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Observation of multiple-valued attractors and crises in a driven nonlinear circuit

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The attractor of the chaos in a driven nonlinear dissipative circuit is observed. The attractor is made of a curve imperfectly folded many times. The crisis effect, the sudden onset of chaos caused by the collision of the subband attractor with the unstable periodic orbit, is verified experimentally.

The work described in this paper concerns analysis of measurements from a simple experiment.[1-4] The voltage from an oscillator, $V(t)=V_0\cos 2\pi f_0 t$, is applied to the series combination of an inductor and a junction diode. The voltage appearing across the diode is a series of rectified pulses. For a low value of V_0, all pulse heights are identical. As the oscillator amplitude is increased, repeated period doubling or bifurcation of the pulse-height pattern occurs. Further increase of V_0 results in appearance of bands of pulse heights and occasional appearance of periodic signals. This feature of the signal has been compared with the properties of one-dimensional maps.[5] The "memory" of the diode associated with the finite lifetime of the carriers is a source of the observed behavior of the signal.[6]

The analysis of the pulse heights in terms of their sequence is described in this paper. By means of simple laboratory equipment, a direct display of the "attractors" is made. It is found that the attractor consists of a curve folded many times which is the characteristic of the two- or higher-dimensional maps.[7] The display of the attractor also enables us to verify an effect called "crisis."[8,9]

The voltage across the diode consists of a sequence of pulses of height $\{a_n\}$ (n stands for the nth pulse). Two sample and hold circuits are used to capture the pulse heights in such a manner that the sequence $\{a_n\}$ can be displayed as points on an x-y oscilloscope at (a_n, a_{n+m}) with m a parameter. Also the horizontal or vertical projections of these points on the diagonal line $y=x$ are displayed. This is accomplished by sampling every mth pulse and also introducing a time delay into the x-axis channel of approximately one-half the time between the pulses to be analyzed. The oscillator frequency applied to the circuit, typically 30 kHz, is about twice the resonance frequency of the circuit. But the choice of f_0 is not critical because the circuit is very dissipative in a range of V_0 where the nonlinear features are observed. The junction diodes we have employed have the carrier lifetime of 50–200 μs. No bifurcations were found if the carrier lifetime was shorter than $1/f_0$.

When the oscillator voltage V_0 is small so that all pulses are identical, a single dot located on the diagonal is seen. In the language of maps, only one fixed point is present. When V_0 is increased, the number of dots observed is doubled at each bifurcation. Figure 1(a) is a relation between a_n and a_{n+1} for a case of eight dots off the diagonal, showing that three bifurcations have taken place. In the language of maps this is the period-8 orbit. Since $a_{n+8}=a_n$, all of the 8 points are on the diagonal line if they were displayed on the a_{n+8}-a_n plane. Figure 1(b) is a picture for V_0 so large that bands of pulse heights appear and the dots degenerate to curve segments. For an even larger V_0, the curve segments merge and form a continuous curve [Fig. 1(c)]; the pulse-height distribution is continuous. At some values of V_0, periodic orbits such as the one shown in Fig. 1(d) appear. At larger values of V_0, a_{n+1} becomes a double- or triple-valued function of a_n [Figs. 1(e) and 1(f)] which cannot be explained by the one-dimensional maps. An indication of a double-valued curve is also seen in the experiment reported in Ref. 3.

Although the frequency spectrum of the signal across the diode is continuous when the pulse-height distribution is continuous, the pulse heights are not random. Their sequence follows the rule which is the curve shown in Fig. 1. If the pulse heights were random, the dots would distribute on the oscilloscope screen uniformly for all choices of m.

The relationship between a_{n+1} and a_n, $a_{n+1}=f_1(a_n)$, consists of curves imperfectly folded many times. The split-

