

Dynamic behavior of nonlinear networks

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(Received 10 March 1983)

We study the global dynamics of nonlinear networks made up of synchronous threshold elements. By writing a master equation for the system, we obtain an expression for the time dependence of its activity as a function of parameter values. We show that with both excitatory and inhibitory couplings, a network can display collective behavior which can be either multiple periodic or deterministic chaotic, a result that appears to be quite general.

Recent advances in large-scale integration have led to the construction of large assemblies of computing elements whose complex dynamic behavior is largely unknown. Examples can be found in information handling structures which range from compact dynamic RAM (random access memories) and logic circuits¹ to extended computer networks.² In all these instances, one deals with large collections of synchronous threshold elements whose states (active or passive) at a given time can in turn excite or inhibit other elements of the network at a later time. Thus, as either the connectivity or the coupling strength between nonlinear elements increases, a sustained activity of the network can, in principle, be excited with suitable initial conditions. If such collective modes do indeed exist, it is of interest to study their nature (e.g., periodic or erratic) and their stability as a function of parameters such as threshold behavior and connectivity. These questions are relevant to cellular automata, neural networks,³⁻⁵ and many-body systems as studied by finite-time step methods such as Monte Carlo techniques.

This paper presents the results of a study of the global dynamics of nonlinear networks made up of threshold elements. By writing a master equation for the system, we obtain an analytic expression for the time dependence of its activity as a function of parameter values. As we will show, for wide ranges of nonlinearity and connectivity a nonlinear network with both excitatory and inhibitory couplings can display collective behavior which can be either multiple periodic or deterministic chaotic. The latter type consists of an intense and sustained erratic activity which results from the relaxation of initial configurations of a system with many degrees of freedom to a strange attractor. This effect, besides providing a plausible explanation for the diffuse reverberations which have been reported in computer simulations of neural networks^{4,5} might also be applicable to some studies of magnetic spin systems as studied by Monte Carlo techniques.

We start by describing the nonlinear elements which make up the network. These are synchronous threshold devices which, when active, send signal pulses to their connected neighbors. The signal pulses (e.g., voltage, magnetic flux, or digital packets) can be either positive (excitatory) or negative (inhibitory), and we will assume that each of the identical elements has the same number of inputs (p excitatory connections plus r inhibitory ones) as outputs. Each threshold device, analogous to the formal neurons of logic networks,⁶ will fire with probability p if after a summing period τ the sum total of its inputs exceeds a threshold

value V_0 . If to each neuron we assign a variable s_i such that it is $+1$ if it fires, and $s_i = -1$ if it does not fire, the threshold behavior is described by specifying a probability function which, following Little, we choose to be given by⁷

$$p(s_i') = [\exp[-\epsilon s_i'(V_i - V_0)] + 1]^{-1}, \tag{1}$$

where s_i' denotes the value of s_i' at $t + \tau$, ϵ determines the width of the threshold region, and V_i is the input potential, which can be written as

$$V_i = \sum_j V_{ij} \frac{s_j + 1}{2}, \tag{2}$$

with V_{ij} denoting the signal (e.g., voltage) input to the i th neuron coming from the j th neuron, and the sum j is over the $r + p$ incoming connections.

A network of N neurons is then described at each instant of time t by an N vector $\alpha = (s_1, \dots, s_N)$ specifying which neurons are active ($s_i = +1$) and which ones are not ($s_i = -1$). If $P(\alpha, t)$ denotes the probability that at time t the network is in state α , its value at $t + \tau$ is determined by the master equation

$$P(\alpha, t + \tau) - P(\alpha, t) = - \sum_{\{\beta\}} \omega(\alpha \rightarrow \beta) P(\alpha, t) + \sum_{\{\beta\}} \omega(\beta \rightarrow \alpha) P(\beta, t), \tag{3}$$

where $\omega(\alpha \rightarrow \beta)$ denotes the transition probability in time τ from a state α to a state β . For the particular case of single-neuron processes it is determined by Eq. (1).⁸ Introducing new variables

