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External noise and the dynamics of Ginzburg-Landau systems far from equilibrium

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The dynamics of Ginzburg-Landau (or "potential") systems driven away from equilibrium and subjected to noise is simulated numerically. It is found that the Fourier spectrum does not always have its peak at the wave number corresponding to the absolute minimum of the potential. This is explained heuristically on the basis of an entropy type of argument.

A great deal of interest has centered in the past few years around the question of long-time evolution in systems far from equilibrium, in particular, the problem of wave-number selection (see Ref. 1 and references therein). In this Rapid Communication, we investigate a restricted class of systems, those for which a so-called Ginzburg-Landau "free energy" can be written down. Examples are provided, e.g., by[2] superconducting films subjected to a steady microwave field or[3] materials under constant nuclear irradiation. It is believed that the final state of such systems is uniquely determined by their free energy V and by the character of their fluctuations—which are not always of purely thermal origin.

Consider a nonequilibrium system which, beyond threshold, may stabilize in any of several local minima of V, $V_i = V(k_i)$, each corresponding to a structure with wave number k_i. External noise[4] induces transitions between minima i,j at a *net* rate $R_{i\to j}$ which, close to a stationary state, is given approximately by[5] $A_{ij}\{\exp[(V_i - V_j)/T] - 1\}$, where T is an effective temperature depending on the noise intensity, and A_{ij} is a kinetic prefactor which embodies such effects as noise statistics, activation energy, etc. For our purposes, A_{ij} may be thought of as indicating an effective attempt frequency for $i \to j$. In the following, we assume a Gaussian white-noise spectrum. When the noise is weak, it is reasonable to expect that A_{ij} is essentially independent of i,j, and that the most probable state is the (unique) one corresponding to the absolute minimum of V. At stronger noise levels, however, the prefactor may acquire a nontrivial k dependence, thus effectively renormalizing the "bare" free energy V.[5] It is the purpose of the present work to study this effect in more detail. We shall be interested mainly in the "high-noise" limit (to be defined later), where the influence of the prefactor actually dominates the transition rate, i.e., entropy effects are the most important ones.

In what follows, we present a one-dimensional model system, simulate its dynamical behavior numerically, and propose a partial explanation of the results, in the form of a rough estimation[1] of the k dependence of the kinetic prefactor. The estimate is based on the idea that A_{ij} is directly related to the noise sensitivity of the pattern's *local* wave number, as explained below.

We consider a model defined by the following free energy:

$$V = \int dx \left(-[\epsilon - (\partial_{xx}^2 + q_0^2)^2]\frac{u^2}{2} + \frac{a}{4}u^4 + \frac{b}{4}(\partial_x u)^4 \right) \quad (1)$$

and

$$\partial_t u = -\frac{\delta V}{\delta u} + \zeta(t) \quad , \quad (2)$$

where u is a real variable and $\zeta(t)$ Gaussian white noise. The boundary conditions are assumed periodic with period $L >> q_0^{-1}$ and the integral in (1) runs over one period. A variant of (1) with $b = 0$ has been used to study Rayleigh-Bénard convection[6]; we include the b term for reasons that will become apparent later. We have integrated numerically the Fourier-transformed Eq. (2) for $\epsilon = 0.95$, $q_0 = 1$, where the system is past threshold [i.e., there exist nontrivial stationary solutions of (2)], and various values of a and b ranging from 0.5 to 500. The integration procedure is described in Refs. 1 and 7. In Figs. 1 to 3, we present typical time-averaged spectra; usually the averaging time was on the order of 400 time units. Figure 1 corresponds to the case $a = 0.5$, $b = 0$; the noise variance is 50%. Although this is very high, the peak in the spectrum remains located very near the absolute minimum of the potential energy V, as denoted by the arrow. The second figure pertains to the case $a = 0.5$, $b = 1$; the noise is about 30%. We see that the spectrum is more complicated, with the peak now displaced slightly from the minimum of the energy. Finally, in Fig. 3 we stimulate the same system, but at a higher noise level of 50%. The spectral peak has shifted even further away from the energy minimum. It would thus appear that the spectrum of a Ginzburg-Landau system driven from equilibrium depends, in a delicate way, on the detailed form of the free energy V as well as on the intensity of external noise.