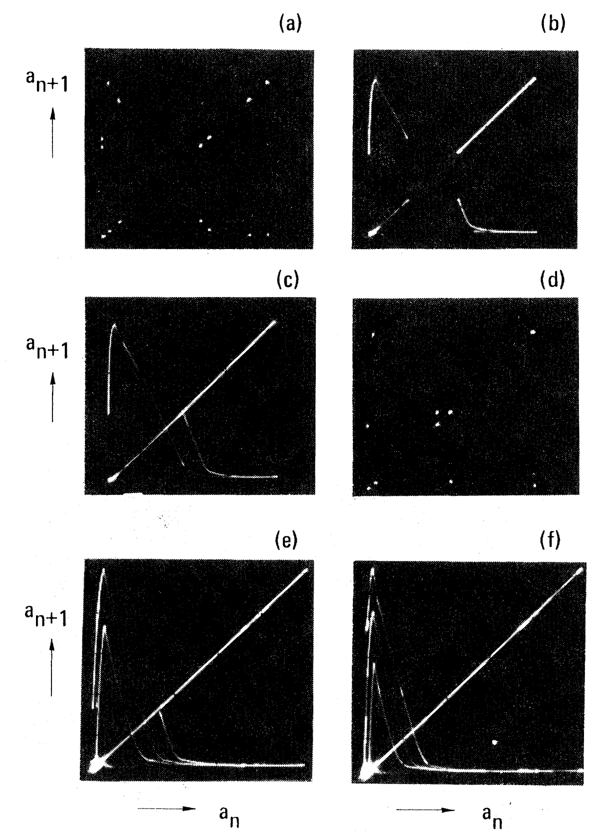

FIG. 1. Relation between the consecutive pulse heights a_n and a_{n+1}. The parameter V_0, the oscillator amplitude, is smallest in (a) and is increased successively in the order (b), (c), ... (a) Period-8 orbit, (b) aperiodic subbands, (c) merged band, (d) period-5 orbit, (e) double-, and (f) triple-valued attractors.

ting associated with the imperfect folding is not resolved in the pictures shown in Fig. 1. The splitting is, however, clearly visible in displays of $a_{n+m}=f_m(a_n)$ for m greater than unity. Examples of this are shown in Fig. 2. As m is increased, the folds spread apart into many connected curves. For very large values of $m(>10)$, the dots nearly uniformly distribute over the oscilloscope screen, but they still form many lines.

The description of the observed pulse-height sequence requires the higher-dimensional map. The physical reason for this is undoubtedly that the memory of the diode stays longer than one oscillator period. If the memory were limited to one period, the height of the $(n+1)$th pulse a_{n+1} would depend only on the height of the nth pulse a_n or

$$a_{n+1}=F(a_n) \quad . \tag{1}$$

If the memory is extended to two periods, one may have

$$a_{n+1}=G(a_n,a_{n-1}) \quad , \tag{2}$$

which may be expressed as a two-dimensional map letting

$$b_{n+1}=a_n \quad . \tag{3}$$

Instead of deriving the functional form of G from the nonlinear circuit analysis, which is complicated, we have looked at a simple model function studied by Henon[10]:

$$a_{n+1}=H(a_n)+\gamma b_n \quad , \tag{4}$$

where H is a peaked function and γ is a small constant. The computational iteration of (3) and (4) produces an imperfectly folded attractor on the a_n-a_{n+1} or the a_n-b_n plane. The Housdorf dimension of the attractor has been confirmed to be larger than one[11]; the attractor is strange. The attractor folds spread apart in the plot of a_n vs a_{n+m} with large m. All of these features are seen in Figs. 1 and 2; the curves in these figures are attractors. Although the dimension of the observed attractor has not been measured, the close resemblance of the observed attractors with the one produced by the Henon map suggests that the observed attractor is strange with dimension between one and two.

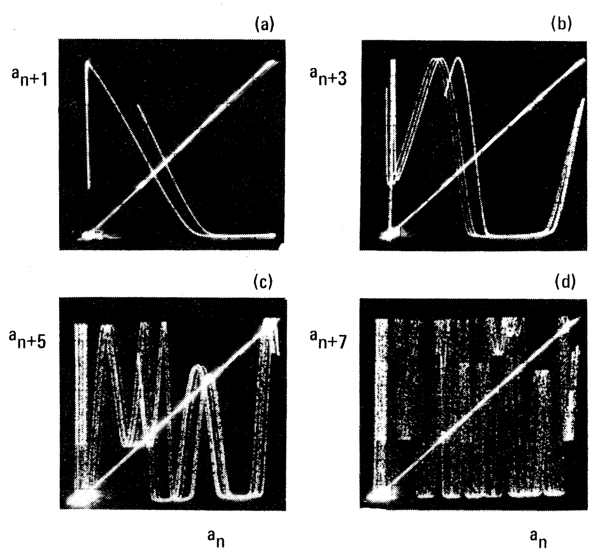

FIG. 2. a_{n+m} as a function of a_n. (a) $m=1$, (b) $m=3$, (c) $m=5$, and (d) $m=7$. V_0 is kept constant.

At the particular values of V_0, sudden transitions between the periodic and the aperiodic orbits appear.[2,3] Figure 1(d) shows one such case; the appearance of the period-5 orbit. The same type of transitions was found in the numerical works.[8,9] Grebogi, Ott, and Yorke[8] have pointed out that a sudden transition from a periodic to an aperiodic state occurs when an unstable periodic orbit collides with the attractor and they called this transition crisis. Following is the experimental verification of this process.

