Possible use of motional magnetic fields to flip spins in a Penning trap

Lowell S. Brown
Department of Physics, FM-15, University of Washington, Seattle, Washington 98195
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The most precise determination of the magnetic moments of the electron and positron come from the University of Washington $g-2$ experiments. A small distortion in the magnetic field is introduced in these experiments so as to provide a mechanism to flip and measure the spin. The experiments would be substantially improved if this inhomogeneous part of the magnetic field were removed. Driving the end caps produces an electric field at the frequency ω_d which is parallel to the magnetic field. The cyclotron motion of the electron at frequency ω_c transforms this electric field into a magnetic field which is perpendicular to the main magnetic field and which alternates at frequency $\omega_c+\omega_d$ in the particle's rest frame. One might think that such an alternating magnetic field would flip the spin, thereby removing the need for the field inhomogeneity. There are, however, subtle relativistic corrections to the spin motion. These nearly cancel the motional magnetic field effect and rule out this scheme of spin flipping.

I. INTRODUCTION AND SUMMARY

The magnetic moments of the electron and positron have been measured with marvelous precision.[1] These experiments employ a Penning trap which consists of a weak electric quadrupole field together with a strong uniform magnetic field. In order to flip the spin and detect the g-2 resonance, a small nonuniformity is introduced in the magnetic field. This inhomogeneous part of the magnetic field, coupled with thermal vibrations of the particle, produces a linewidth[1,2] which limits the accuracy of the experiments. They would be improved substantially if some other method were devised to flip the spin in a uniform magnetic field.

Motional magnetic fields might appear to provide an alternative mechanism to flip the spin.[3] The constant electric quadrupole field is produced by two end-cap electrodes at a common potential which are aligned along the magnetic field and a ring electrode at a different potential which circles the magnetic axis. Suppose that in addition an alternating voltage is applied between the end caps. This gives an additional electric field $\vec{\mathscr{E}}(t)=\vec{\mathscr{E}}_0\cos\omega_d t$. In general, the electric field will be spatially varying with both radial and axial components. Near the center of the trap, which is the desired operating point, the radial fields become negligible and $\vec{\mathscr{E}}_0$ can be taken to be a constant vector along the magnetic field $\vec{B}$. The main motion of the particle for our purposes is about a cyclotron circle at the angular frequency $\omega_c=eB/mc$, where m is the electron mass. Thus there is small alternating magnetic field $\vec{b}_\perp(t)=c^{-1}\vec{v}(t)\times\vec{\mathscr{E}}(t)$ in the particle's rest frame which is perpendicular to the large constant magnetic field. This alternating field has a frequency component at $\omega_c+\omega_d$ and so a spin-flipping resonance occurs at $\omega_c+\omega_d=\omega_s=\frac{1}{2}g\omega_c$ or when the drive frequency equals the anomaly frequency $\omega_d=\frac{1}{2}(g-2)\omega_c=a\omega_c$. On resonance, the rate at which the flipping proceeds is given by $\Omega_\perp=\frac{1}{2}(eg/2mc)(v/c)\mathscr{E}_0$. In the $g-2$ experiments the particle is in a low-lying Landau state $\frac{1}{2}mv^2\approx\hbar\omega_c$ with $\omega_c/2\pi\approx100$ GHz, which gives $v\approx10^6$ cm/sec. Since $2mc^2/e\approx10^6$ V, we see that to obtain a usable rate, $\Omega_\perp/2\pi\approx1$ Hz requires a rather strong electric field $\mathscr{E}_0\approx6$ V/cm.

The scenario we have just presented is, unfortunately, not the whole story. Other effects conspire to reduce the spin-flip rate to an unacceptable value.

(i) The constant electric quadrupole potential also acts on the particle. If this field is strong, the particle is bound to a point where the total electric field vanishes, and the motional magnetic field disappears. That is, the total oscillating electric field is given by $E_2z(t)+\mathscr{E}(t)$, where E_2 is the amplitude of the quadrupole field and the z axis is taken to be along the magnetic field. In the limit where E_2 is large,

$$z(t)=-\mathscr{E}(t)/E_2$$

and the total field vanishes. In general, this effect produces an overall factor of $\omega_d^2(\omega_d^2-\omega_z^2)^{-1}$ in $\Omega_\perp$, where ω_z is the axial oscillation frequency.

