Nonlinearly evolved Faraday rotation due to strong radiation in a plasma

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Faraday rotation (FR) has a nonlinear part which can be excited by a strong elliptically polarized electromagnetic wave in a plasma, even in the absence of a static magnetic field. It is due to the different dispersion rates of the strong left- and right-circular polarization components of the wave (the corresponding refractive indices are denoted by n_L and n_R) and induces weak nonuniformities in a magnetized plasma. As a result the study of the nonlinear characteristics requires the WKB type of nonlinearly correct first-harmonic field solutions. This nonlinear evolution problem has been investigated in this paper. The new results, depending on both the field intensity and the nonlinearly induced spatial inhomogeneity, are (1) the nonlinearly correct expressions for n_L and n_R, (2) the amplitude evolution equations in which the coefficients are slowly varying with respect to x (the Cartesian coordinate parallel to the direction of propagation) and their WKB solution, (3) the expression for the nonlinearly correct FR angle, and (4) a numerical estimation on the interaction of a plasma with a CO_2 pulsed laser in which it has been shown that the nonlinear increment of FR is opposite in sign to the value found in the linearized approximation of the field equations. The new results have been discussed in some special cases and some comments are given on enlarging the scope of further work on the FR of strong waves.

I. INTRODUCTION

Faraday rotation (FR) is a magneto-optical effect of birefringence. It occurs when the refractive indices n_L and n_R of the left- and right-circular polarization components of a wave are different for the same frequency; so the phase velocity (and also the group velocity) is different for the two polarizations. As a consequence, the direction of the resultant field of the two circular polarizations is different from that without the external biasing magnetic field and has been experimentally observed in crystals and plasmas.[1] In plasmas the effect occurs because electrons gyrate opposite to ions. Materials in a spontaneously magnetized state (e.g., ferrites) also exhibit FR. Their original internal fields (related to Larmor precession of electrons) may be considered as generating agents of the Faraday effects.

For strong electromagnetic waves in plasmas and other media the FR is modified by the nonlinearly induced precessional rotation[2] (PR), which is actually the self-generating rotation of the reference frame in which the field equations are described with respect to the laboratory frame of the observer. The former reference frame has been called by Chakraborty[3] (1977) the principal polarization reference frame (PPRF). The FR of the standard theory is derivable from the linearized field solutions. So it is that which is obtained in the PPRF whereas in the laboratory frame of the observer the FR is a combination of this FR and nonlinearly induced FR which is due to the self-induced nonlinear precessional angle of rotation of the PPRF. Therefore, the magnitude of the observed FR in the laboratory frame will depend on the sense of rotatation of PR and the FR in PPRF.

The recorded FR can even be in the opposite sense to that predicted by the standard linear theory if the latter is less than the magnitude of the nonlinearly generated angle of PR of the PPRF and in the opposite sense. In the end of Sec. III C an example illustrating this possibility is given. The referred PR of the polarization ellipse is such that the eccentricity of the ellipse remains constant due to conservation of the photon angular momentum.

The well-known theory of FR is that for a plane-polarized wave of very weak field intensity. In fact, the infinitely small amplitude wave approximation for the linearized solution of the field equations is used in the theory. But this value of FR is considerably modified for strong waves. Although there is much application of effects on FR of strong waves, we are not aware of any appropriate theoretical investigation on this modification. The most important factor influencing the value of FR is the appearance of the nonlinearly induced effect of PR of strong elliptically polarized waves in material media including plasmas.[2-8] But the self-induced PR effect was not a very familiar topic of theoretical research until very recently; therefore, it seems that the theoretical investigation of the nonlinear modification of the FR effect along the line motivated in this paper has not been attempted earlier. Here, for the first time, the expression for the nonlinearly correct FR angle has been obtained when an intense elliptically polarized electromagnetic wave propagates parallel to the direction of a homogeneous applied magnetic field in a plasma slab.

An important aspect of the Faraday effect in plasmas is the induced magnetization[9–13] (inverse Faraday effect) (IFE) produced by a circularly polarized wave. It is a consequence of the gyration of plasma electrons, which for left-circular polarization is parallel and for right-circular polarization is antiparallel to the direction of wave propagation. The theory developed in this paper will be useful for the study of (a) nonlinearly correct evolution of FR, (b) the complementary effect of nonlinear modification of IFE, and (c) of nonlinear modification of synchrotron radiation from plasmas.

We consider here the effects (a) and (b). However, the important effect (c), namely, the nonlinear evolution of synchrotron radiation due to the PR effects, should be considered elsewhere in the near future.

In a plasma, strong electromagnetic radiation induces several nonlinear effects including PR. In a collisionally damped magnetized plasma slab, the difference in the wave numbers K_L and K_R gives rise to weak inhomogeneities even when the plasma is assumed homogeneous before the application of the strong field. Moreover, in a magnetized plasma slab, the $\vec{v}\times\vec{B}$ force can give the particles a constant drift velocity[14]; this drift changes the constant part of the density from its field-free value, and thus modifies the electrostatic field through Poisson's equation.[2] The plasma density is therefore slowly and continuously varying due to nonlinearities and so the WKB approximation[15–16] has to be used. With the help of the WKB solution the expression for the nonlinearly induced inverse Faraday effect (i.e., IFE; it is created inside the plasma due to precessing electromagnetic waves) can also be obtained. In unmagnetized collision-free plasmas the nonlinearly developed nonuniformity vanishes, and so the WKB solution is not necessary, but the enlisted nonlinear effects, which include the intensity-induced PR, the nonlinearly induced birefringence, the nonlinear FR, and the IFE, do not vanish.