It is natural to surmise that the prefactor A_{ij} plays a major role in explaining our results. We now evaluate approximately the kinetic part of A_{ij}, i.e., the rate of attempts at es-

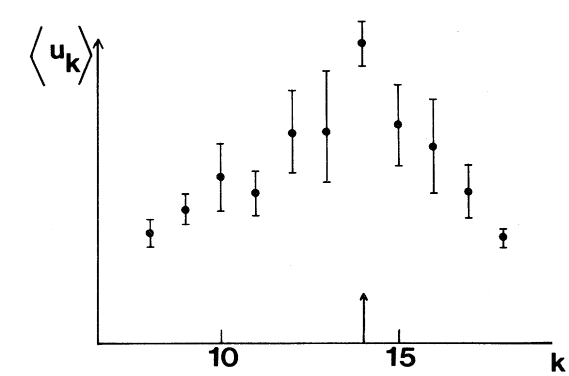

FIG. 1. Time averaged Fourier spectrum for model [(1),(2)] with $\epsilon = 0.95$, $q_0 = 1$, $a = 0.5$, $b = 0$. External noise variance is 50% of largest peak amplitude; bars indicate standard deviation of amplitude. The wave number k is in units of 2π divided by the length. Arrow denotes the absolute minimum of the free energy V [see Eq. (1)].

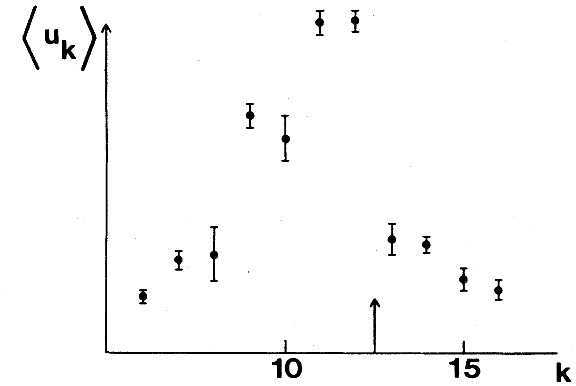

FIG. 2. Same as Fig. 1, but with $b=1$, and noise variance 30%. Note that the spectral peak has shifted slightly away from the minimum of V.

cape from i. In order to do so, we assume the existence of two time scales $t_{\text{noise}} \ll t_{\text{phase}}$, where t_{noise} is the average time between noise-triggered events $i \to j$ (or "phase slips"[5]), while t_{phase} is the usual phase diffusion time scale (see, e.g., Ref. 8). The above inequality clearly requires that the noise be strong enough. We are interested in the dynamics on a scale t_{noise}. Let us now consider a *large* subregion of the system, with length $L' \gg q_0^{-1}$; a local and instantaneous "wave number" q may be defined as follows. With the quasicontinuous Fourier spectrum in the subregion $u_k = U_k e^{i\phi_k}$ (and $u_{-k} = u_k^*$), we introduce $P_p(\{u_k\}) = \partial U_k/\partial k\,|_{k=p}$; q is then given, for instance, by

$$P_q(\{u_k\}) = 0 \ , \qquad (3)$$

with the range of q properly restricted for unicity. As the system evolves, so will q. Restricting ourselves to variations δq such that $L'\delta q \ll 1$, we therefore set to zero the total time derivative of (3):

$$0 = \frac{\partial P_q}{\partial t} + \frac{dq}{dt}\frac{\partial P_k}{\partial k}\bigg|_{k=q} = \frac{\partial}{\partial q}\frac{dU_q}{dt} + \frac{dq}{dt}\frac{\partial P}{\partial q} \ . \qquad (4)$$

Our basic assumption is that fluctuations of q, as *defined* by (4), are attempts at escaping from the minimum of V with wave number q. Equation (4) will now be made more explicit by using (2), which can be rewritten

$$\frac{du_q}{dt} = \left(i\frac{d\phi_q}{dt} + \frac{dU_q}{dt} \right) e^{i\phi_q} = f_q(\{u\}) + \zeta_q(t)$$

$$= f_q(\{U_{nq}\})e^{i\phi_q} + \sum_l B_l e^{i\phi_l} + \zeta_q(t) \ . \qquad (5)$$

The last decomposition rests on the fact that the mode q and its harmonics are phase locked—thence the common phase factor $\exp(i\phi_q)$—and represent a set of hydrodynamic variables. The B_l terms are contributions from the modes u_k with $k \neq nq$. These are of two types. First, those with $|k-nq| \ll q$, the sidebands of the spectral peaks; the dynamics of these modes is described by phase-diffusion-type equations.[8] Since we assume a time scale $t \ll t_{\text{phase}}$, the sole effect of the terms involving the sideband modes in Eq. (5) will be to effectively cancel the contribution from $id\phi_q/dt$ in the left-hand side. As for the other modes, they are *not* strongly phase locked to u_q, and we shall assume the noise is intense enough so that their phases are essentially random, leading to near cancellation. This approximation is valid at those times when the system does not come close to a saddle point of V.[9] Equation (5) thus reduces to $dU_q/dt = f_q(\{U_{nq}\}) + \zeta_q$ and (4) becomes

$$-\frac{\partial}{\partial q}[f_q(\{U_{nq}\}) + \zeta_q] = \frac{dq}{dt}\frac{\partial^2 U_q}{\partial q^2} \ . \qquad (6)$$

