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Feynman path summation for the Dirac equation: An underlying one-dimensional aspect of relativistic particle motion

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It is observed that the Feynman path summation for the one-dimensional Dirac equation can be projected into three spatial dimensions to yield a path-summation formula for physical spin-$\frac{1}{2}$ particles of nonzero mass. Since the three-space projection matrix is independent of time and does not involve the particle's mass, relativistic motion governed by the Dirac equation has an underlying one-dimensional aspect.

The Dirac equation for one spatial dimension is expressible as

$$\left(\frac{\partial}{\partial t}-im\tau_1+\tau_3\frac{\partial}{\partial \xi}\right)\psi(\xi,t)=0\ , \tag{1}$$

where ξ denotes the spatial coordinate, $\psi(\xi,t)$ is a two-component wave function, m is the particle's mass, physical units are chosen such that $\hbar=c=1$, and

$$\tau_1\equiv\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ . \tag{2}$$

Assuming that the initial value $\psi(\xi,0)$ is prescribed, the general solution to (1) is given by

$$\psi(\xi,t)=\int_{-\infty}^{\infty}K(\xi-\xi',t;m)\psi(\xi',0)\,d\xi' \tag{3}$$

in which the propagation kernel

$$K(\xi,t;m)=\left(\mathbb{1}\frac{\partial}{\partial t}+im\tau_1-\tau_3\frac{\partial}{\partial\xi}\right)\times\int_0^{\infty}\frac{\sin[(k^2+m^2)^{1/2}t]\cos(k\xi)}{\pi(k^2+m^2)^{1/2}}\,dk \tag{4}$$

satisfies

$$\left(\mathbb{1}\frac{\partial}{\partial t}-im\tau_1+\tau_3\frac{\partial}{\partial\xi}\right)K(\xi,t;m)=0\ \text{ for } t>0 \tag{5}$$

subject to the initial condition

$$K(\xi,0;m)=\mathbb{1}\delta(\xi)\ , \tag{6}$$

where $\mathbb{1}$ is the 2×2 identity matrix.

Feynman and Hibbs[1] have noted a path-summation expression for the propagation kernel (4),

$$K(\xi,t;m)=\lim_{\epsilon\to0}\sum_{R=0}^{\infty}A_R(\xi,t;\epsilon)(im\epsilon)^R\ , \tag{7}$$

$$A_R(\xi,t;\epsilon)\equiv\epsilon^{-1}\begin{pmatrix}N_{++} & N_{+-}\\ N_{-+} & N_{--}\end{pmatrix}\ . \tag{8}$$

In (8) $N_{lf}=N_{lf}(\xi/\epsilon,t/\epsilon,R)$ is the number of paths composed of t/ϵ constant-velocity steps from the space-time origin (0,0) to (ξ,t) with $(t+\xi)/2\epsilon$ steps forward $(\Delta\xi=\Delta t=\epsilon)$, $(t-\xi)/2\epsilon$ steps backward $(\Delta\xi=-\Delta t=-\epsilon)$, and R reversals (i.e., changes in the sign of successive $\Delta\xi$), where the subscripts f and l denote the signs of the first and last steps, respectively, for the class of space-time paths. It follows from the recurrence formulas satisfied by N_{lf} and expansion for small ϵ that (7) satisfies the propagation kernel equations (5) and (6),[2] and hence the right-hand side of (7) equals the right-hand side of (4).

Paths that enter the Feynman summation (7) have $d\xi/dt=\pm1$ during each step interval: The particle moves with a lightlike shuttle motion, forward or backward along the ξ axis, and the double-valueness that makes (7) and (8) appear as 2×2 matrices stems from the initial and final values of $d\xi/dt=\pm1$. Such paths are appropriate quantum mechanically in view of the Heisenberg operator equation $d\xi/dt=\tau_3$ which follows from (1). Similarly, the three-dimensional Dirac equation yields a Heisenberg velocity operator[3]

$$\frac{d\vec{x}}{dt}=\vec{\alpha}\equiv\vec{\sigma}\times\tau_3 \tag{9}$$

whose components in each spatial direction have eigenvalues equal to ±1. However, the three components of (9) are noncommuting dynamical variables, and thus only one of the velocity components dx_1/dt, dx_2/dt, or dx_3/dt is diagonalizable at a certain instant of time. Thus, the propagation kernel for the three-dimensional Dirac equation [shown below in (14)] cannot be obtained from a dimensional extension of (7) and (8) in which the paths are defined on a cubical lattice in $\vec{x}$ space with two components of $d\vec{x}/dt$ equal to zero and the third component equal to ±1 during each step interval.

With the distance coordinate ξ in place of t, the 2×2 matrix propagation kernel for Weyl's mass-zero neutrino wave function satisfies

$$\left(\mathbb{1}\frac{\partial}{\partial\xi}+\vec{\sigma}\cdot\vec{\nabla}\right)W(\vec{x},\xi)=0\ , \tag{10}$$

$$W(\vec{x},0)=\mathbb{1}\delta^{(3)}(\vec{x})\ , \tag{11}$$

in which $\vec{\sigma}$ denotes the Pauli spin-$\frac{1}{2}$ matrices. Since[4]

$$\delta^{(3)}(\vec{x}) = (2\pi r^2)^{-1}\delta(r) = -(2\pi r)^{-1}\delta'(r)$$

with $r \equiv |\vec{x}|$, the solution to (10) and (11) is given by

$$W(\vec{x},\xi) = \frac{1}{2\pi}\left\{\mathbb{1}\frac{\partial}{\partial\xi} - \vec{\sigma}\cdot\vec{\nabla}\right\}[(\operatorname{sgn}\xi)\delta(r^2-\xi^2)]$$
$$= \frac{1}{4\pi}\left\{\mathbb{1}\frac{\partial}{\partial\xi} - \vec{\sigma}\cdot\vec{\nabla}\right\}[r^{-1}\delta(r-\xi) - r^{-1}\delta(r+\xi)] . \tag{12}$$