In the simple types of nonlinear wave interactions in plasmas the nonlinear increment to FR (or the PR) of the polarization ellipse of a wave and other complementary effects of strong electromagnetic radiation are generally smaller by many orders of magnitude compared to those in material media, because they would vanish altogether for free electrons in a plasma in the electronic dipole approximation.[17] These nonlinear effects would hardly be accessible to experimental observations at optical frequencies. Actually, the nonlinear optical characteristics in plasma processes obtained by Arons and Max[2] and others[3–5] are small and imperceptible in laboratory experiments even with powerful laser beams (neodymium-glass laser, $\lambda=1.06$ μm, power flux$\sim 10^{16}$ W/cm^2). In some subsequent investigations by us[6,7] it has been shown that the nonlinear effects (the PR and hence the nonlinear part of the FR) can be much enhanced in a magnetized plasma to detectably large values in some physically possible near-resonant interactions and phase-matching conditions. In a magnetized electron plasma the wave frequency approaches the electron gyration frequency Ω_e $(=eH_0/mc)$ and thus the wave exchanges maximum energy with the electrons leading to significant nonlinear responses.[6(a)] These responses have also been shown to be enhanced for two-wave (including standing-wave) interactions[6(b)] where the beat frequency of two traveling waves approximates or exactly equals the characteristic plasma frequency (i.e., $\omega_1-\omega_2\approx\omega_p$). When the ion motion is also considered in a magnetized plasma, another low-frequency near-resonant interaction, occurring in the neighborhood of $\omega=\Omega_i$ (Ω_i is the ion cyclotron frequency), was investigated[7] where the PR and other complementary effects are large enough to be detectable.

High-intensity laser fields induce relativistic and other nonlinear effects in the motion of electrons. Even for radiation intensity$\sim 10^{22}$ W/cm^2 the ion motion remains well below the electron velocity and so has negligible effect on propagation. The power of the laser fields is restricted below the threshold power limit for the suppression of self-focusing, self-trapping, and SRS mechanisms. The threshold power to initiate these effects depends mainly on the plasma density, pulse duration, and laser frequency.[18] In a dense plasma (number density$\sim 5\times10^{18}$/cm^3), the threshold power (W_{th}) for a CO_2 gas laser (wavelength $\lambda=10.6$ μm, $\omega=1.78\times10^{14}$/sec) would be as high as $\sim 10^{18}$ W/cm^2. In a plasma slab below this threshold power limit [i.e., when dimensionless amplitudes α and β are less than unity where $\alpha,\beta=e(a,b)/m\omega c$ and the field amplitudes a, b, etc., are explained in Sec. III] the nonlinear propagation of an intense radiation has solutions which imply that the major axis of the vibration ellipse of a wave undergoes self-precession.

We find that a laser-plasma interaction for which fluxes (P) of a CO_2 pulsed laser of as much as 3×10^{11} W/cm^2 would be suitable for the estimation of precessional effects (since $P \ll W_{\text{th}}$). In such a case $\alpha^2 \approx \beta^2 \approx 1.4\times 10^{-5}$, which is very small compared to unity and our cold-plasma results are applicable. For a weak magnetic field of $\sim 10^3$ G while the FR in radians per cm of propagation of the wave obtained from the linearized field solution is $\sim 7.5\times 10^{-2}$, the corresponding increment due to the nonlinear effects turns out to be $\sim -10^{-6}$.

The basic equations are given in Sec. II. We have obtained here (in Sec. III) (1) the nonlinearly correct expressions for the refractive indices n_L and n_R, (2) the amplitude evolution equation in which the coefficients are slowly varying with respect to the space coordinate (parallel to the direction of wave propagation and the applied homogeneous magnetic field) and its WKB solution, (3) the expression for the nonlinearly correct FR angle, and (4) a numerical estimation in the case of interaction of a plasma with a CO_2 pulsed laser in which it has been shown that the nonlinear increment to the FR angle is in the opposite sense to its value evaluated in the linearized approximation. In Sec. IV we have discussed the results and their implications in some simple cases and we have commented on enlarging the scope of further work on the FR of strong waves in plasmas and other media.

II. STATEMENT OF THE EQUATIONS

The plasma is cold and consists of a mixture of electron fluid and immobile heavy ions providing the neutralizing static uniform background of positive charges, therefore the macroscopic equations of momentum transfer and conservation of charged particles for electrons are used here. The medium is assumed to be lossy (collision frequency ν) and the momentum loss in any direction is proportional to the momentum vector per unit volume in that direction. All field components are written as the sum of an exponential function for a plane wave and its complex conjugate (c.c.). The thermal velocities are assumed to be negligible ($v_{\text{th}} \ll c$) here.

The basic field equations are

$$\frac{\partial N}{\partial t} + \vec{\nabla}\cdot(N\vec{v}) = 0\,, \tag{2.1}$$

$$\left[\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right]\left[\frac{\vec{v}}{(1-v^2/c^2)^{1/2}}\right] = -\frac{e\vec{E}}{m} - \frac{e(\vec{v}\times\vec{H})}{mc} - \nu\vec{v}\,, \tag{2.2}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}\,, \tag{2.3}$$

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} - \frac{4\pi N\vec{v}}{c}\,, \tag{2.4}$$

$$\vec{\nabla}\cdot\vec{E} = -4\pi e(N-N_0),\quad \nabla\cdot\vec{H} = 0\,, \tag{2.5}$$

where $\vec{E}$ is the electric field. $\vec{H}$ is the magnetic field, $\vec{v}$ and N are the average velocity and number density of electrons, ν is the collision frequency of the electrons, and N_0 is the constant number density of the ions.

