Remarks on the renormalization group in statistical fluid dynamics

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A variant of the renormalization group is applied to the problem of randomly forced fluids studied by Forster, Nelson, and Stephen [Phys. Rev. A 16, 732 (1977)] and others. Amplitude factors (thought to be nonuniversal by some authors) are evaluated and shown to have universal values. Comparisons with closures are made. The possibility of a breakdown of self-similarity and/or universality due to intermittency effects is discussed.

INTRODUCTION

Forster, Nelson, and Stephen[1] and others[2-4] have shown that certain problems in statistical fluid dynamics which involve strong nonlinearities are amenable to renormalization-group (RG) techniques. Such machinery has been worked out when the turbulence at scales smaller than a given scale l acts on scales larger than (or equivalent to) l like an eddy viscosity; this reduces the effective coupling constant (based on the renormalized eddy viscosity) to a value that allows perturbation calculations (in particular, of the eddy viscosity acting on even larger scales). For the procedure to bootstrap, the effective coupling must eventually (i.e., at increasingly large scales) be driven to zero ("trivial" case) or to a nonvanishing small value ("nontrivial" case). This technique can be applied to three-dimensional incompressible flow subject to random forcing with a power-law spectrum

$$F(k)=2Dk^{3-\epsilon} . \tag{1}$$

Here $F(k)$ is the amount of energy injected per wave number. When ϵ is positive and small, the resulting energy spectrum for the fluid is

$$E(k)\propto k^{1-2\epsilon/3} . \tag{2}$$

Note that a simple dimensional argument[5] produces the result (2), which may be also obtained within low-order self-consistent closure approximations such as the direct interaction approximation (DIA) (see, e.g., p. 745 and Note 33 in Ref. 1 or Ref. 6). Kraichnan[6] even argues that (i) the RG method is in no way superior to closure, and (ii) one cannot rule out a breakdown (due to intermittency effects) of the self-similar RG solution that is reminiscent of the breakdown of the self-similar Kolmogorov solution.[7]

We found that some progress can be made on these issues by deriving more quantitative information than provided by (2). Indeed, previous calculations made no attempt to obtain the constant in front of the power law; in a different physical context (e.g., critical phenomena), it often happens that such constants are nonuniversal.[8,9] Here, however, constraints like Galilean invariance make the problem more restricted; thus the constant is calculable and universal, at least with respect to the small scale dynamics. This calculation will be done hereafter, using an RG procedure with variable ultraviolet (uv) cutoff, which is somewhat simpler than the Wilson-type technique used by Forster *et al.* (in particular, no rescaling is needed).

THE PROBLEM

We start with the forced three-dimensional Navier-Stokes equation for an incompressible fluid; in Fourier space it reads

$$\begin{aligned}(-i\omega+\nu_0k^2)V_l(\vec{k},\omega) &= f_l(\vec{k},\omega)-\tfrac{1}{2}iP_{lmn}(\vec{k})(2\pi)^{-4}\\ &\quad\times\int_{R^3\times R} d^3q\,d\Omega\, V_m(\vec{q},\Omega)V_n(\vec{k}-\vec{q},\omega-\Omega)\ ;\end{aligned} \tag{3}$$

here ν_0 is the molecular viscosity, and

$$P_{lmn}(\vec{k})=k_mP_{ln}(\vec{k})+k_nP_{lm}(\vec{k}) , \tag{4a}$$

$$P_{lm}(\vec{k})=\delta_{lm}-k_lk_m/k^2 . \tag{4b}$$

The only nonvanishing cumulant of the zero mean stationary Gaussian force f is the two-point correlation

$$\langle f_m(\vec{k},\omega)f_n(\vec{k}\,',\omega')\rangle = P_{mn}(\vec{k})(2\pi)^4\delta(\vec{k}+\vec{k}\,')\times\delta(\omega+\omega')[F(k)/4\pi k^2] , \tag{5}$$

where $F(k)$, the energy injection spectrum, follows the power law (1), with $\epsilon>-1$; this is the generalization of model B of Forster *et al.* studied in Refs. 2 (model "R"), 3, and 5; the borderline $\epsilon=-1$ corresponds to model A of Ref. 1. The purpose here is to calculate the statistical properties of the solution of (3) at a fixed wave number and a fixed viscosity $\nu_0>0$ when $\epsilon\downarrow 0$.[10]

