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Functional renormalization-group theory of universal $1/f$ noise in dynamical systems

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The functional renormalization group is used to show that a wide class of dynamical systems displays spectra with a $1/f^\delta$ divergence. Cutoffs introduced by stochastic or deterministic perturbations have universal features as well. Numerical support is presented.

The wide occurrence of $1/f$ noise in a large variety of physical systems[1] poses one of the most intriguing problems of statistical physics. In spite of considerable theoretical efforts,[2,3] a general theory that could encompass $1/f^\delta$ divergencies in a number of realizations is still lacking. In this Rapid Communication we show that a class of maps of the interval display $1/f^\delta$ noise down to zero frequencies, and we tie the exponent δ to the properties of the map in a universal fashion via the renormalization-group approach.[4] Furthermore, we show that the occurrence of random noise and other relevant perturbations of the map results in a cutoff of the $1/f^\delta$ divergence, and that the cutoff has also universal features.

Our starting point is the observation of Manneville[5] that a particular map which displays intermittency displays also $1/f$ noise. Intermittency means the occurrence of a signal which alternates randomly between long regular (laminar) phases (so-called intermissions) and relatively short irregular bursts. A rapid progress in the understanding of intermittency has been achieved recently.[6-10] To our purposes, it is most important that the phenomenon of intermittency has been shown to be universal, with universality classes which depend only on the properties of the one-dimensional Poincaré map, close to a particular point.[9,10] The same functional renormalization-group equations employed by Feigenbaum for studying the period doubling cascade[4] pertain to this phenomenon as well (with a mere change of boundary conditions).[9,10] Here we use this fact to construct a universal theory of $1/f$ noise.

Consider then the class of maps of the interval which for $0 < X_n << 1$ reads

$$X_{n+1} = f(X_n) = X_n + u\,|X_n|^z \quad , \tag{1}$$

where z_n is the observed variable at time $t = n$. For concreteness see the map of Fig. 1. We stress however that it will be shown below that Eq. (1) should pertain only to the vicinity of $X = 0$ and should be in fact interpreted as an expansion of f there. u is thus the coefficient of expansion and the exponent z determines the universality class.

The signal generated by this class of maps is shown in Fig. 1(b).[5] Motions on the interval $X < a$ in Fig. 1(a) are responsible to the laminar phases and motions on the interval $X > a$ are responsible for the bursts. Following Manneville[5] we simplify the mathematical analysis of the signal by idealizing it as the signal in Fig. 1(c). Here we consider a train of signals of zero duration and of size unity. Since we are interested in the long-time properties of the correlation function this idealization is reasonable.

Consider then the correlation function $R(m)$,

$$R(m) = \sum_{X(0)=0}^{1} \sum_{X(m)=0}^{1} W(X(m),X(0))X(m)X(0) \quad , \tag{2}$$

where $W(X(m),X(0))$ is the joint probability to see a signal $X(0)$ (zero or one) at the 0th iteration and a signal $X(m)$ at the mth iteration. Writing the joint probability as the conditional times the marginal we find