The subscript 1 will be added to the fields of the first (or the linear) approximation, subsequently in this paper we will use the following complex combinations of the fields:

$$\begin{aligned} &E_{1y} \pm iE_{1z} = E_{\pm},\quad H_{1y} \pm iH_{1z} = H_{\pm}\,,\\ &v_{1y} \pm iv_{1z} = v_{\pm}\,. \end{aligned} \tag{2.6}$$

The subscript 2 is added to the fields of second order and the subscript 3 to the fields of third order. Differential equations for E_+ and E_- correct up to the third order are obtained from the field equations (2.1) to (2.5). These are

$$D\left[\frac{\ddot{E}_+}{c^2} - \frac{\partial^2 E_+}{\partial x^2} + \frac{\omega_p^2}{c^2}E_+\right] + (\nu - i\Omega)\left[\frac{\ddot{E}_+}{c^2} - \frac{\partial^2 E_+}{\partial x^2}\right] = \frac{4\pi e N_0}{c^2}D\left[-v_{2x}\left[\frac{\partial v_+}{\partial x} + \frac{ie}{mc}H_+\right] - D\left[\frac{v_+^2 v_-}{2c^2}\right] + (D+\nu-i\Omega)\left[\frac{N_2 v_+}{N_0}\right]\right], \tag{2.7}$$

$$D\left[\frac{\ddot{E}_-}{c^2} - \frac{\partial^2 E_-}{\partial x^2} + \frac{\omega_p^2}{c^2}E_-\right] + (\nu + i\Omega)\left[\frac{\ddot{E}_-}{c^2} - \frac{\partial^2 E_-}{\partial x^2}\right] = \frac{4\pi e N_0}{c^2}D\left[-v_{2x}\left[\frac{\partial v_-}{\partial x} - \frac{ieH_-}{mc}\right] - D\left[\frac{v_-^2 v_+}{2c^2}\right] + (D+\nu+i\Omega)\left[\frac{N_2 v_-}{N_0}\right]\right], \tag{2.8}$$

where $D \equiv \partial/\partial t$ and overdots also denote time derivatives.

III. NONLINEAR WKB SOLUTIONS IN A MAGNETIZED PLASMA SLAB

To isolate the space and time parts of the field variables we now put

$$\frac{eE_+}{m\omega c}=A_+e^{-i\omega t}+B_+e^{i\omega t}\,,$$
$$\frac{eE_-}{m\omega c}=A_-e^{-i\omega t}+B_-e^{i\omega t}\,. \tag{3.1}$$

Since $eE/m\omega c$ is a dimensionless number (less than unity for the laser-produced electric field), the spatially dependent parts are A_+,B_+ and A_-,B_-. Then we can write

$$\beta e^{i\theta_L}=A_+(x)e^{-i\omega t},\quad \bar{\alpha}e^{-i\bar{\theta}_R}=B_+(x)e^{i\omega t}\,,$$
$$\alpha e^{i\theta_R}=A_-(x)e^{-i\omega t},\quad \bar{\beta}e^{-i\bar{\theta}_L}=B_-(x)e^{i\omega t}\,, \tag{3.2}$$

where the dimensionless amplitude parameters α, β, $\bar{\alpha}$, $\bar{\beta}$ are given by

$$(\alpha,\bar{\alpha},\beta,\bar{\beta})=\frac{e(a,\bar{a},b,\bar{b})}{m\omega c}\,. \tag{3.3}$$

θ_L and θ_R are the phases of the forward-going waves of left- and right-circular polarizations and the amplitudes A_+, A_-, B_+, B_- are functions of x.

For these solutions the electric field is assumed in the form

$$E_+=\bar{a}e^{-i\bar{\theta}_R}+be^{i\theta_L},\quad E_-=ae^{i\theta_R}+\bar{b}e^{-i\bar{\theta}_L}\,. \tag{3.4}$$

These will be used to evaluate the nonlinear terms in the right-hand sides of (2.7) and (2.8). The nonlinearly developed first harmonic parts of these fields will be determined by substituting (3.1) in the right-hand sides of (2.7) and (2.8). Then we find that the nonlinear equations for the fields of the left- and right-circular polarizations reduce to the following four second-order (normal form) ordinary differential equations:

$$\frac{d^2A_+}{dx^2}+K_L^2A_+=0\,, \tag{3.5a}$$
$$\frac{d^2B_+}{dx^2}+\bar{K}_R^2B_+=0\,, \tag{3.5b}$$
$$\frac{d^2A_-}{dx^2}+K_R^2A_-=0\,, \tag{3.5c}$$
$$\frac{d^2B_-}{dx^2}+\bar{K}_L^2B_-=0\,, \tag{3.5d}$$