PROCEDURE AND RENORMALIZED VISCOSITY

Let us consider an "artificial" problem with an $O(1)$ ultraviolet cutoff. It is known that by a suitable modification (renormalization) of the viscosity in (3), this new problem (with cutoff) becomes equivalent (up to irrelevant terms in the $\epsilon\downarrow 0$ limit) to the original one (for wave numbers less than the cutoff, of course). For this it is convenient to first consider an infinitesimal change from a cutoff Λ to $\Lambda-\delta\Lambda$, which is equivalent to an infinitesimal increase $\delta\nu$ of the viscosity ν. One starts from the problem with cutoff Λ and viscosity ν; one uses the standard method of perturbatively calculating the Fourier amplitudes in the wave-number shell $[\Lambda-\delta\Lambda,\Lambda]$ and substituting into the equation for the modes $k<\Lambda-\delta\Lambda$. The expansion parameter is the Reynolds number (reduced coupling constant)

$$R(\Lambda)=D^{1/2}\nu^{-3/2}\Lambda^{-\epsilon/2} \tag{6}$$

which measures the relative strength of the nonlinear and viscous terms near the cutoff Λ; this number must be small for the expansion to be valid. The eliminated modes generate an additional eddy viscosity $\delta\nu$ with

$$\frac{\delta\nu}{\nu}=HR^2(\Lambda)\frac{\delta\Lambda}{\Lambda}\ , \tag{7}$$

where

$$H=\frac{1}{10\pi^2} \tag{8}$$

in the limit $\epsilon\downarrow 0$. It is now possible to make a *finite* change in the cutoff from Λ to a large value Λ_0. This requires the use of a cutoff-dependent viscosity $\nu(\Lambda)$ which is decreased as Λ is increased. From (7) and (6), $\nu(\Lambda)$ must satisfy the differential RG equation

$$\frac{d\nu(\Lambda)}{d\Lambda}=-HD\nu^{-2}\Lambda^{-1-\epsilon}\ . \tag{9}$$

Integrating we obtain

$$\nu^3(\Lambda)-\nu^3(\Lambda_0)=3HD\epsilon^{-1}(\Lambda^{-\epsilon}-\Lambda_0^{-\epsilon})\ . \tag{10}$$

In (10), one may take the $\Lambda_0\rightarrow\infty$ limit (ϵ is positive), in which the viscosity $\nu(\Lambda_0)$ tends to the prescribed molecular value ν_0. If we finally only retain the dominant term in the $\epsilon\downarrow 0$ limit, we find the renormalized viscosity at wave number Λ,

$$\nu(\Lambda)\simeq(3H)^{1/3}D^{1/3}\epsilon^{-1/3}\Lambda^{-\epsilon/3}\ , \tag{11}$$

which is large in the sense that it involves ϵ to a negative power. Equivalently, the renormalized Reynolds number

$$R(\Lambda)\simeq(3H)^{-1/2}\epsilon^{1/2} \tag{12}$$

is small, making the procedure self-consistent.

ENERGY SPECTRUM

Let us denote the energy spectrum at wave number k, for viscosity ν, forcing strength D and cutoff Λ, by $E(k;\nu,D,\Lambda)$. Because of the above-stated equivalence, we have

$$E(k;\nu_0,D,\ \Lambda_0=\infty)=E(k;\nu(\Lambda),D,\Lambda)\ , \tag{13}$$

provided that $\nu(\Lambda)$ and ν_0 are related by (10). In the $\epsilon\downarrow 0$ limit the renormalized Reynolds number is small and we may calculate $E(k;\nu(\Lambda),D,\Lambda)$ perturbatively, i.e., by the linear approximation

$$E(k;\nu(\Lambda),D,\Lambda)\simeq\frac{2Dk^{3-\epsilon}}{2\nu(\Lambda)k^2}\ , \tag{14}$$

where $k\lesssim\Lambda=O(1)$. Using (11), (13), and (14), we find

$$E(k;\nu_0,D,\ \Lambda_0=\infty)\simeq D^{2/3}(3H)^{-1/3}\epsilon^{1/3}k^{1-2\epsilon/3} \tag{15}$$

($\epsilon\downarrow 0$). We see that the amplitude in front of the power law $k^{1-2\epsilon/3}$ has been obtained explicitly in terms of ϵ and D.

