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Self-replicating attractor of a driven semiconductor oscillator

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The measured attractor of a driven *R-L*-varactor system is found to be self-replicating and inconsistent with a simple logistic map. A calculation based on a standard model of a varactor diode is in qualitative agreement with the data.

It is well established that a driven series *RLC* circuit in which the capacitor is replaced by a varactor diode can exhibit period doubling, chaotic bands, and periodic windows at subharmonics of the fundamental drive frequency.[1,2] If the system is highly dissipative — that is, for large resistance — the measured period-doubling routes to chaos are consistent with the theory of Feigenbaum[3] and the order of appearance of the periodic windows agrees with the predictions of Metropolis, Stein, and Stein.[4] This would suggest that the underlying attractor of this system is quite similar to the logistic equation $x_{n+1}=4ax_n(1-x_n)$. However, for systems with smaller values of series resistance the true attractor is more complex although quite regular in its structure.

The system studied is a series combination of a sine wave voltage source, a resistor, an inductor, and a varactor diode. Circuit parameters are given in Table I and defined below. The amplitude of the voltage source, V_d, is digitally controlled: The digital range of 2750 to 3150 corresponds to a drive amplitude range of 1 to 10 V. Figure 1 is a bifurcation diagram of the current flowing through the circuit as a function of V_d: For each value of drive amplitude the current is sampled once per cycle for many cycles at the negative-slope zero crossing of the driving sine wave.

The structure is quite complex but exhibits the usual features of period doubling, chaotic bands, and periodic windows. A return map (Fig. 2) of the current measured in the chaotic band at 2955 just below the period-4 window has a great deal more structure than a simple parabola. This is clarified by forming a phase-space portrait in three dimensions[5] from the points $I(t)$, $I(t+T)$, $I(t+2T)$, rotating it, and then projecting it onto a plane [Fig. 3(a)]. This structure slowly evolves as the drive voltage is raised to bring the system from the period-4 window to the period-5 window. It becomes the attractor of Fig. 3(b) in the chaotic band at 3059, just below the period-5 window. The self-replicating nature of the attractor is quite evident in a comparison of these two figures. The attractor in the period-5 window lies on the five exposed tips of this "plant." The attractor measured in the chaotic band below the period-3 window has similar structure but with appropriately fewer roots, vertical

TABLE I. Measured circuit parameters.

I_s	2.8 pA	V_t	34 mV
C_t	82 pF	ϕ	0.6 V
C_d	56×10^{-6} pF	R	25 Ω
γ	0.44	L	50 μH
f_0	2.5 MHz	V_d	0–10 V

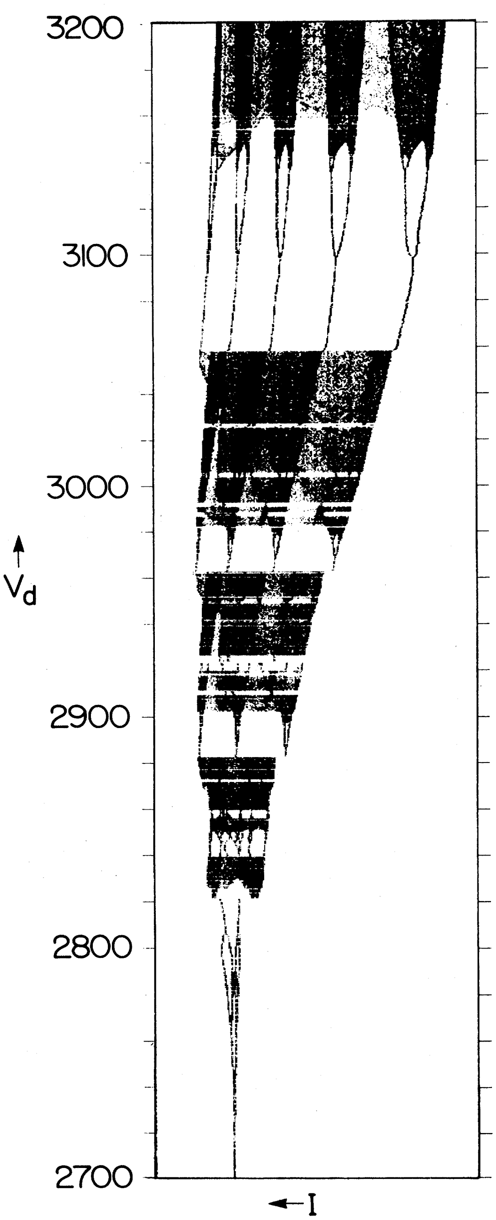

FIG. 1. Measured bifurcation diagram of the circuit with the parameters of Table I. Approximately one million data points are plotted. See text for explanation of V_d.