where the nonlinear dispersions relations are

$$n_L^2=\frac{K_L^2c^2}{\omega^2}=1-\frac{X}{1+Y+iZ}+\frac{X}{2}\Bigg[\frac{\beta\bar{\beta}e^{ip_L}}{(1+Y+iZ)^2[(1+Y)^2+Z^2]}+\frac{2\alpha\bar{\alpha}e^{ip_R}}{(1+Y+iZ)^2[(1-Y)^2+Z^2]}$$
$$-\frac{\alpha\bar{\alpha}e^{ip_R}[(1+Y+iZ)(n_L+n_R)-2\bar{n}_R(Y+iZ)]}{(4+2iZ-X)(1+Y+iZ)(1-Y-iZ)}\left[\frac{n_L}{1-Y+iZ}+\frac{n_R}{1+Y+iZ}\right]\Bigg]$$
$$+\frac{i\alpha\bar{\alpha}v_Rce^{ip_R}}{\omega(1+Y+iZ)}\left[\frac{\bar{n}_R}{1-Y+iZ}-\frac{n_R}{1-Y-iZ}\right]$$
$$+\frac{i\beta\bar{\beta}v_Lce^{ip_L}}{\omega(1+Y+iZ)}\left[\frac{\bar{n}_L}{1+Y+iZ}-\frac{n_L}{1+Y-iZ}\right], \tag{3.6}$$

and

$$n_R^2=\frac{K_R^2c^2}{\omega^2}=1-\frac{X}{1-Y+iZ}+\frac{X}{2}\Bigg[\frac{\alpha\bar{\alpha}e^{ip_R}}{(1-Y+iZ)^2[(1-Y)^2+Z^2]}+\frac{2\beta\bar{\beta}e^{ip_L}}{(1-Y+iZ)^2[(1+Y)^2+Z^2]}$$
$$-\frac{\beta\bar{\beta}e^{ip_L}[(1-Y+iZ)(\bar{n}_L+\bar{n}_R)+2(Y-iZ)n_L]}{(4+2iZ-X)(1-Y-iZ)(1+Y-iZ)}\left[\frac{\bar{n}_L}{1-Y+iZ}+\frac{\bar{n}_R}{1+Y+iZ}\right]\Bigg]$$
$$+\frac{i\alpha\bar{\alpha}v_Rce^{ip_R}}{\omega(1-Y+iZ)}\left[\frac{n_R}{1-Y-iZ}-\frac{\bar{n}_R}{1-Y+iZ}\right]$$
$$+\frac{i\beta\bar{\beta}v_Lce^{ip_L}}{\omega(1-Y+iZ)}\left[\frac{n_L}{1+Y-iZ}-\frac{\bar{n}_L}{1+Y+iZ}\right], \tag{3.7}$$

in which ν_L and ν_R are, respectively, the imaginary parts of K_L and K_R,

$$p_R=\theta_R-\bar{\theta}_R,\ \ p_L=\theta_L-\bar{\theta}_L,\ \ X=\frac{\omega_p^2}{\omega^2}\ ,$$
$$Y=\Omega/\omega,\ \ Z=\nu/\omega,\ \ \omega_p^2=4\pi N_0e^2/m\ , \tag{3.8}$$

and N_0 is the number density of electrons and ν is the collision frequency.

A. General solution

For convenience we write

$$K_L=K_L^0+\delta K_L,\ \ K_R=K_R^0+\delta K_R\ ,$$
$$\bar{K}_L=\bar{K}_L^0+\delta\bar{K}_L,\ \ \bar{K}_R=\bar{K}_R^0+\delta\bar{K}_R\ , \tag{3.9}$$

where $K_{R,L}^0$ and $\bar{K}_{R,L}^0$ are the values of $K_{R,L}$ and $\bar{K}_{R,L}$, respectively, in the linearized approximation and so are constants, and $\delta K_{R,L}$ and $\delta\bar{K}_{R,L}$ are the nonlinear increments. Obviously $|\delta K_L| \ll |K_L^0|$, $|\delta K_R| \ll |K_R^0|$, etc., and δK_L and δK_R are slowly varying functions of the space coordinate in the direction of wave propagation (the x direction here), in the presence of constant magnetic field in that direction, in a collisionally damped plasma. So putting $n_L=n_L^0+\delta n_L$ and $n_R=n_R^0+\delta n_R$ into (3.6) and (3.7) and ignoring δn_L^2 and δn_R^2 we can evaluate δn_L and δn_R from (3.6) and (3.7). These quantities are proportional to the intensity-dependent nonlinearly excited parts in the right-hand sides of (3.6) and (3.7).

Since the nonlinear terms in the right-hand sides of Eqs. (2.7) and (2.8) contain the fields of both right- and left-circular polarizations, the difference between the values of K_L and K_R is responsible for the appearance of factors like $\exp(ip_R)$ and $\exp(ip_L)$ in the intensity-dependent nonlinearly excited parts in the right-hand side of (3.6) and (3.7). Consequently, K_L and K_R become slowly varying functions of x though K_L^0 and K_R^0 are constants, and the nonlinear propagation problem becomes a problem of wave process in slowly varying plasmas in this case. Since the wave numbers are weakly inhomogeneous, the wave solutions should be of the order of the WKB type, correct up to first order of derivatives of K_R and K_L.