$$\tilde{J}_{ij} \equiv \frac{\epsilon V_{ij}}{2}, \quad \tilde{H} = \frac{\epsilon}{2} \sum_j V_{ij} - 2V_0,$$

and

$$E_i \equiv \sum_{j=1} \tilde{J}_{ij} s_j + \tilde{H}, \quad \omega(\alpha \rightarrow \beta)$$

can be written as

$$\omega_L(s_L) = \exp(-E_L s_L) / \sum_{s_L'} \exp(E_L s_L'). \tag{4}$$

The quantity of interest is the activity of the network which we will define as

$$A = (\langle s_k \rangle + 1)/2, \tag{5}$$

with the expectation value of the k th spin at time t given by

$\langle s_k \rangle \equiv \sum_{\{s\}} s_k P(s, \dots, s_N, t)$ where $\{ \}$ implies a sum over the 2^N possible values of the set $\{s_1, \dots, s_N\}$, and $\langle s_k \rangle$ ranges from -1 to $+1$. Using Eq. (3) the expectation value can be written in terms of the following equation of motion

$$\langle s_k \rangle_{t+\tau} - \langle s_k \rangle_t = -2 \langle s_k \omega_k(s_k) \rangle_t, \quad (6)$$

with $\omega_k(s_k)$ given by Eq. (4). It now remains to calculate the right-hand side of this equation. In the limit of small excess voltages ($V_i - V_0$ small) or broad threshold regimes (small ϵ), it is possible to generate a series expansion in powers of E_k for Eq. (4). To third order in E_k , we obtain

$$\omega_k(1 - E_k s_k + E_k^3 s_k^3/2), \quad (7)$$

which can in turn be used to evaluate the sum in Eq. (6). In the spirit of a mean-field theory, we shall assume that $\langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle$ if $i \neq j$, $\langle s_j \rangle = \langle s \rangle$, and write $\sum_j \tilde{J}_{kj} = \tilde{J}$, $\sum_j \tilde{J}_{kj}^2 = \tilde{J}^2$, and $\sum_j \tilde{J}_{kj}^3 = \tilde{J}^3$ with

$$\tilde{J} \equiv p\tilde{J}_1 - r\tilde{J}_2, \quad (8a)$$

$$\tilde{J}^2 \equiv p\tilde{J}_1^2 + r\tilde{J}_2^2, \quad (8b)$$

$$\tilde{J}^3 \equiv p\tilde{J}_1^3 - r\tilde{J}_2^3, \quad (8c)$$

where J_1 and J_2 are the strengths of the excitatory and inhibitory potentials, respectively. A lengthy but straightforward calculation then gives the following equation of motion

$$\begin{aligned} \langle s \rangle_{t+\tau} = & \tilde{H}(1 - \tilde{J}^2 - \tilde{H}^3/3) + (\tilde{J} - \tilde{J}\tilde{J}^2 + 2\tilde{J}^3/3 - \tilde{H}^2\tilde{J}) \langle s \rangle_t \\ & - \tilde{H}(\tilde{J}^2 - \tilde{J}^2) \langle s \rangle_t^2 - (\tilde{J}^3/3 - \tilde{J}\tilde{J}^2 + 2\tilde{J}^3/3) \langle s \rangle_t^3. \end{aligned} \quad (9)$$