$$R(m) = C_m(1|1)\,W(X(0)=1) \quad , \tag{3}$$

where $C_m(1|1)$ is the conditional probability to see a

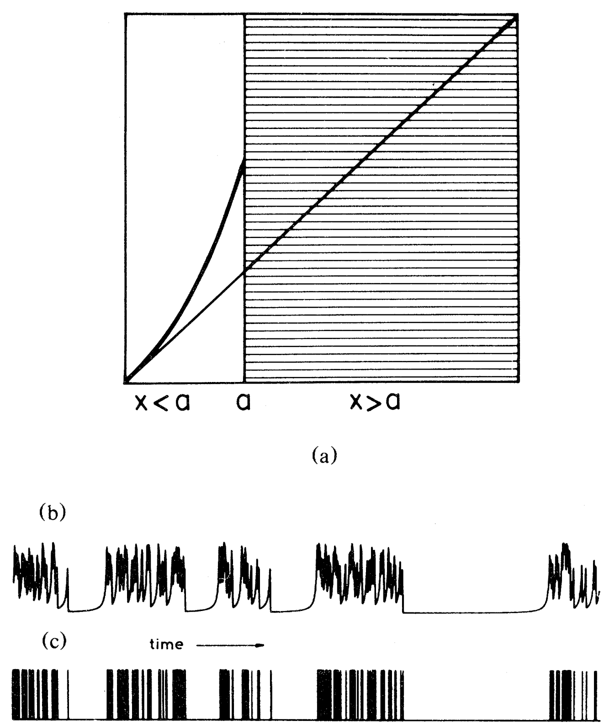

FIG. 1. (a) A map of the interval which for $X < a < < 1$ conforms with Eq. (1). For $X > a$ the map is arbitrary except that it should lead to chaotic motion such that the trajectory is reinjected to the interval $X < a$. (b) A typical (Ref. 5) signal obtained from iterating the map of Fig. 1. The intermissions are associated with motions on the interval $X < a$, and the bursts come from motions on the interval $X > a$. (c) The idealization of the signal (a).

nonzero signal at the mth iteration if there were such a signal at the zeroth iteration. We thus see that the issue of the existance of $1/f$ noise will be determined by the long-time (large m) properties of C_m. We derive now an equation for C_m in terms of $P(k)$ where $P(k)$ is the probability for the occurrence of an intermission of length k. This equation would read

$$C_m = \sum_{k=0}^{m} C_{m-k} P(k) \quad , \tag{4}$$

with the initial conditions $C_0 = 1$, and $P(0) = 0$. The reason for Eq. (4) is that we can have a signal at the mth iteration only by having a signal at the $(m-k)$th iteration plus an intermission of length k. For large m we can pass to continuous time and Eq. (4) together with its initial conditions transforms to

$$C(t) = \int_0^t d\tau\, C(t-\tau) P(\tau) + \delta(t) \quad . \tag{5}$$

Upon Laplace transformation we find

$$C_s = \frac{1}{1 - P_s} \quad , \tag{6}$$

where $f_s = \int_0^\infty e^{-st} f(t)\, dt$. To calculate the spectrum

$$S_f = \int_0^\infty C(t) \cos(2\pi f t)\, dt$$

we shall evaluate $C_{s \to -2\pi i f} + C_{s \to 2\pi i f}$ at the end.

The probability for an intermission $P(l)$ is now found as follows: the size l at an intermission is determined by the starting point X_0 on the laminar side of the map, and the point a, $l = l(X_0, a)$. Conversely, X_0 is a function of l, $X_0 = X_0(l,a)$. The probability that the trajectory is reinjected between X_0 and $X_0 + dX_0$ is denoted by $\hat{P}(X_0)\, dX_0$ and can therefore be written as

$$\hat{P}(X_0)\, dX_0 = \hat{P}(X_0(l,a)) \left| \frac{dX_0}{dl} \right| dl \equiv P(l)\, dl \quad . \tag{7}$$

The function $X_0(l,a)$ can be now found via the renormalization-group method. According to this approach the intermittent behavior is determined by the eigenfunctions and eigenvalues of the doubling operator

$$Tf(X) = \alpha f\left(f\left(\frac{X}{\alpha} \right) \right) \quad , \tag{8}$$

where α is a rescaling factor and $f(x)$ is a function *which for* $X \to 0$ behaves like the function of Eq. (1) [i.e., corresponding to the boundary conditions $f(0) = 0$ and $f'(0) = 1$]. In the absence of relevant perturbations (to be discussed below), a repeated application of T leads to a fixed point $f^*(X)$:

$$T^n f(X) = \alpha^n f^{2^n}(X/\alpha^n) \underset{n \to \infty}{\longrightarrow} f^*(X) \quad . \tag{9}$$

Here both

$$\alpha = 2^{1/(z-1)}$$

and

$$f^*(X) = X[1 - (z-1) u X^{z-1}]^{-1/(z-1)}$$

depend only on z which determines the universality class.[9,10]