The nonlinearly correct WKB solutions of (3.5a)—(3.5d) are

$$A_+=\frac{A_+^0}{K_L^{1/2}}\exp\left[i\int_0^l K_L dx\right]\ , \tag{3.10a}$$

$$B_+=\frac{B_+^0}{\bar{K}_R^{1/2}}\exp\left[-i\int_0^l \bar{K}_R dx\right]\ , \tag{3.10b}$$

$$A_-=\frac{A_-^0}{K_R^{1/2}}\exp\left[i\int_0^l K_R dx\right]\ , \tag{3.11a}$$

$$B_-=\frac{B_-^0}{\bar{K}_L^{1/2}}\exp\left[-i\int_0^l \bar{K}_L dx\right]\ , \tag{3.11b}$$

where the amplitudes $A_\pm^0$ and $B_\pm^0$ are constants correct up to the WKB approximation and the limits of integration are 0 and l because the magnetized plasma slab exists in $0<x<l$. Substituting these solutions into (3.1) we obtain

$$E_+=\frac{m\omega c}{e}\left[\frac{A_+^0e^{i\theta_L}}{K_L^{1/2}}+\frac{B_+^0e^{-i\bar{\theta}_R}}{\bar{K}_R^{1/2}}\right]\ , \tag{3.12}$$

$$E_-=\frac{m\omega c}{e}\left[\frac{A_-^0e^{i\theta_R}}{K_R^{1/2}}+\frac{B_-^0e^{-i\bar{\theta}_L}}{\bar{K}_L^{1/2}}\right]\ , \tag{3.13}$$

where

$$\theta_{R,L}=\int_0^l K_{R,L}dx-\omega t\ ,$$
$$\bar{\theta}_{R,L}=\int_0^l \bar{K}_{R,L}dx-\omega t\ . \tag{3.14}$$

Using the first relation of (2.6) we evaluate E_y and E_z from (3.12) and (3.13) and find that

$$\tan\phi=\frac{E_z}{E_y}=\frac{i\left[\dfrac{B_-^0e^{-i\bar{\theta}_L}}{\bar{K}_L^{1/2}}+\dfrac{A_-^0e^{i\theta_R}}{K_R^{1/2}}-\dfrac{A_+^0e^{i\theta_L}}{K_L^{1/2}}-\dfrac{B_+^0e^{-i\bar{\theta}_R}}{\bar{K}_R^{1/2}}\right]}{\left[\dfrac{B_-^0e^{-i\bar{\theta}_L}}{\bar{K}_L^{1/2}}+\dfrac{A_-^0e^{i\theta_R}}{K_R^{1/2}}+\dfrac{A_+^0e^{i\theta_L}}{K_L^{1/2}}+\dfrac{B_+^0e^{-i\bar{\theta}_R}}{\bar{K}_R^{1/2}}\right]}\ . \tag{3.15}$$

The wave vector which lies in the yz plane makes the angle ϕ with the y axis at the distance x from the origin along the direction of propagation (x axis). At $x=0$ we can write

$$K_R=K_L=K\ ,\ \ \bar{K}_R=\bar{K}_L=\bar{K}\ , \tag{3.16}$$

where K and $\bar{K}$ are given by

$$(K^2,\bar{K}^2)c^2=\omega^2-\frac{\omega\omega_p^2(\omega\mp i\nu)}{\omega^2+\nu^2} . \tag{3.17}$$

If, moreover, we put $\phi=\phi_0$ when $x=0$ we find that

$$\tan\phi_0=\frac{i\left[\dfrac{B_-^0e^{-i\bar{\theta}}}{\bar{K}^{1/2}}+\dfrac{A_-^0e^{i\theta}}{K^{1/2}}-\dfrac{A_+^0e^{i\theta}}{K^{1/2}}-\dfrac{B_+^0e^{-i\bar{\theta}}}{\bar{K}^{1/2}}\right]}{\left[\dfrac{B_-^0e^{-i\bar{\theta}}}{\bar{K}^{1/2}}+\dfrac{A_-^0e^{i\theta}}{K^{1/2}}+\dfrac{A_+^0e^{i\theta}}{K^{1/2}}+\dfrac{B_+^0e^{-i\bar{\theta}}}{\bar{K}^{1/2}}\right]} , \tag{3.18}$$

where

$$\theta=Kx-\omega t , \quad \bar{\theta}=\bar{K}x-\omega t . \tag{3.19}$$

Hence, $\phi=0$ if $B_-^0=B_+^0$, $A_+^0=A_-^0$ and $\phi=\pi/2$ if $B_-^0=-B_+^0$, $A_-^0=-A_+^0$. Obviously ϕ_0 is the angle the wave vector makes with the y axis at the origin and the net angle of rotation of the wave vector as the wave travels the distance x inside the magnetized slab is $\phi-\phi_0$. This angle of rotation consists of two parts ϕ_l and ϕ_{nl} where ϕ_l is independent of the field intensity and so obtainable from the linearized solution of the field equations (2.1) to (2.5); the nonlinear field-intensity-dependent part is ϕ_{nl}. These will be discussed in Sec. III C.

The nonlinearly correct evaluation of the angle of Faraday rotation is $\phi-\phi_0$. If we write

$$\tan(\phi-\phi_0)=P/Q$$

we find that

$$P=2i\left[\frac{B_+^0B_-^0e^{-i(\bar{\theta}_L+\bar{\theta})}}{(\bar{K}_L\bar{K})^{1/2}}+\frac{A_+^0B_-^0e^{-i(\bar{\theta}_L-\theta)}}{(\bar{K}_LK)^{1/2}}+\frac{A_+^0A_-^0e^{-i(\theta_R+\theta)}}{(K_RK)^{1/2}}-\frac{A_+^0A_-^0e^{i(\theta_L+\theta)}}{(K_LK)^{1/2}}-\frac{A_+^0B_-^0e^{i(\theta_L-\bar{\theta})}}{(K_L\bar{K})^{1/2}}\right.$$
$$\left.-\frac{A_-^0B_+^0e^{-i(\bar{\theta}_R-\theta)}}{(\bar{K}_RK)^{1/2}}+\frac{A_-^0B_+^0e^{i(\theta_R-\bar{\theta})}}{(K_R\bar{K})^{1/2}}-\frac{B_+^0B_-^0e^{-i(\bar{\theta}_R+\bar{\theta})}}{(\bar{K}_R\bar{K})^{1/2}}\right] , \tag{3.20}$$