Our result is universal in the sense that—under the assumptions made—it does not depend on the molecular (bare) viscosity and on the small scale forcing. Similar universality holds in some of the nontrivial regimes of the passive scalar problem,[1] namely, those in which all the interactions are sufficiently "local" (see also Ref. 11). Recall that in connection with turbulence, local is generally used to mean "between comparable scales." Presumably, the reason why amplitudes are universal (in the above sense) for nontrivial Navier-Stokes–type problems is that the number of relevant renormalized couplings is equal to the number of reduced couplings with nontrivial asymptotic value. This number is 1 for Navier-Stokes (NS) and 2 for NS plus passive scalar, where viscosity and diffusivity are renormalized and the reduced couplings are the Reynolds and the Prandtl number. In contrast, for model A of Ref. 1, on the nontrivial side ($\epsilon=2-d>0$), there are two renormalized couplings (viscosity and forcing) but still one reduced coupling (Reynolds number). A calculation similar to the above one gives the renormalized viscosity and forcing

$$\nu(\Lambda)\simeq(8\pi)^{-1/2}(D_0/\nu_0)^{1/2}\epsilon^{-1/2}\Lambda^{-\epsilon/2}\ , \tag{16a}$$

$$D(\Lambda)=(D_0/\nu_0)\nu(\Lambda)\ . \tag{16b}$$

Relation (16b) is of a fluctuation-dissipation type which introduces a dependence upon the molecular viscosity ν_0; thus the large scale amplitudes are not universal. Of course, whatever the model, lack of universality also occurs in the trivial or marginal cases ($\epsilon\leq 0$) where no finite limit is obtained when the cutoff Λ_0 goes to infinity; at the crossover of model R, the calculation gives

$$E(k)\simeq D^{2/3}(3H)^{-1/3}k[\ln(\Lambda_0/k)]^{-1/3}\ , \tag{17}$$

where a small-scale influence comes through the logarithm correction.

COMPARISON WITH CLOSURES

The result (15) is identically reproduced by the direct interaction approximation.[12] For Markovian closures such as the eddy-damped quasinormal Markovian approximation (EDQNM)[13] or the test-field model (TFM),[14] the situation is somewhat different because the latter have adjustable constants. Such constants can be unequivocally determined to be compatible with the RG calculation. The RG result also suggests an improvement for closures of the Obukhov-Heisenberg type[13] (see also Ref. 15). In such closures the energy flux through wave number k is constructed from an eddy viscosity $\nu_e(k)$, the latter being self-consistently determined in terms of the energy spectrum, e.g., as

$$\nu_e(k)=A[E(k)/k]^{1/2}\ . \tag{18}$$

The RG result with its correct ϵ dependence can be reproduced by making the constant A dependent on the local exponent of the spectrum. A suitable choice is

$$A(k)=(3H)^{1/2}\left\{-\tfrac{3}{2}k\left[\frac{\partial}{\partial k}\ln\left[\frac{E(k)}{k}\right]\right]\right\}^{-1/2}\ , \tag{19}$$

which may be of interest in connection with subgrid-scale modeling.[16]

INTERMITTENCY

The role of barely local interactions versus local interactions was discussed by Kraichnan.[6] For small positive ϵ, the dynamics of the regime described by the RG involves "barely local" interactions, i.e., interactions over a range of wave numbers whose logarithm is $O(1/\epsilon)$. Kraichnan warns about the possibility that local interactions, which are nowhere included in the RG analysis, might upset the result. However, we observe that the characteristic dynamical time for the barely local interactions is

$$\tau_{\rm BL}(k) \simeq [k^2 \nu(k)]^{-1} \simeq \epsilon^{1/2}(3H)^{-1/2}[k^3 E(k)]^{-1/2} , \tag{20}$$

which is smaller by a factor $\epsilon^{1/2}$ than the local eddy turnover time. Hence for small ϵ, barely local interactions dominate over local ones and the RG result can hardly be upset by the inclusion of local interactions. It seems, therefore, unlikely that a local spontaneous buildup of intermittent fluctuations can lead to a breakdown of the self-similar RG result.

