Small-signal gain in lethargic and conventional laser amplifiers

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The lethargic regime of a swept-gain laser amplifier is investigated theoretically to determine the procedure for passing over to the conventional limit T_1 (level lifetime) $>> T_2$ (phase memory time) of laser amplifiers. The calculation is done with a model in which the transit time of the atoms t_0, is less than T_1, which is soluble in closed form. The conventional limit is taken as $t_0 \rightarrow \infty$. The general and conventional limits involve contradictory conditions of operation; nonetheless, both agree with experiment. This apparent contradiction is resolved by our finding that the conventional limit is a singular case.

INTRODUCTION

Research into short-wavelength lasers[1] has shown that maximum gain per atom is achieved in swept-gain amplifiers.[2-5] Some earlier work dealt with amplifiers that are equivalent to a swept-gain case.[6] A gain pulse, moving with the velocity of light c, is found to give an asymptotically decreasing gain coefficient due to "laser lethargy,"[2,3] first verified in the free-electron laser (FEL),[7,8] and subsequently in laser amplifiers.[9] Lethargic gain is incompatible with laser operation since the light pulse lags behind the gain, thus utilizing a decreasing fraction of it. In the FEL, an exponential gain is achieved by what is equivalent to having a gain pulse that moves at a velocity less than that of light (called a "delayed" velocity or boundary).[8]

An earlier paper gives an analytic solution for small-signal gain in a swept-gain amplifier with a delayed boundary.[10] It dealt with a transit-time broadened laser amplifier, i.e., the atomic decay times are long compared to their transit time across the laser beam. The model describes a gain pulse moving at velocity v, with atoms[1] moving transversely to the laser pulse, so inhomogeneous broadening can be neglected. Note that transit-time broadening dominates the Stanford FEL.[7]

Here we generalize, straightforwardly, the previous work to include homogeneous broadening by transforming our equations back to the previous case.[10] The results can then be used to compare with the FEL in the warm-beam limit.[11] We are not concerned with this comparison, but rather with the connection between the analysis of the delayed boundary problem and the conventional analysis of small-signal gain[12] in the case where both the level lifetime T_1 and the transit time t_0 are much longer than the phase memory time T_2. We consider the case $T_1 >> t_0$, and go to the limit $t_0 >> T_2$.

The transition between the lethargic and conventional cases is not trivial. The conventional analysis[12] applies for $v \geq c$. The optical pulse grows exponentially, and moves at a pulse velocity $V < c$. With lethargy, the pulse can only have exponential gain for $v \leq c$, and then only for a pulse velocity $V = v \leq c$. Hence the conventional analysis meets neither requirement for the lethargy analysis. The conventional analysis is valid but singular, applying rigorously only in the limit $T_1 \rightarrow \infty$, $t_0 \rightarrow \infty$. In our analysis of the general case we find that the answer depends on how the limit is taken. Our result connects with the conventional one when the limit is taken so that for all t_0 (or T_1) the velocity v gives maximum gain. By our construction, the conventional result is shown to be the largest gain achievable in the limit $T_1 >> t_0 >> T_2$. This conclusion maintains consistency in our understanding of laser gain.

METHOD OF SOLUTION

The amplifier is described by the equations

$$\frac{\partial E(\mu,z)}{\partial z} = \alpha' P(\mu z), \quad \frac{z}{\beta} \leq \mu \leq \frac{z}{\beta} + t_0 \ . \tag{1}$$

$$\frac{\partial P(\mu,z)}{\partial \mu} = -\frac{P(\mu,z)}{T_2} + E(\mu,z), \quad \frac{z}{\beta} \leq \mu \leq \frac{z}{\beta} + t_0 \ , \tag{2}$$

which are the Maxwell-Bloch equations[13] for this case in the small-signal limit. The slowly varying field amplitude E is in units of the Rabi frequency, and P is the dimensionless slowly varying amplitude of the polarization. The variable $\mu = t - z/c$ is the retarded time (z denotes position and t denotes time) and α' is a coefficient related to the gain.[14]

The difficulty in the analysis is to impose the appropriate boundary conditions.[10] The swept-gain boundary is imposed at the retarded time $\mu = z/\beta$, where β is given by $v^{-1} = \beta^{-1} + c^{-1}$. The atoms enter the beam at $\mu = z/\beta$ with $P(\mu,z) = 0$, and exit at $\mu = z/\beta + t_0$, where t_0 is the transit time. The region $z/\beta \leq \mu \leq z/\beta + t_0$ has active atoms and nonzero gain. In the region $\mu \leq z/\beta$ the field is nonzero, since it propagates out of the region of gain with a velocity of $c > v$. In the region $\mu \geq z/\beta + t_0$ the field is zero, provided one has a zero field at $z = 0$ for $\mu > t_0$. For $t_0 << T_1$, and weak fields, the inversion remains at its initial value of unity. Because the field is unknown at $\mu = z/\beta$, one cannot apply a boundary condition on the field at that time, as is done in the conventional analysis. Instead the field is prescribed at the time $\mu = z/\beta + t_0$, for all z, in which case it reads $E(\mu,z) = 0$.