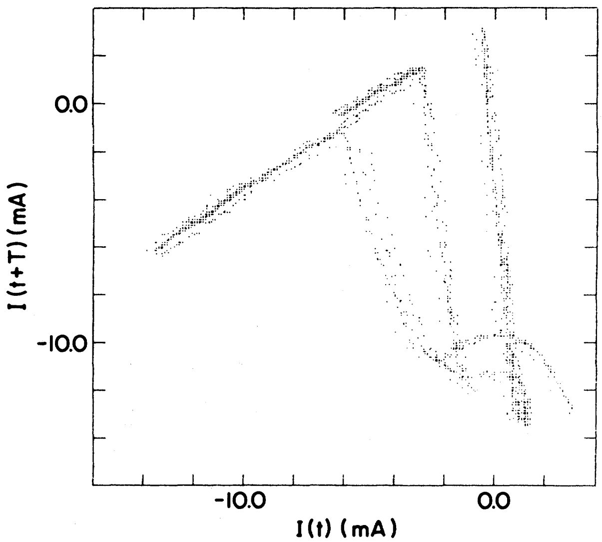

FIG. 2. Return map of the system measured just below the period-4 window, $V_d = 2955$. The driving period is T.

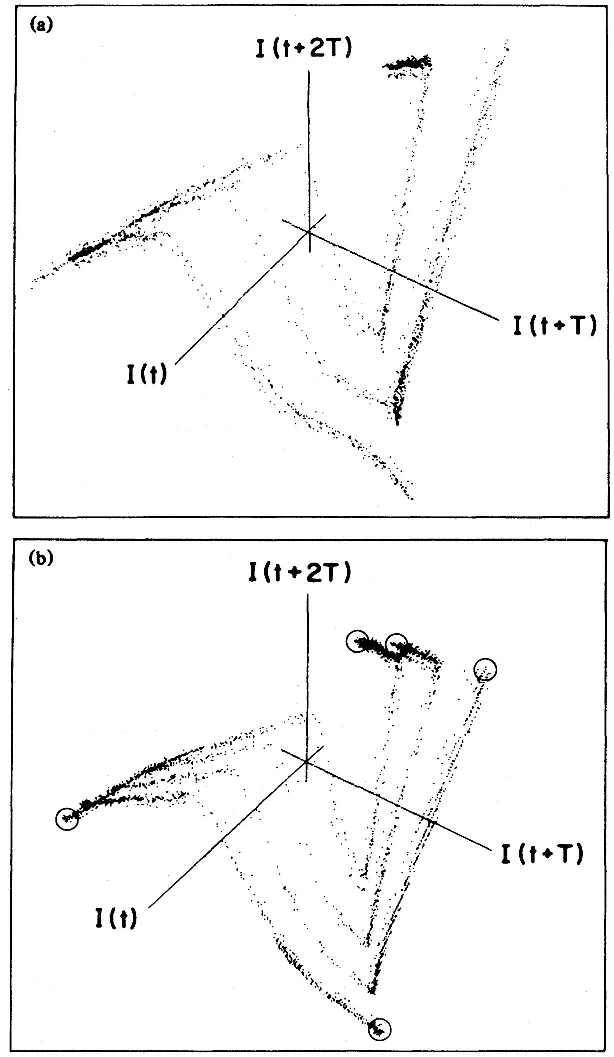

FIG. 3. Projection of the three-dimensional attractor (a) just below the period-4 window and (b) just below the period-5 window. The open circles indicate the location of the stable period-5 attractor.

stems, and leaves. If one decreases the resistance of the circuit, more major periodic windows and their plantlike attractors are brought within range of the available drive voltage. This explains why the highly dissipative systems are well described by a logistic equation: If the resistance is large, only one root segment of the attractor is accessible to a large range of drive voltages and thus the attractor is essentially an inverted parabola.

