single transition at the betatron resonance followed by repeated transitions at the cyclotron resonance. The particular role of $f=\frac{1}{2}$ and the need for very small sizes strongly suggest the use of superconducting cylinders, rather than wire-wound solenoids. Since the flux trapped in a superconducting cylinder is quantized in units $hc/2e$, it should be possible to make the betatron resonance appear and disappear as the trapped flux changes from even to odd multiples of $hc/2e$. Our discussion has developed criteria for the radius of the solenoid and for the strength of the magnetic field, but not for the mass of the ion. The mass of the ion determined the frequency range of the resonances, but does not seem otherwise restricted. Thus we suggest the use of singly charged ions in weak magnetic fields of a few gauss or less. Under these conditions, the solenoid radii can be on the scale of a few micrometers or more.

ACKNOWLEDGMENTS

We acknowledge conversations on this subject with K. Harvey, L. Sander, and W. L. Williams. We are grateful for the assistance of J. Yen during part of this work. This research was supported in part by a grant from the National Science Foundation PHY-79-21560.

---

[1]Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959).

[2]W. Ehrenburg and R. E. Siday, Proc. Phys. Soc. London, Sect. B62, 8 (1949).

[3]T. T. Wu and C. N. Yang, Phys. Rev. D 12, 3845 (1975). The same physical effects can be described without the use of gauge theory. Two alternatives are discussed by S. Mandelstam, Ann. Phys. (N.Y.) 19, 1 (1962); G. Casati and I. Guareri, Phys. Rev. Lett. 42, 1578 (1979). We will use the gauge viewpoint, however, since it is the most widely accepted formalism.

[4]R. C. Jaklevic, J. J. Lambe, A. H. Silver, and J. E. Mercereau, Phys. Rev. Lett. 12, 274 (1964).

[5]R. G. Chambers, Phys. Rev. Lett. 5, 3 (1960); H. A. Fowler, L. Marton, J. A. Simpson, and J. A. Suddeth, J. Appl. Phys. 32, 1153 (1961); G. Mollenstedt and W. Bayh, Naturwissenschaften 49, 81 (1962); W. Bayh, Z. Phys. 169, 492 (1962).

[6]G. Schaal, C. Jonsson, and E. F. Krimmel, Optik (Stuttgart) 24, 529 (1966).

[7]Objections to the interpretation of electron interference experiments have been raised by P. Bocchieri and A. Loinger, Nuovo Cimento 47A, 475 (1978); P. Bocchieri, A. Loinger, and G. Siragusa, *ibid.* 51A, 1 (1979); and answered by U. Klein, Lett. Nuovo Cimento 25, 33 (1979); A. Zeilinger, *ibid.* 25, 333 (1979); D. Bohm and B. J. Hiley, *ibid.* 52A, 295 (1979).

[8]For a recent review, see Second International Conference PMFC, National Bureau of Standards, Gaithersburg, Maryland, June, 1981 (unpublished).

[9]M. Peshkin, I. Talmi, and L. J. Tassie, Ann. Phys. (N.Y.) 12, 426 (1961); M. Kretzschmar, Z. Phys. 185, 97 (1965); M. Peshkin, Phys. Rep. 80, 375 (1981).

[10]B. S. Deaver and W. M. Fairbank, Phys. Rev. Lett. 7, 43 (1961); R. Doll and M. Nabauer, *ibid.* 7, 51 (1961); N. Byers and C. N. Yang, *ibid.* 7, 46 (1961).

[11]W. A. Little and R. D. Parks, Phys. Rev. Lett. 9, 9

(1962).
[12]L. Onsager, Philos. Mag. 43, 1006 (1952).
[13]J. R. Gilleland and A. Rich, Phys. Rev. A 5, 38 (1972).
[14]L. Landau and E. Lifshitz, *Quantum Mechanics* (Addison-Wesley, Cambridge, MA, 1958), Chap. 16.
[15]L. D. Landau, Z. Phys. 64, 629 (1930); M. H. Johnson and B. A. Lippmann, Phys. Rev. 76, 82B (1949).
[16]Magnus, Oberhettinger, and Soni, *Special Functions of Mathematical Physics,* 3rd ed. (Springer, New York, 1966), Chap. 6. A similar equation has been given by R. P. Feynman, *Lectures on Physics* (Addison-Wesley, Cambridge, MA, 1960), II-25.30.