These equations and boundary conditions enable us to find a set of functions, labeled by n, in which we can expand[15] any arbitrary initial field at $z = 0$, provided of course that it is zero for $\mu \geq t_0$. These solutions take the form of a pulse that has an invariant shape in the rest frame of the gain. These have been called "supermodes" by Dattoli and Renieri,[15] who were the first to apply such an analysis to swept-gain devices. Supermodes have the form $f_n(\mu - z/\beta)\exp(\eta_n z)$, with a similar expression for P. Only modes with positive gain survive for large z; modes with negative gain are ignored (n runs over values $\eta_n > 0$). A general field takes the form

$$E = \sum_n c_n f_n(\mu - z/\beta)\exp(\eta_n z) \ . \tag{3}$$

Solution for general T_2

The solution for general T_2 is now found by a substitution that takes this problem over to the one solved previously.[10] Define $\bar{E}$ and $\bar{P}$ by

$$E=\bar{E}\exp\left[-\frac{\mu}{T_2}\right],\quad P=\bar{P}\exp\left[-\frac{\mu}{T_2}\right]\ . \tag{4}$$

Upon substitution into Eqs. (1) and (2), one obtains equations for $\bar{E}$ and $\bar{P}$ that are identical to Eqs. (1) and (2), except that the term containing T_2 in Eq. (2) is missing. This transforms the problem into the one solved in Ref. 10. We denote by α_n the gain found in Ref. 10 for the case $1/T_2=0$, and we define γ_n by

$$\alpha_n(\beta)=2\left[\left[\frac{\alpha'}{\beta}\right](1-\gamma_n^2/\beta\alpha')\right]^{1/2}\ . \tag{5}$$

As found in Ref. 10, the boundary condition on $\bar{E}$ leads to the condition

$$\sin(t_0\gamma_n)=\frac{\gamma_n}{\alpha'\beta}\ , \tag{6}$$

which determines the acceptable values of γ_n. For convenience, let us define $\tau=\mu-z/\beta$, which is the time in the rest frame of the gain medium. Then the analysis of Ref. 10 together with Eq. (4) gives

$$P=\sum_n c_n\exp\left[\left[\alpha_n(\beta)-\frac{1}{\beta T_2}\right]z\right]\times\exp\left[-\frac{\tau}{T_2}+\frac{\beta\tau\alpha_n}{2}\right]\sin(\gamma_n\tau)\ . \tag{7}$$

We analyze P rather than E since the formula is somewhat less cumbersome. In the limit $t_0 >> T_2$, the pulse evolves to become much wider than T_2, in which case E can be obtained from Eq. (2) by dropping the time derivative, giving $E\cong P/T_2$.

In the limit that t_0 goes to infinity, Eq. (6) shows that γ_n vanishes as $1/t_0$. Therefore γ_n is obtainable by multiplying the right-hand side of Eq. (6) by t_0/t_0, and taking the limit as $t_0\rightarrow\infty$, keeping $\gamma_n t_0$ constant. The right-hand side of Eq. (6) vanishes in that limit, giving

$$\gamma_n=\frac{n\pi}{t_0}\ . \tag{8}$$

Equations (5)–(8) give the general solution for the amplifier in the limit $t_0 >> T_2$. In Fig. 1 we show the gain obtained from Eq. (7)

$$g_n(\beta)=\alpha_n(\beta)-\frac{1}{\beta T_2}\ , \tag{9}$$

as a function of $1/\beta$. For this plot we have taken $\alpha'=1$ and $T_2=1$, which is no restriction since they can be removed from the problem by scaling μ and z. The gain is computed for $n=1$, and α_n is evaluated using a first-order Taylor's expansion of Eq. (5) in γ_n^2. The position of the medium velocity for the conventional analysis, i.e., $v=c$, or $1/\beta=0$, is shown explicitly in Fig. 1. One can see immediately that there are two difficulties in the passage to the conventional case. First, the gain for $1/\beta=0$ is zero for all finite t_0; secondly, when t_0 is increased, the point of maximum gain is found to move away from $1/\beta=0$ towards $1/\beta=1$. In the general analysis, the exponentially growing pulse has the same velocity as the gain medium, while in the conventional anlaysis,[12] these velocities are different. We show below that the limit of the optimum medium velocity that is found in Fig. 1 agrees with the pulse velocity obtained from the conventional analysis.