We found, however, that under certain circumstances intermittency can upset the universality of the result. This is related to Landau's objection to the universality of the Kolmogorov constant for the inertial range spectrum

$$E(k) \simeq C_{\rm Kol} \langle \epsilon \rangle^{2/3} k^{-5/3} , \quad \nu_0 \downarrow 0 , \quad k \to \infty . \tag{21}$$

Here $\langle \epsilon \rangle$ denotes, as usual, the mean energy input, which has nothing to do with our previous ϵ (crossover parameter). Landau's objection to the universality of $C_{\rm Kol}$ (which is discussed, e.g., in Sec. 25 of Ref. 15) goes roughly as follows. Assume that the energy input is modulated by a random constant $m > 0$ (e.g., by having a collection of identical wind tunnels with different upstream velocities). For a given realization we have

$$E(k) \simeq C_{\rm Kol} m^{2/3} \langle \epsilon \rangle^{2/3} k^{-5/3} . \tag{22}$$

Hence, superaveraging over all the realizations, we have

$$\langle E(k) \rangle_{\rm super} \simeq C_{\rm Kol} \langle m^{2/3} \rangle_{\rm super} \langle \epsilon \rangle^{2/3} k^{-5/3} . \tag{23}$$

Alternatively, one can calculate the energy spectrum using the superaveraged energy input to obtain

$$\langle E(k) \rangle_{\rm super} \simeq C_{\rm Kol} \langle m \rangle_{\rm super}^{2/3} \langle \epsilon \rangle^{2/3} k^{-5/3} , \tag{24}$$

which is inconsistent with (23) unless m is sharp. This Landau argument appears much less academic if we think not of several wind tunnels but of distant points of a single wind tunnel in which the upstream velocity at the grid fluctuates over distances large when compared to the mesh. This kind of objection has led Kolmogorov to develop his intermittent version of the inertial range theory.[17]

Turning to the RG result (15), the interesting observation is that the forcing strength D appears with the $\frac{2}{3}$ exponent in the energy spectrum and thus that the Landau objection applies in the same form. However, this time, the constant must be universal. In fact, the RG analysis becomes unapplicable when one includes in the forcing a modulation factor $\sqrt{m}$ (to give an m factor in the forcing spectrum). The reason is as follows. The renormalization $\delta\nu$ of the viscosity, following from a modification $\delta\Lambda$ in the cutoff, is actually a random quantity with a mean and fluctuations. If we assume that the force has only short-range correlations,[18] then the fluctuations of $\delta\nu$ at wave number k are weaker than the mean by a factor $(k/\Lambda)^{-3/2}$. This follows by counting the number N of regions of diameter $\sim\Lambda^{-1}$ in a region of diameter $\sim k^{-1}$ and applying a $1/\sqrt{N}$ statistical factor. If, however, there are long-range correlations, such as result from the inclusion of the $\sqrt{m}$ random modulation, then the above analysis becomes invalid. Indeed, the fluctuations in $\delta\nu$ are now as big as the mean and the RG calculation breaks down. We conclude that the Landau-type nonuniversality can occur only if we consider forces with long-range correlations (in the above sense).

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[9] D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena,* International Series in Pure and Applied Physics (Mc Graw-Hill, New York, 1978).

[10] On the trivial side ($\epsilon < 0$), dynamics are linear and (nonuniversal) amplitudes may be computed by various techniques, including RG.

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[13] See, e.g., S. A. Orszag, in *Fluid Dynamics,* edited by R. Balian and J.-L. Peube (Gordon and Breach, New York, 1977), p. 235.

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[17] A. N. Kolmogorov, J. Fluid Mech. 13, 82 (1962).

[18] More specifically we assume that the bandpassed filtered force around wave number Λ (with, say, an octave width) has no appreciable correlations extending over distances much larger than Λ^{-1}. This holds, e.g., for Gaussian forces.