Among the features of the bifurcation diagram (Fig. 1) are the discontinuities shown at the boundaries of many of the periodic windows. Some windows contract inwards and have a range of currents less than the neighboring chaotic regions, such as the period-3 windows at 2870 and 2910, and others expand outwards, especially the major period-3, -4, and -5 windows at 2880, 2960, and 3060, respectively. This behavior is well described by boundary and interior crises of the attractor.[6] Such discontinuities are often indicative of hysteresis in the system. In fact, a bifurcation diagram made with the control voltage decreasing rather than increasing looks nearly the same but shows the abrupt chaotic to periodic transition boundaries shifted downward in position causing some features to be lost entirely. These shifted transitions show that the system has multiple basins of attraction. The hysteresis seen in the bifurcation diagrams is also coupled to the slope discontinuities of the attractor (Fig. 3). Three points of each major window always occur in association with slope discontinuities of the attractor. Each inwardly contracting window that has been examined was found to have at least one point lying on a nondifferentiable point of the attractor. The measured attractor and the hysteresis of the data are incompatible with the published model of Rollins and Hunt.[7]

In order to model this system, we have used the standard model of a varactor diode as a nonlinear capacitance in parallel with a nonlinear conductance.[8] The nonlinear differential capacitance $C(V_v) = dQ_v/dV_v$ was modeled as the transition capacitance for $V_v > -\phi$,

$$C(V_v) = C_t(1 + V_v/\phi)^{-\gamma} ,$$

and as the diffusion capacitance for $Vv < -\phi$,

$$C(V_v) = C_d \exp(-V_v/V_t) ,$$

where V_v and Q_v, the voltage across and the charge stored by the varactor, are positive for reverse bias conditions. Each capacitance is only significant in its range of definition and makes a negligible contribution to the diode capacity outside of it. These equations are integrated to find $Q_v(V_v)$ and thus $V_v(Q_v)$. The thermal voltage V_t is adjusted slightly to ensure continuity of $V_v(Q_v)$ at $Q_0 = Q_v(-\phi)$. The conductance is derived from the usual I-V characteristic of a pn junction,

$$I_{rv} = I_s\{1 - \exp[-V_v(Q_v)/V_t]\} ,$$

where the sign convention is as above. These equations can be combined with the usual circuit equations for a resistor R and an inductor L to give a second-order differential equa-

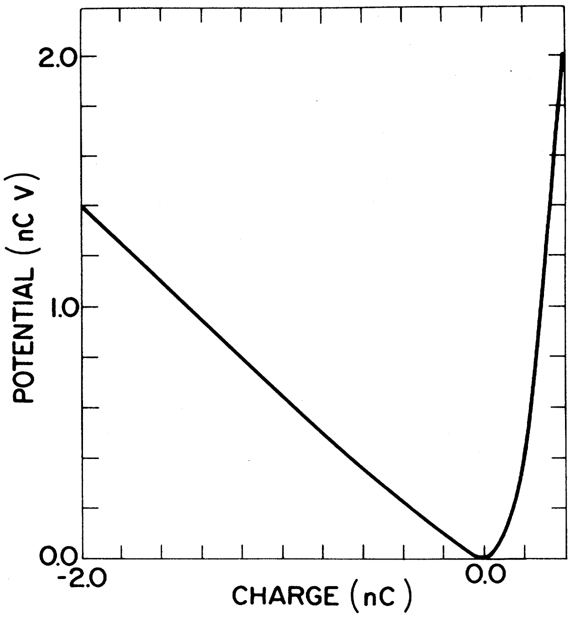

FIG. 4. Calculated potential function of the nonlinear circuit.

tion for the stored charge Q_v,

$$L\frac{d^2Q_v}{dt^2}+\left[L\frac{dI_{rv}}{dQ_v}+R\right]\frac{dQ_v}{dt}+V_v(Q_v)+RI_{rv}(Q_v)=V_d\sin(2\pi f_0t)\ . \quad (1)$$

This is the equation of a driven oscillator with nonlinear friction and force terms. The friction term is well described by a step function whose value is given by the resistor for $Q_v>Q_0$ and which steps up to a new value below Q_0. The force term, dominated by $V_v(Q_v)$, can be integrated giving the highly asymmetric potential well of Fig. 4. The potential behaves as $Q_v^{2.8}$ for large positive charge and as $a_0|Q_v|+a_1|Q_v|\ln|Q_v|$ for large negative values. These two components of the potential are due to the transition capacitance and the diffusion capacitance, respectively. The calculated bifurcation diagram for the measured circuit parameters of Table I is shown in Fig. 5. As can be seen, the model has all the features of the data and the attractor (not shown) has the same plantlike structure. Repeating the computation with I_{rv} set to zero, thus removing the varactor conductance, produces similar results, showing that the asymmetric potential is the cause of the periodic and chaotic behavior.