Equation (9), together with Eq. (5), describes the dynamical behavior of the network to third order in $\epsilon(V_i - V_0)$. Although a detailed study of the solutions of the equation for different parameter values will be given elsewhere, we will now describe several important regimes in the global behavior of the activity. Consider first a network with only purely excitatory (i.e., ferromagneticlike) or inhibitory (i.e., antiferromagneticlike) interactions. In such a limit, either \tilde{J}_1 or \tilde{J}_2 are zero, and $\tilde{J}^2 \gg \tilde{J}^2$. It then follows that Eq. (9) becomes $\langle s \rangle_{t+\tau} = \tilde{H} \pm |\tilde{J}| \langle s \rangle_t + |\tilde{J}|^3 \langle s \rangle_t^3/3$. The solutions of this equation are such that for $J < 1$ all initial values of the spin configuration relax to the fixed point $\langle \bar{s} \rangle \approx \tilde{H}$ and therefore $A \approx (1 + \tilde{H})/2$. This static limit is similar to the dynamics of Ising-like models using a continuous approximation to the master equation, Eq. (3).⁹ If \tilde{H} is vanishingly small (i.e., the average input equals the threshold), we conclude that the long-time activity of a network with no competing interactions consists of a simple time-independent fixed point in which almost half of the neurons are firing at a given time. Thus the network carries no information. This result shows that all that is needed in order to have a sustained activity in a network with no competing interactions is a smooth nonlinear threshold.¹⁰

The second and most interesting case is the one in which the network has the same amount of randomly distributed excitatory and inhibitory connections (the neuron glass limit). In this limit \tilde{J} is negligible compared to \tilde{J}^2 or \tilde{J}^3 , and Eq. (9) becomes (in the limit of $\tilde{H} \rightarrow 0$)

$$\langle s \rangle_{t+\tau} = \tilde{R} (\langle s \rangle_t - \langle s \rangle_t^3), \quad (10)$$

where $\tilde{R} = (2\tilde{J}^3/3 - \tilde{J}\tilde{J}^2)$, and which for realistic values of the parameters displays nontrivial dynamics. Consider a

network with an initial activity A . For values of \tilde{R} in the range $-2 < \tilde{R} < 2$ its long-time behavior corresponds to a stable, time-independent fixed point whose magnitude increases monotonically with \tilde{R} . At $\tilde{R} = 2$ a pitchfork bifurcation takes place whereby initial configurations now relax to a periodic attractor with period 2τ . Therefore, the activity will not only be large and finite, but consist of collective modes which oscillate periodically in time. As \tilde{R} is increased even further, one encounters a full cascade of period doubling bifurcations at each of which the period of the asymptotic oscillations lengthens by a factor of 2. Thus, a computation of a property of the network (such as its energy) at fixed \tilde{R} would reveal an apparent finite set of configurations whose number depends on the period of the activity.¹¹ For values beyond $\tilde{R} = 2.3$ a new regime is encountered. This regime is characterized by an intense and erratic activity in which initial configurations of the network relax towards a strange attractor. Figure 1 shows an example of the time evolution of the activity of the network for $\tilde{R} = 2.34$, as generated by Eqs. (5) and (10) after an initial transient of 1000 time steps. As can be seen, the collective behavior of the system is chaotic, with a peak-to-peak intensity which can be determined by the first and second iterates of Eq. (10) evaluated at the maximum of its right-hand side. Furthermore, this deterministic erratic behavior is characterized by a broad-band component of its power spectrum which grows like $[\tilde{R} - (\tilde{R})_c]^\sigma$ with $\sigma \approx 1.525$.¹² As the value of \tilde{R} is further increased, the system may also exhibit laminar regimes which in some cases will be preceded by intermittent behavior,¹³ followed by an even more chaotic activity at larger values of \tilde{R} .

The effects just described provide a plausible explanation for the diffuse reverberations which have been observed in simulations on neural networks,⁴ and the reports of sustained oscillations in similar systems.¹⁴ Also, since the appearance of such collective modes in our theory seems to be a direct consequence of both finite-time steps and the nonlinearity of the threshold function, we expect them to be pervasive in many other network systems with the same characteristics, such as in Monte Carlo simulations of spin-glasses.¹⁵

Finally, we would like to mention that up to this point we have neglected explicit spontaneous fluctuations of the neurons other than those provided by the initial conditions. Such fluctuations (thermal, noise, or inputlike) can be of relevance to more complicated networks¹⁶ and to the dynamics of many-body systems when studied using finite-time step techniques. They can be straightforwardly included.