$$Q=2\left[\frac{B_-^0A_+^0e^{-i(\bar{\theta}_L-\theta)}}{(\bar{K}_LK)^{1/2}}+\frac{B_-^0B_+^0e^{-i(\bar{\theta}_L+\bar{\theta})}}{(\bar{K}_L\bar{K})^{1/2}}+\frac{A_-^0A_+^0e^{i(\theta_R+\theta)}}{(K_RK)^{1/2}}+\frac{A_-^0B_+^0e^{i(\theta_R-\bar{\theta})}}{(K_R\bar{K})^{1/2}}+\frac{A_+^0B_-^0e^{i(\theta_L-\bar{\theta})}}{(K_L\bar{K})^{1/2}}\right.$$
$$\left.+\frac{A_+^0A_-^0e^{i(\theta_L+\theta)}}{(K_LK)^{1/2}}+\frac{B_+^0B_-^0e^{-i(\bar{\theta}_R+\bar{\theta})}}{(\bar{K}_R\bar{K})^{1/2}}+\frac{B_+^0A_-^0e^{-i(\bar{\theta}_R-\theta)}}{(\bar{K}_RK)^{1/2}}\right] . \tag{3.21}$$

B. Solution in some simple cases

The linearized dispersion relations for the left- and right-circular polarization components, obtained from Eqs. (3.6) and (3.7), are

$$n_L^{02}=\frac{K_L^{02}c^2}{\omega^2}=1-\frac{X}{1+Y+iZ} ,$$
$$n_R^{02}=\frac{K_R^{02}c^2}{\omega^2}=1-\frac{X}{1-Y+iZ} . \tag{3.22}$$

These are Eqs. (4.10.1) and (4.10.2) of Krall and Trivelpiece[19] and are not identical only when $\Omega\neq0$; when $\Omega=0$ we find that $K_L^0=K_R^0$ and birefringence cannot develop in the linearized approximation.

In the absence of a static magnetic field and collisional loss the birefringence still occurs. This is evident from the nonlinear dispersion relations (3.6) and (3.7) which reduce to

$$\frac{K_L^2c^2}{\omega^2}=n_L^2=1-X+\frac{X}{2}\left[2\alpha^2+\beta^2-\frac{4n^2\alpha^2}{4-X}\right],$$
$$\frac{K_R^2c^2}{\omega^2}=n_R^2=1-X+\frac{X}{2}\left[\alpha^2+2\beta^2-\frac{4n^2\beta^2}{4-X}\right]. \qquad (3.23)$$

This case has been considered by Arons and Max.[2]

In the absence of collisional loss we have $p_R=0$ and $p_L=0$ and the wave numbers K_R and K_L are constant; therefore the WKB solution is not necessary, and Eqs. (3.6) and (3.7) reduce to

$$n_L^2=1-\frac{X}{1+Y}+\frac{X}{2}\left[\frac{\beta^2}{(1+Y)^4}+\frac{2\alpha^2}{(1-Y^2)^2}-\frac{\alpha^2[(1+Y)(n_L+n_R)-2n_RY]}{(1-Y^2)(4-X)}\left[\frac{n_L}{1-Y}+\frac{n_R}{1+Y}\right]\right], \qquad (3.24)$$

$$n_R^2=1-\frac{X}{1-Y}+\frac{X}{2}\left[\frac{\alpha^2}{(1-Y)^4}+\frac{2\beta^2}{(1+Y^2)^2}-\frac{\beta^2[(1-Y)(n_R+n_L)+2n_LY]}{(1-Y^2)(4-X)}\left[\frac{n_L}{1-Y}+\frac{n_R}{1+Y}\right]\right]. \qquad (3.25)$$

For an order-of-magnitude estimation in Sec. III C we would consider (3.24) and (3.25) for (1) a strong magnetic field at low wave frequencies so that $\omega\pm\Omega$ can be replaced simply by $\pm\Omega$ and (2) a weak magnetic field at high wave frequencies so that $(1\pm Y)^{-1}$ approximately becomes $1\mp Y$ and $1\pm Y^2$ is replaced by unity.

Case 1. Strong magnetic field at low frequencies. In this case $|\Omega|\gg\omega$ and therefore (3.24) and (3.25) reduce to

$$n_L^2=1-\frac{X}{Y}+\frac{X}{2Y^2}\left[\frac{2\alpha^2+\beta^2}{Y^2}+\frac{\alpha^2(n_L-n_R)^2}{4-X}\right] \qquad (3.26)$$

and

$$n_R^2=1+\frac{X}{Y}+\frac{X}{2Y^2}\left[\frac{\alpha^2+2\beta^2}{Y^2}-\frac{\beta^2(n_L-n_R)^2}{4-X}\right]. \qquad (3.27)$$