In conclusion, we have shown that a driven *R-L*-varactor system has a complicated, though regular, multidimensional self-replicating attractor. We conjecture that this structure repeats itself indefinitely as the control voltage is increased to arbitrarily large values and that a new branch is added to the attractor only when a period doubling transition to chaos has occurred. This macroscopic repetition of the attractor is quite different from the well-known microscopic self-similarity of a multidimensional map like the Henon attractor.[9] This behavior is modeled by a driven second-order differential equation that is derived from the physics of *p-n* junctions.

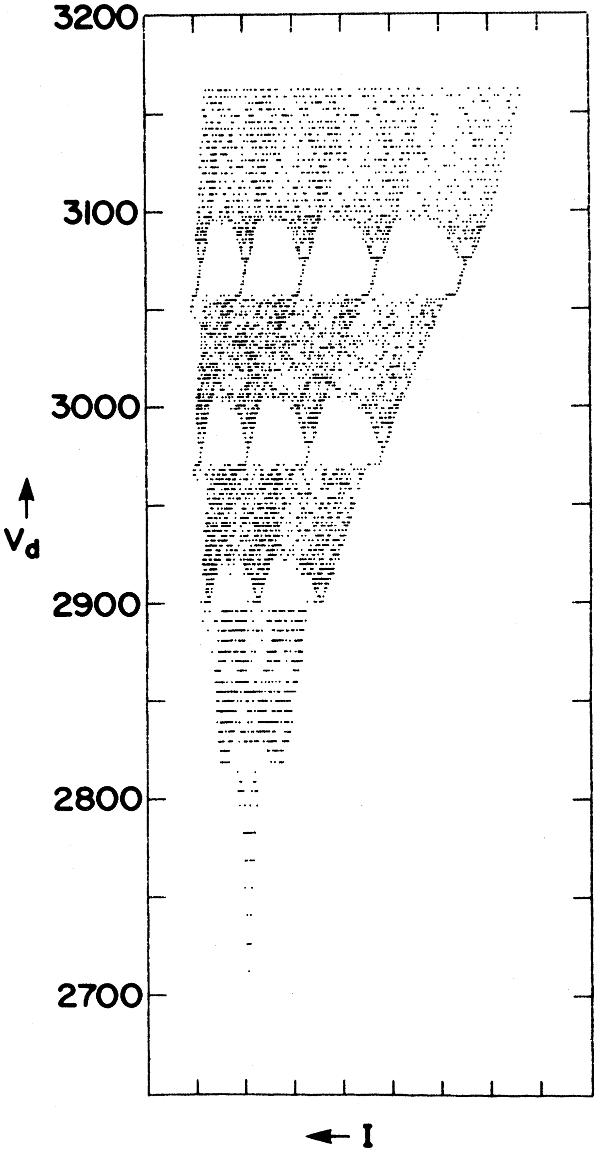

FIG. 5. Bifurcation diagram calculated by solving the nonlinear differential equation (1). About 10 000 points are plotted.

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[1]Paul S. Linsay, Phys. Rev. Lett. 47, 1349 (1981).

[2]James Testa, Jose Perez, and Carson Jefferies, Phys. Rev. Lett. 48, 714 (1982).

[3]M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978).

[4]N. Metropolis, M. L. Stein, and P. R. Stein, J. Comb. Theory A 15, 25 (1973).

[5]N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw, Phys. Rev. Lett. 45, 712 (1980).

[6]Celso Grebogi, Edward Ott, and James A. York, Phys. Rev. Lett. 48, 1507 (1982); Physica D (in press).

[7]R. W. Rollins and E. R. Hunt, Phys. Rev. Lett. 49, 1295 (1982).

[8]*Electronic Designer's Handbook,* 2nd ed., edited by L. J. Giacoletto (McGraw-Hill, New York, 1977).

[9]M. Henon, Commun. Math. Phys. 50, 69 (1976).