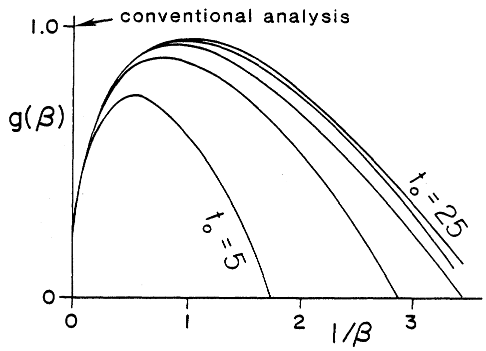

FIG. 1. Gain $g(\beta)$ as a function of $1/\beta$ for $t_0=5$, 10, 15, 20, and 25. The gain approaches a maximum of $g=1$ at $\beta=1$ in these dimensionless units which are obtained by setting $\alpha'=1$, and $T_2=1$. The conventional case is shown at $1/\beta=0$ and $g=1$ (note that the velocity of the optical pulse is at $1/\beta=1$ in this case). When the symbol g is used without an argument it refers to the condition of the global maximum gain given by $g=\alpha' T_2$.

Connection to the conventional analysis

To pass to the limit $t_0\rightarrow\infty$ it is necessary to take some prescription for choosing the velocity at each step, since Fig. 1 shows that the gain is otherwise undetermined. The natural choice is to take the velocity that gives rise to the highest gain since this gives the optimum operation of the amplifier in the small-signal regime. In the limit $t_0\rightarrow\infty$ the gain is given from Eqs. (5), (8), and (9) as

$$g(\beta)=2\sqrt{\alpha}/\sqrt{\beta}-1/\beta T_2\ ,$$

and its maximum occurs at $\beta=1/\alpha' T_2^2$. The maximum gain is $g=\alpha' T_2$, which is the conventional result for amplifier gain in this limit.[12] Now substitute these expressions into Eq. (7) to obtain (keeping only the lowest-order terms in γ_n)

$$P=\exp(gz)\ \sum_n c_n\exp[-\gamma_n^2(gT_2^2z+\tau T_2/2)]\sin(\gamma_n\tau)\ . \tag{10}$$

Let us quickly review the conventional analysis in order to compare it with this expression. The analysis is carried out by Fourier transform of Eqs. (1) and (2) with the boundary conditions $P(\mu=0,z)=0$ and $E(\mu=0,z)=0$, for all z, i.e., for $v=c$, $1/\beta=0$. We denote by $C_0(\omega)$ the Fourier transform of $E(\mu,z=0)$ and expand in ω to obtain[12]

$$E(\mu,z)=\frac{1}{2\pi}\int d\omega\, C_0(\omega)\exp(i\omega\mu)\exp\left[\frac{\alpha' z T_2}{1+i\omega T_2}\right]\ .$$

$$\approx\exp(gz)\left[\frac{1}{2\pi}\right]\int d\omega\, C_0(\omega)\exp[-gz(\omega T_2)^2]\exp[i\omega(\mu-gT_2z)]\ . \tag{11}$$

The various factors in Eqs. (10) and (11) are arranged to correspond to each other. The first shows that they have the same gain. The last factor is clear if one rewrites the expression for β obtained above in terms of g to give $1/\beta = gT_2$, and if one recollects that $\tau = \mu - z/\beta = \mu - gT_2 z$. Hence the last factor in Eq. (11) describes a pulse moving with the velocity $1/gT_2$ in the retarded frame. In Eq. (10) the pulse velocity equals the velocity of the gain medium, which is $\beta = 1/gT_2$. The remaining factor in Eq. (11) describes the phenomenon of gain narrowing.[12] The width of the spectrum of the electromagnetic field gets narrower as $1/\sqrt{z}$ with propagation. This factor is present in Eq. (10) as well, but there is an additional term of the form $\gamma_n^2 \tau T_2/2$. The factor multiplying γ_n^2 in Eq. (10) can be written as $T_2(z/\beta + \frac{1}{2}\tau)$. For $\tau << t_0$, $z \to \infty$, this extra term is negligible and the expressions (10) and (11) give the same pulse shape. When finally the pulse width approaches t_0, the term $\tau T_2/2$ reminds us that our approximate analysis no longer applies.

CONCLUSION

We have investigated the properties of gain in cases in which the conventional boundary condition $v = c$ leads to nonexponential gain due to laser lethargy, and a boundary condition $v < c$ is needed for exponential growth. We have shown how to connect this case to the conventional analysis,[12] which is seen to emerge as a singular case. The procedure for going over to the conventional analysis is nonunique. To obtain the result, one must go to the limit of a homogeneously broadened amplifier by a process that keeps the gain at its maximum. In that case, the optimal gain velocity is found to be the pulse velocity of the conventional analysis, and the value of the gain is the value found in the conventional analysis. Gain narrowing is also seen, but the approximations that connect the two solutions break down if the pulse gets too wide, which is precisely that one would anticipate.

The analysis does not imply a need to reinvestigate all cases in which the conventional analysis has been used to predict experimental results. The analysis is relevant, in a practical sense, mainly for swept-gain lasers, and not necessarily for laser amplifiers. For a typical neodymium glass amplifier the proper boundary velocity is so close to c that corrections can be ignored. In addition, the small-signal signal applications of amplifiers almost always involve transients, while the present analysis applies to the asymptotic gain. There are still puzzling features connected with the way the general approach works in the case of transients. Hopefully further study will clear up this problem.

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