Let us consider the attractor in the a_n-a_{n+5} plane. The period-5 orbit appears to be five dots on the diagonal line. At a value of V_0 slightly smaller than the value, say V_1, where the transition to the period-5 orbit appears, five bunches of extrema of the attractor curves appear near the diagonal line. These extrema are created by the unfolding effect described previously in this paper. None of the extrema of the attractors crosses the diagonal line when $V_0 < V_1$. Figure 3(a), which is the expanded picture of

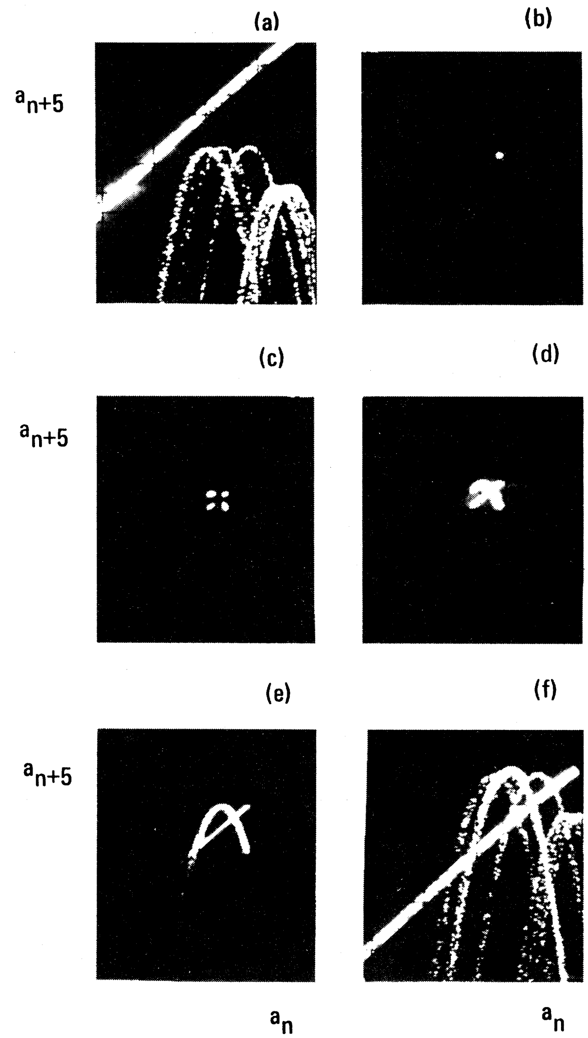

FIG. 3. Part of the plot of a_{n+5} as a function of a_n showing the feature of the crisis on the period-5 orbit. V_0 is increased in the order (a), (b), ... (a) Aperiodic orbit before period-5 orbit appears, (b) period-5 orbit, (c) period-10 orbit, (d) period-5 subband, (e) the unstable orbit, and (f) recovery of aperiodic orbit.

one of five regions, shows this. At this particular region the extrema are maxima. As V_0 increases, the maxima move upward, and one of the several maxima touches the diagonal line. This occurs at the five different regions at the same value of V_0. Each of the five tangential points is the marginally stable double orbit. If V_0 is slightly larger than V_1, then the double orbit splits into two orbits; one is unstable (derivative of the attractor curve at the tangential point, $da_{n+5}/da_n > 1$) and the other one is stable ($|da_{n+5}/da_n| < 1$). This stable orbit appears as a dot on the diagonal line as shown in Fig. 3(b). The unstable orbit is not observable. The increase of V_0 makes the stable period-5 orbit unstable, i.e., $da_{n+5}/da_n < -1$. But this is nonlinearly stable, so that the dot in Fig. 3(b) bifurcates into two off-diagonal dots as shown in Fig. 3(c). As V_0 increases further, these two dots eventually become a segment of a curve which is one of five minisubbands shown in Fig. 3(d). The subband expands until the left-hand-side end of it hits the unstable orbit. The unstable orbit is the left-hand-side crossing point of the attractor and the diagonal line shown in Fig. 3(e). Thus the band hits the unstable orbit when it hits the diagonal line. This collision of the attractor with the unstable orbit occurs at the five different regions on the diagonal line at the same value of V_0. The full-band aperiodic orbit then suddenly appears [Fig. 3(f)]. We have seen the same process occurring in the case of the period-3 orbit. The attractor of the subband in this case is a double-valued function consisting of one continuous curve. The expansion of the band occurs when one of the ends of the curve collides with the unstable orbit.