The duration of an intermission can be obtained from the equation $X(l, X_0) = a$. Focusing on intermissions of length

$$l = 2^n >> 1 \tag{10}$$

we then solve

$$X(2^n, X_0) = \alpha^{-n} f^*(\alpha^n X_0) = 1 \tag{11}$$

or

$$X_c(l = 2^n, a) = \frac{a}{[1 + (z-1) u \alpha^{n(z-1)}]^{1/(z-1)}} \tag{12}$$

which for large n leads to $X_0 \sim \alpha^{-n}$. Using Eq. (12) and the form of α we find in Eq. (7)

$$P(l) = \hat{P}(\text{const}\, l^{-1/(z-1)})\, l^{-z/(z-1)} \underset{l \to \infty}{=} \hat{P}(0)\, l^{-1/\nu} \quad , \tag{13}$$

where

$$\frac{1}{\nu} = \frac{\log\alpha}{\log 2} + 1 = \frac{z}{z-1} \quad .$$

We thus see that $P(l)$ is universal and that in order to obtain intermissions of arbitrary long duration it is sufficient to have a finite probability of reinjection close to $X_0 = 0$.

To calculate the spectrum S_f we have to evaluate P_s. Using the fact that $P(0) = 0$ we evaluate then

$$P_s = \int_1^\infty e^{-sl} P(l)\, dl = \int_1^\infty e^{-sl} l^{-1/\nu}\, dl \Big/ \int_1^\infty l^{-1/\nu}\, dl \tag{14}$$

and substitute the result in Eq. (6). The integrals are available[11] and lead to the final result

$$S_f \underset{f \to 0}{\sim} \begin{cases} f^{-1/(z-1)} , & z > 2 \ , \\ \dfrac{1}{f|\log f|^2} , & z = 2 \ , \\ f^{-(2z-3)/(z-1)} , & \frac{3}{2} < z < 2 \ , \\ |\log f| , & z = \frac{3}{2} \ , \\ \text{const} , & z < \frac{3}{2} \ . \end{cases} \tag{15}$$

Equation (15) is the central result of this paper.

The final issue is the effect of perturbations of the map on the spectrum. The effects of both deterministic and stochastic perturbations on the intermittency problem have already been studied,[7-10] with the net result that the average intermission length, as well as all the other moments of $P(l)$, become finite. The average intermission length, $\langle l \rangle$, has been shown[9,10] to scale like

$$\langle l \rangle \sim \epsilon^{-\nu} g(\sigma/\epsilon^\mu) \quad , \tag{16}$$

where ϵ is the shift of the map from tangency at $X = 0$, and σ is the amplitude of random noise present. g is a universal scaling function and $\mu = (z+1)/z$. For our purposes it is sufficient to know that for $s\langle l \rangle << 1$

$$P_s = \int_1^\infty e^{-sl} P(l)\, dl \simeq 1 - s\langle l \rangle + \tfrac{1}{2} s^2 \langle l^2 \rangle \quad . \tag{17}$$

Upon substitution in Eq. (6) we find for $f \to 0$

$$S_f = \frac{1}{2\pi \langle l \rangle} \delta(f) + \frac{1}{2} \frac{\langle l^2 \rangle}{\langle l \rangle^2} \quad . \tag{18}$$

The δ function contribution can be eliminated by considering $\hat{C}_m = C_m - C_\infty$ and we thus conclude that the spectrum becomes now constant for small frequencies. The above analysis breaks down for frequencies $f_c \sim \langle l \rangle^{-1}$ or for