Case 2. Weak magnetic field at high frequencies:

$$n_L^2=1-X(1-Y)+\frac{X}{2}\left[2\alpha^2+\beta^2-\frac{\alpha^2(n_L+n_R)^2}{4-X}\right]-2Y\left[X\beta^2+\frac{\alpha^2(n_L^2-n_R^2)}{4-X}\right] \qquad (3.28)$$

and

$$n_R^2=1-X(1+Y)+\frac{X}{2}\left[\alpha^2+2\beta^2-\frac{\beta^2(n_L+n_R)^2}{4-X}\right]+2Y\left[X\alpha^2+\frac{\beta^2(n_L^2-n_R^2)}{4-X}\right]. \qquad (3.29)$$

C. Quantitative implications of the results

For an order-of-magnitude estimation the expressions for the FR angle can be written simply as

$$\phi=\left[\frac{K_L-K_R}{2}\right]l\ , \qquad (3.30)$$

where l is the length of the slab. We can write ϕ as the sum $\phi_l+\phi_{nl}$ where ϕ_l is the contribution to FR from the linearized field solutions and is independent of the field intensity; ϕ_{nl} is due to the intensity-induced nonlinear effects. K_L and K_R are solved approximately from (3.24) and (3.25) in the different limiting cases.

In the case of unmagnetized and collisionless plasmas from (3.23) we easily find that

$$\phi_l=0\ ,\quad \phi_{nl}=\frac{X\omega(\alpha^2-\beta^2)l}{8c\sqrt{1-X}}\ . \qquad (3.31)$$

For case (1), a strong magnetic field at low wave frequency, Eqs. (3.26) and (3.27) give

$$\phi_l=-\frac{X\omega}{4cY}l\ ,$$
$$\phi_{nl}=\frac{X\omega}{8cY^2}\left[\frac{\alpha^2-\beta^2}{Y^2}+\frac{(\alpha^2+\beta^2)(n_L^0-n_R^0)^2}{4-X}\right]l\ . \qquad (3.32)$$

Since $Y^2\gg1$, and $n_L^0\simeq1-X/2Y$ and $n_R^0\simeq1+X/2Y$, in this case we find that

$$\phi_{nl} \simeq \frac{X^3\omega(\alpha^2+\beta^2)l}{128cY^4(4-X)} + \frac{X\omega(\alpha^2-\beta^2)l}{8cY^4} . \quad (3.33)$$

Again, since $X<1$,

$$\phi_{nl} \simeq \frac{X\omega(\alpha^2-\beta^2)}{8cY^4} l . \quad (3.34)$$

For case (2), weak magnetic field at high frequencies, we find that

$$\phi_l = \frac{\omega XYl}{2c\sqrt{1-X}} ,$$
$$\phi_{nl} = \frac{\omega Xl}{2c\sqrt{1-X}} [\tfrac{1}{4}(\alpha^2-\beta^2)-(\alpha^2+\beta^2)Y] . \quad (3.35)$$

The first term of ϕ_{nl} of (3.35) is obviously identical with the ϕ_{nl} of (3.31). The general results may be considered for a laser-plasma interaction. For instance, in a dense plasma ($N_0 \sim 5\times10^{18}/\text{cm}^3$) a pulsed CO_2 laser (wavelength 10.6 micrometer, $\omega = 1.78\times10^{14}/\text{sec}$) having an intensity as high as 3×10^{11} W/cm^2 would be suitable for the estimation of the nonlinear increment to FR. In such a case $\alpha^2 \approx \beta^2 \approx 1.4\times10^{-5}$ (which is very small compared to unity) and for a weak magnetic field (10^3 G), $\Omega \approx 1.78\times10^{10}/\text{sec}$, our cold-plasma results are applicable.

From Eq. (3.35) we get (in rad/cm)

$$\frac{\phi_{nl}}{l} \approx -10^{-6} , \quad \frac{\phi_l}{l} \approx 7.5\times10^{-2} . \quad (3.36)$$

It is interesting to note that in this case the nonlinearly correct FR is in the opposite sense to that of the FR obtained from the linearized field solution.

VI. CONCLUDING REMARKS

We have the following.

(a) Equations (3.6) and (3.7) give the nonlinearly correct expressions for the refractive indices n_L and n_R for the left- and right-circular polarization components of the wave. In these formulas the space-dependent factors are $\exp(ip_L)$ and $\exp(ip_R)$ which vary slowly but continuously in lossy plasmas due to the dependence of p_L and p_R on field intensity and collision frequencies. However, in nonlossy plasmas these factors vanish. For this reason the wave numbers K_L and K_R are also slowly varying quantities. As a result the normal form ordinary second-order differential equations for the variation of amplitudes have slowly varying coefficients. The WKB solution, being appropriate for these equations, has been used.

With the help of the WKB solution for the electric field, an expression for the FR angle has been obtained. From this expression the contribution of the linearized solution of the field equation has been separated and the intensity-dependent contribution due to the nonlinear increment to the field variables has also been obtained. These two parts have been quantitatively determined and compared in some simple cases. In the case of interaction of dense magnetized plasmas with a CO_2 pulsed laser the nonlinear contribution to the FR angle is shown [vide Eq. (3.36)] to be in the opposite sense with respect to the contribution from the linearized approximation of the field equations.

(b) The theory of FR has also been applied to tokamaks. Submillimeter laser techniques for measuring the poloidal magnetic field of tokamaks produced by the driving current are being developed today (cf. Lax[20]). Faraday rotation measurements use the same submillimeter lasers used for the interferometers. The technique involves a rotating polarizer which modulates the beam at a frequency of the order of 1000 Hz. This permits discrimination between the interferometer and FR signals. An analyzer measures the angle of rotation of polarization. Very small angles can be measured to determine the poloidal field patterns and the current profiles.