The latter 2×2 *projection matrix* maps scalar functions of ξ into 2×2 matrix-valued functions of $\vec{x}$ by direct multiplication and integration over ξ. In particular, (12) enables one to express the 4×4 propagation kernel for the three-dimensional Dirac equation as a Feynman summation based on (7), viz.,

$$\mathscr{K}(\vec{x},t;m) = \lim_{\epsilon\to 0}\sum_{R=0}^{\infty}\int_{-\infty}^{\infty} W(\vec{x},\xi)\otimes A_R(\xi,t;\epsilon)(im\epsilon)^R d\xi , \tag{13}$$

where $\otimes$ signifies the direct product of the indicated 2×2 matrices. In view of (7) and (12), the Feynman summation (13) is equivalent to

$$\mathscr{K}(\vec{x},t;m) = \int_{-\infty}^{\infty} W(\vec{x},\xi)\otimes K(\xi,t;m)\,d\xi$$
$$= -(4\pi r)^{-1}\mathbb{1}\otimes\left[\frac{\partial K(r,t;m)}{\partial r} + \frac{\partial K(-r,t;m)}{\partial r}\right] - (4\pi)^{-1}\vec{\sigma}\cdot\vec{\nabla}\otimes[r^{-1}K(r,t;m) - r^{-1}K(-r,t;m)] . \tag{14}$$

In (13) each path in (ξ,t) space-time is weighted with the additional projection-matrix factor $W(\vec{x},\xi)$ and a final "summation" is performed by integrating over all ξ.

To see that (14) is the propagation kernel for the three-dimensional Dirac equation, one makes use of (10), (5), and the definition part of (9) to obtain

$$\vec{\alpha}\cdot\vec{\nabla}\mathscr{K}(\vec{x},t;m) = -\int_{-\infty}^{\infty}\frac{\partial W(\vec{x},\xi)}{\partial\xi}\otimes\tau_3 K(\xi,t;m)\,d\xi = \int_{-\infty}^{\infty} W(\vec{x},\xi)\otimes\tau_3\frac{\partial K(\xi,t;m)}{\partial\xi}\,d\xi$$
$$= \int_{-\infty}^{\infty} W(\vec{x},\xi)\otimes\left[-\mathbb{1}\frac{\partial}{\partial t} + im\tau_1\right]K(\xi,t;m)\,d\xi . \tag{15}$$

Hence, it follows from the first and last members of (15) that

$$\left[\mathbb{1}\frac{\partial}{\partial t} + \vec{\alpha}\cdot\vec{\nabla} + im\beta\right]\mathscr{K}(\vec{x},t;m) = 0 , \tag{16}$$

where $\beta \equiv -\mathbb{1}\otimes\tau_1$. Moreover, the second member of (14) in combination with (6) and (11) implies that

$$\mathscr{K}(\vec{x},0;m) = \mathbb{1}\otimes\mathbb{1}\delta^{(3)}(\vec{x}) . \tag{17}$$

Equations (16) and (17) are the defining relations for the propagation kernel which gives the time evolution of the four-component Dirac wave function, and thus this propagation kernel is expressed by the Feynman summation (13).

Since the three-space projection matrix $W(\vec{x},\xi)$ in (13) is independent of time and does not involve the particle's mass, relativistic motion governed by the Dirac equation appears to have a fundamental one-dimensional aspect: In the path summation (13), it is the lightlike shuttle motion along the ξ axis which generates the time evolution of the four-component Dirac wave function in $\vec{x}$ space.

The extension of this Feynman path summation to include electromagnetic interaction is straightforward. For the primary case of a static magnetic field described by the vector potential $\vec{A} = \vec{A}(\vec{x})$, (10) is superseded by

$$\left[\mathbb{1}\frac{\partial}{\partial\xi} + \vec{\sigma}\cdot(\vec{\nabla} - ie\vec{A})\right]W(\vec{x},\xi) = 0 \tag{18}$$

and it follows from the second member of (14), (18), and (5) that (16) obtains with $(\vec{\nabla} - ie\vec{A})$ in place of $\vec{\nabla}$. The propagation kernel equation for general electromagnetic interactions then follows unambiguously from the requirement of invariance under gauge transformations $\vec{A}\to(\vec{A}+\nabla\chi)$, $A_0\to(A_0+\partial\chi/\partial t)$, and $\mathscr{K}\to(\exp ie\chi)\mathscr{K}$ with $\chi = \chi(\vec{x},t)$ arbitrary.

[1]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), pp. 35 and 36.

[2]See, for example, G. Rosen, *Formulations of Classical and Quantum Dynamical Theory* (Academic, New York, 1969), pp. 118–122.

[3]See, for example, P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, London, 1947), pp. 260–262.

[4]The Dirac function $\delta(r)\equiv\delta(-r)$ has the normalization $\int_0^{\infty}\delta(r)\,dr = \frac{1}{2}$ for $r \geqq 0$.