$$f_c \sim \epsilon^\nu g^{-1}(\sigma/\epsilon^\mu) \quad . \tag{19}$$

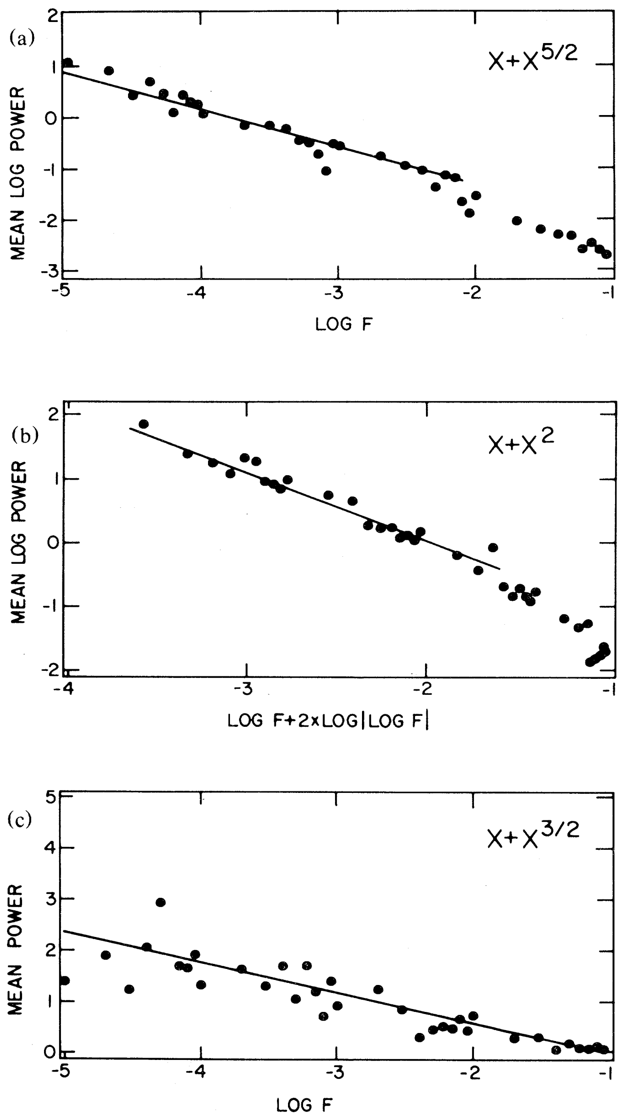

FIG. 2. Power spectra obtained by averaging ten runs of 500 000 iterations each for three values of z: (a) $z=\frac{5}{2}$, (b) $z=2.0$, (c) $z=\frac{3}{2}$. Agreement with Eq. (15) is obtained.

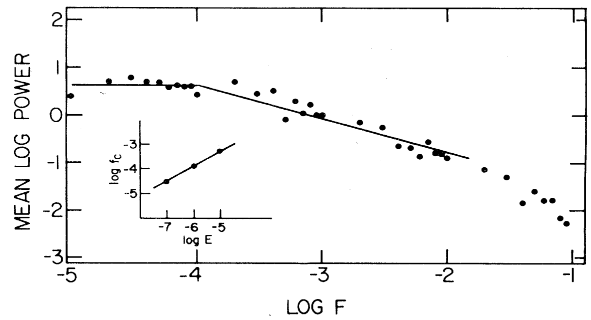

FIG. 3. The cutoff in the $1/f^{\delta}$ divergence due to shift from tangency. $z=\frac{5}{2}$ with $\epsilon=10^{-6}$ is shown. In the inset $\log f_c$ vs $\log\epsilon$ is drawn for $z=\frac{5}{2}$. Agreement with Eq. (19) is obtained.

Above f_c the power law Eq. (15) will be regained.

In Fig. 2 we present numerical support to Eq. (15). In Fig. 3 we show the cutoff frequencies introduced by a shift from tangency. Equation (19) is verified.

Finally we comment on the observability of these spectra in other nonlinear systems. The intermittent signals found near tangent bifurcations give rise to spectra which differ from those analyzed here. The reason for the difference is that in those cases the limit $\sigma\to 0$, $\epsilon\to 0$ provides a marginally stable fixed point which is attractive from one side ("type I intermittency," cf. Ref. 6). The analysis of spectra in these cases will be published in a forthcoming paper. The intermittency mechanism discussed above is of type II or III according to the classification of Ref. 6. In the limit $\sigma\to 0$, $\epsilon\to 0$ the system has a purely repellent marginally fixed point. This type of intermittency has been observed and analyzed in a mathematical model for chemical reactions.[12] Since the mechanisms of intermittency of types II and III are generic,[6] it is to be expected that the $1/f^{\delta}$ noise spectra calculated above will be observed in nonlinear dynamical systems.

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