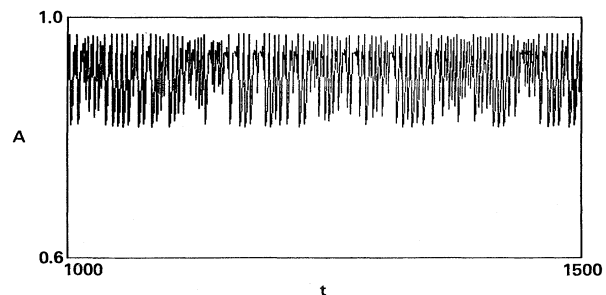


FIG. 1. Activity of a threshold network with competing interactions as a function of time (in units of the summing period τ) after a transient of 1000 iterations. $\tilde{R} = 2.34$.

ed by writing in place of Eqs. (1) and (4) the transition rate that appears in, for example, studies of kinetic Ising⁹ models, i.e.,

$$\omega_k(s_k) = \frac{1}{2}(1 - s_k \tanh \beta E_k) \quad (11)$$

with $\beta = (k_B T)^{-1}$ and k_B the Boltzmann constant. Using

this formula, we have computed the spin dynamics of such systems and obtained results which are similar to those of Eq. (9). This leads us to believe that these studies might also throw some light on the spin-glass problem.

This work was supported in part by a grant from the Office of Naval Research, No. N00014-82-0699.

¹A good description of such systems is provided by C. Mean and L. Conway, *Introduction to VLSI Systems* (Addison-Wesley, Reading, MA, 1980), Chaps. 5-7.

²See, for instance, L. Kleinrock, *Queuing Systems, Vol. II: Computer Applications* (Wiley, New York, 1976), Chaps. 5 and 6.

³See *Essays on Cellular Automata*, edited by A. W. Burks (University of Illinois Press, Urbana, 1970).

⁴M. Rochester, J. H. Holland, L. H. Haibt, and W. L. Duda, *IRE Trans. Inf. Theory* IT-2, 80 (1956). Notice that this work, dealing with neural networks of biological interest, stresses transient behavior after an initial stimulus. Our treatment deals with global dynamics, which applies to asymptotic behavior as well.

⁵A thorough review of neural networks is given by R. J. McGregor and E. R. Lewis, *Neural Modelling* (Plenum, New York, 1977).

⁶W. S. McCullough and W. Pitts, *Bull. Math. Bio.* 5, 115 (1943).

⁷W. A. Little, *Math. Bio.* 19, 101 (1974); G. L. Shaw and K. J. Ronney, *Phys. Lett.* 74A, 146 (1979). This choice is not unique and similar functional forms yield the same dynamics.

⁸Processes including simultaneous transitions of many neurons can be calculated by performing a multiple product of Eq. (1) by itself. Using Eq. (3) one then obtains a dynamical equation for the activity which, to third order in E_L , yields the same dynamics as

Eqs. (6) and (7).

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¹⁰Earlier studies of neural networks in which sharp thresholds were used claimed that competing interactions are needed in order to obtain stable behavior. See, for example, W. R. Ashby, H. von Foerster, and C. C. Walker, *Nature* 196, 561 (1962).

¹¹This observation might be of relevance to a Monte Carlo study of a neural network by J. J. Hopfield, *Proc. Natl. Acad. Sci. USA* 79, 2554 (1982).

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¹⁴D. R. Smith and C. H. Davidson, *J. Assoc. Comput. Mach.* 9, 268 (1962); P. A. Anninos, *Kybernetik* 11, 11 (1972).

¹⁵See, for example, A. J. Bray and M. A. Moore, *J. Phys. C* 11, 1187 (1978); D. Stauffer and K. Binder, *Z. Phys. B* 24, 97 (1979); A. P. Young and S. Kirpatrick, *Phys. Rev. B* 25, 440 (1982).

¹⁶For the case of brain neural networks, such effects have been discussed by G. Shaw and R. Vasudevan, *Math. Biosci.* 21, 207 (1974).