Finally, averaging over the noise,

$$-\frac{\partial}{\partial q}[f_q(\{U_{nq}^s\})] \simeq \left\langle \frac{dq}{dt}\frac{\partial^2 U_q}{\partial q^2} \right\rangle , \qquad (7)$$

where we have taken

$$\left\langle \frac{\partial f_q(\{U\})}{\partial q} \right\rangle \simeq \frac{\partial f_q(\langle\{U\}\rangle)}{\partial q} \simeq \frac{\partial f_q(\{U^s\})}{\partial q} \ ,$$

with U^s a stationary solution of (2) without noise. This is accurate provided the average is made over times such that q changes only little.[10] The right-hand side of (7) can now be taken as a measure of the escape attempt rate. We expect that, when noise is large, this rate will rule the dynamics; if, e.g., the attempt rate should vanish for some $q = q_s$, we anticipate a particularly noise-resistant state which would become populated preferentially, i.e., at a given time, most subregions would lie close to $q = q_s$. Note that the left-hand side of (7) depends solely on the stationary amplitudes U^s. It is thus easy to evaluate numerically, and does indeed exhibit a zero in the range of interesting wave numbers. We now examine the computed dynamical average spectra in the light of our prediction that $q = q_s$ should be "selected." In doing this, we must keep the following in mind. The dynamical simulations are costly and, in fact, had to be restricted in such a way that $k_{\max}$, the largest wave number taken into account, *always* satisfied $k_{\max} < 2q$, q denoting the main spectral peak; not too far from threshold, the neglected peaks at $2q$, etc., remain small anyway. With (7) effectively limited to $n=1$, it is not difficult to see that the $b(\partial_x u)^4$ term in (1) plays a crucial role: When $b=0$, equating the left-hand side of (7) to zero yields $q_s = q_{\min}$, the wave number at which V also has its absolute minimum. Figure 1 confirms that, when $b=0$, the maximum in the spectrum seems to remain located at $q_{\min}$. On the other hand, with $b \neq 0$, we have $q_s \neq q_{\min}$; Figs. (2) and (3) indicate that, in this instance, progressively larger noise results

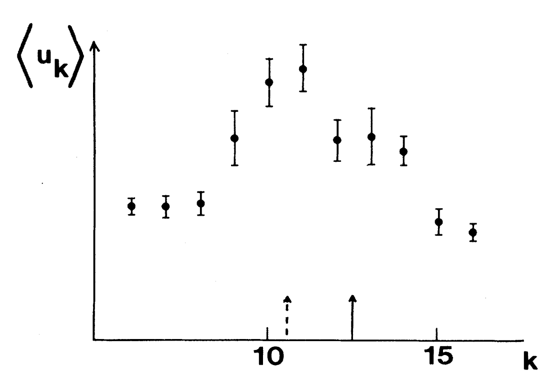

FIG. 3. Same as Fig. 2, but with noise variance of 50%. The spectral peak appears very close to the state with *least noise sensitivity*, as given by equating Eq. (7) to zero, and denoted here by the dashed arrow.

in the peak shifting away from $q_{\min}$ toward q_s (interrupted arrow). This seems to be true—within numerical accuracy—for all the cases which we have tested. Note that, remarkably, (7) depends in no way on the existence of a potential V: Only the equations of motion, $f_q(\{u\})$, not the potential, enter the derivation. It is thus interesting to mention here that results similar to those described above seem to hold in dynamical simulations of *nonpotential* systems as well[7,11]; altogether, these various results would appear to strengthen the case for the analysis that leads us to Eq. (7).

To summarize, we have evaluated, by numerical simulation, the average spectra of several driven Ginzburg-Landau systems under noise. The spectra are difficult to calculate analytically, but we believe that, at least, the location of the first spectral peak can be understood both in the low- and in the high-noise limits. The analysis presented here for the high-noise case actually seems to carry over to the much more complicated but widespread case of non-Ginzburg-Landau systems.

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[9]Indeed, it is not difficult to see that the saddle-point configurations (Ref. 5) are *coherent* superpositions involving modes to which we here ascribe *random* phases.

[10]Equation (7) could be made more precise by taking account of the renormalization due to amplitude fluctuations at fixed q: Far enough from threshold, however, this effect will be slight.

[11]In the simulations of Ref. 7, the effects of harmonics $n = 2, 3$ *were* taken into account. Our experience with nonpotential systems has been that selection in this case requires very low noise levels (a few percent) and is therefore much sharper in terms of peak narrowness.