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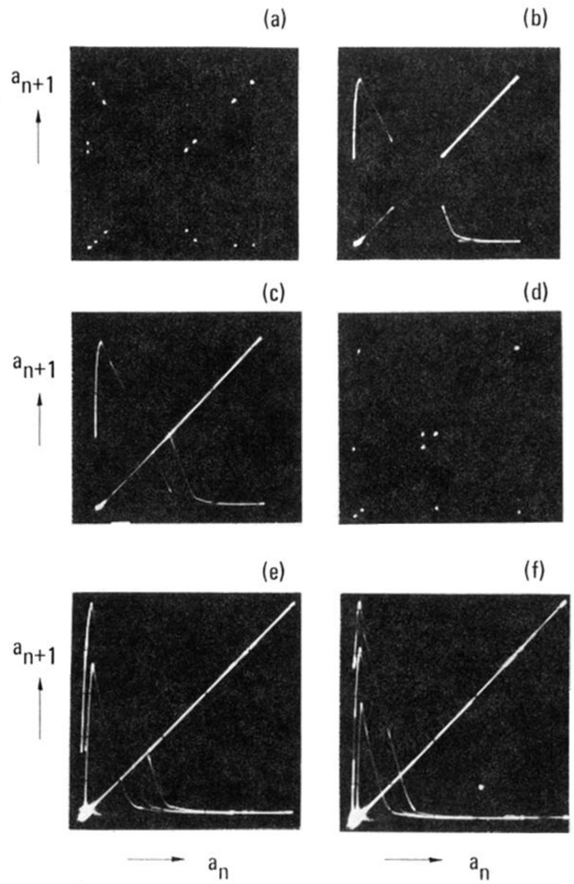

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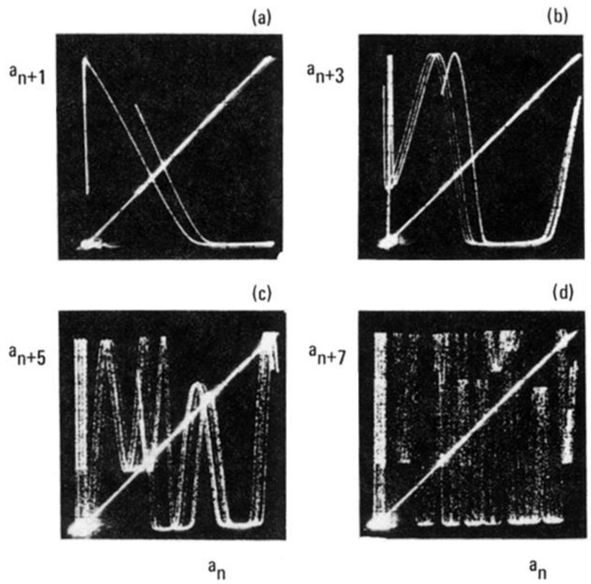

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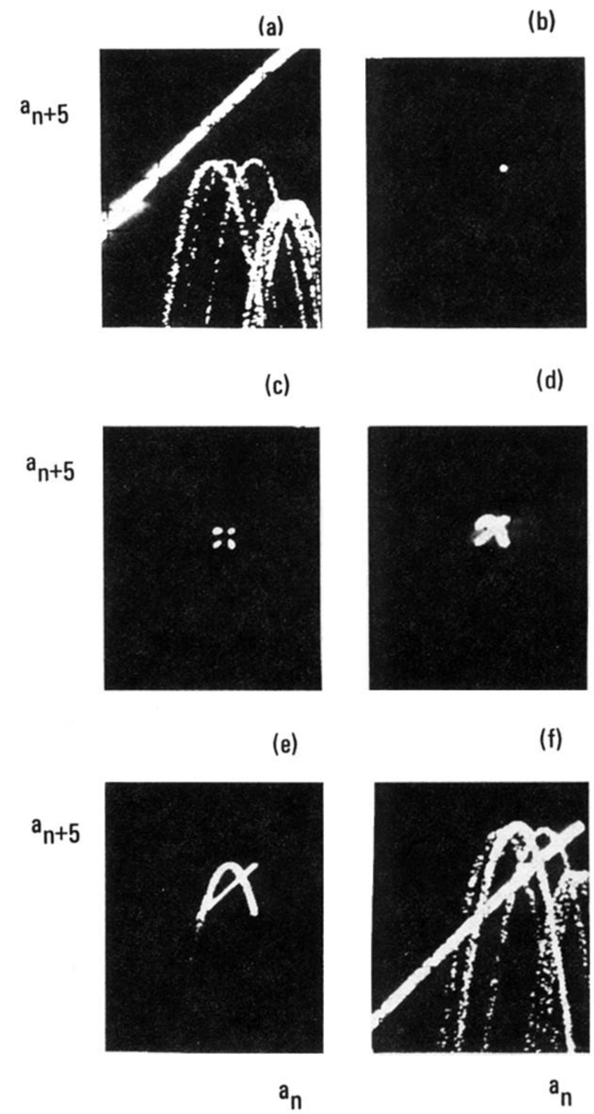

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