(c) Heald and Wharton[21] have pointed out that FR measurements with waves beamed through controlled fusion plasmas stimulated the development of microwave diagnostics as a standard measuring technique. The contemporary experiments involving lasers or other coherent sources from the visible to the millimeter-wave range use FR as one of the experiments for diagnostic purposes in hot, dense, magnetically confined fusion plasmas. The fields being strong in these experiments, the earlier theory should be replaced by the nonlinearly correct theory developed in this paper, which shows the possibility of even a rotation in the opposite sense owing to the predominance of the nonlinear correction in some cases.

(d) The generation of the induced magnetic field, the so-called IFE, due to the PR of strong waves, should be important in experiments with laser-induced plasmas. Recently, some very useful experimental research has been reported on the detection of IFE. For enlarging the scope of research on the nonlinearly induced FR, this research is summarized and discussed here.

The theory of inverse Faraday effect in solids induced by electromagnetic radiation was first developed by Pershan *et al.*[9] Pomeau and Quemeda[10] have considered the magnetization induced in plasmas by circularly polarized microwaves. Deschamps *et al.*[11] have observed such a magnetic field by using a pulsed microwave signal (3000 MHz) supplied by a klystron delivering a few megawatts during 12 μsec with a repetition frequency of 10 Hz. For circular polarization using signals of maximum amplitude 3 V, induced magnetic field ($\sim 10^{-2}$ G) has been observed by these authors.

Later Steiger and Woods,[12] and Talin *et al.*[13] have studied IFE in plasmas induced by circularly polarized electromagnetic radiation. Talin *et al.*[13] have pointed out the interesting possibility of experiments on IFE in neutral gases where the induced magnetization can be detected by the Zeeman shift of the atomic lines of a noninterfering probe gas. These authors have also noted the fact that in near-spectral resonances, due to the existence of strongly frequency-dependent susceptibility, the IFE is greatly enhanced and is strong enough to be detected. It will therefore be interesting to study the nonlinearly correct angular momentum using the WKB solution in a magnetized plasma. The nonlinear increment to the angular momentum would give rise to a spatially (slowly) varying increment in the magnetization and so in the nonlinearly induced IFE. This self-generated magnetic field can be a possible source of an externally given static magnetic field and for the consequent experimentally observable effects.

(e) In laser-induced plasmas a large-scale inhomogeneity and an induced magnetization are inevitable in addition to other complications. The theory developed here for FR due to a nonlinearly induced weak nonuniformity should be useful for the estimation of this magnetization. Therefore, it seems that a good possibility exists of enlarging the scope of the study initiated here and bringing the theory close to experimental investigations. In the future we hope to channel the theoretical investigation in these directions.

[1]L. A. France and E. R. Williams, J. Atmos. Terr. Phys. 38, 957 (1976).

[2]J. Arons and C. E. Max, Phys. Fluids 17, 1983 (1974).

[3]B. Chakraborty, Phys. Rev. A 12, 2153 (1975).

[4](a) B. Chakraborty, Phys. Rev. A 16, 1297 (1977); (b) B. Chakraborty and S. K. Chandra, Phys. Fluids 21, 2336 (1978).

[5]A. Decoster, Phys. Rep. 47, 285 (1978).

[6](a) B. Charkraborty, Manoranjan Khan, and B. Bhattacharyya, Plasma Phys. 24, 585 (1982); (b) Manoranjan Khan and B. Chakraborty, J. Appl. Phys. 50, 7977 (1979).

[7](a) B. Bhattacharyya and B. Chakraborty, Physica (Utrecht) 97C, 257 (1979); (b) B. Bhattacharyya, Phys. Rev. A 27, 568 (1983).

[8](a) B. Chakraborty, Manoranjan Khan, and B. Bhattacharyya, Phys. Lett. 78A, 454 (1980); (b) B. Chakraborty, Manoranjan Khan, and B. Bhattacharyya, Phys. Rev. A 23, 345 (1981).

[9]P. S. Pershan, J. P. Van der Ziel, and L. D. Malmstrom, Phys. Rev. 143, 574 (1966); P. S. Pershan, Phys. Rev. 130, 919 (1963).

[10]Y. Pomeau and D. Quemada, C. R. Acad. Sci. Ser. B 264, 517 (1967).

[11]J. Deschamps, M. Fitaire, and M. Lagoutte, Phys. Lett. 25, 1330 (1970).

[12]A. D. Steiger and C. H. Woods, Phys. Rev. A 5, 1467 (1972).

[13]B. Talin, V. P. Kaftandjan, and L. Klein, Phys. Rev. A 11, 648 (1975).

[14](a) T. J. M. Boyd and J. J. Senderson, *Plasma Dynamics* (Nelson, London, 1969); (b) E. S. Sarachik and G. T. Schappert, Phys. Rev. D 1, 2738 (1970).

[15]D. A. Tidman, Phys. Rev. 117, 362 (1960).

[16](a) B. Chakraborty, J. Math. Phys. 11, 2570 (1970); (b) M. R. Khan and S. N. Paul, Appl. Sci. Res. 33, 209 (1977).

[17]N. Nloembergen and Y. R. Shen, Phys. Rev. 141, 298 (1966).

[18]P. Kaw and J. Dawson, Phys. Fluids 13, 472 (1970).

[19]N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973).

[20]B. Lax, Laser Focus 18, 70 (1982).

[21]M. A. Heald and C. B. Wharton, *Plasma Diagnostics with Microwaves* (Wiley, New York, 1965).