(ii) Relativistic corrections are more important. They arise both from the kinematic Thomas precession and from relativistic velocity-dependent torques. These are subtle effects which have no simple physical explanation. The mathematics is described in detail in the following section. The result is that the relativistic corrections largely cancel the simple motional field effect and replace the overall g factor with the anomaly $a=\frac{1}{2}(g-2)$ in the formula for $\Omega_\perp$. Since $a\approx 10^{-3}$, this reduces the spin-flip rate to an unusable value.

(iii) The axial alternating field $\vec{\mathscr{E}}(t)$ produces, via the Maxwell displacement current, a perpendicular alternating magnetic field. This magnetic field also causes spin flips at a rate comparable to that given by the relativistically corrected motional field effect, with the rate involving an overall factor of the anomaly a.

The net result of all of these various effects is to produce a spin rotation (Rabi) frequency given by

$$\vec{\Omega}=\frac{ea(g+4)}{4mc^2}E_{\parallel}\hat{k}\times\vec{v} \; . \tag{1}$$

Here $E_{\parallel}$ is the total electric field which acts on the particle along the direction of the magnetic field denoted by $\hat{k}$. Expressing the quadrupole part of this axial electric field $E_{\parallel}$ in terms of the natural frequency of the axial oscillation ω_z gives

$$E_{\parallel}=\frac{m}{e}\omega_z^2 z(t)+\mathscr{E}(t) \; . \tag{2}$$

The axial coordinate of the particle $z(t)$ obeys the equation of motion

$$m\frac{d^2}{dt^2}z=-eE_{\parallel} \; . \tag{3}$$

(We use the charge of the electron $-e$.) The general solution is the superposition of a free oscillation plus a driven motion. If the drive is applied for a sufficiently long time, the free oscillations will be damped by mechanisms omitted in Eq. (3). With a sinusoidal driving field

$$\mathscr{E}(t)=\mathscr{E}_0\cos\omega_d t \; , \tag{4}$$

the long-term motion is given by

$$z(t)=\frac{1}{\omega_d^2-\omega_z^2}\frac{e}{m}\mathscr{E}(t) \; . \tag{5}$$

In this case the spin rotation frequency reads

$$\vec{\Omega}=\frac{ea(g+4)}{4mc^2}\frac{\omega_d^2}{\omega_d^2-\omega_z^2}\mathscr{E}\hat{k}\times\vec{v}$$
$$=\frac{a(g+4)}{4c^2}\omega_d^2 z\hat{k}\times\vec{v} \; . \tag{6}$$

Clearly, the rate is increased if the axial frequency ω_z is adjusted to lie near the drive frequency ω_d (which in turn must be near the anomaly frequency $a\omega_c$). However, this adjustment also increases the amplitude of the axial oscillation $z(t)$ of the particle, and this amplitude must be kept reasonably small so as to avoid field nonlinearities.

II. CALCULATION

Including terms of order v^2/c^2, the electron (or positron) spin $\vec{S}$ obeys the equation of motion

$$\frac{d}{dt}\vec{S}=\vec{\Omega}\times\vec{S} \; , \tag{7}$$

where

$$\vec{\Omega}=\frac{e}{2mc}\left[\left(g-\frac{v^2}{c^2}\right)\vec{B}-(1+2a)\vec{v}\times\vec{E}/c -a(\vec{v}\cdot\vec{B})\frac{\vec{v}}{c^2}\right] \; . \tag{8}$$

This result follows from the covariant equation of motion for the spin four-vector s^μ by using an instantaneous Lorentz boost to write s^μ in terms of the spin variable $\vec{S}$ which is defined in the particle's rest frame.[4] The result also follows from performing[5] a Foldy-Wouthuysen transformation on the Dirac Hamiltonian. We shall take the magnetic field B to be constant and to point along the positive z axis specified by the unit vector $\hat{k}$. The electric field $\vec{E}$ has a time-independent part $\vec{E}_Q$ derived from the electric quadrupole potential plus a spatially uniform but time-dependent part $\vec{\mathscr{E}}(t)$ which is aligned along $\hat{k}$. We shall decompose vectors into their parts which are parallel ($\parallel$) and perpendicular ($\perp$) to the magnetic field direction $\hat{k}$. For example,

$$\vec{v}=v_{\parallel}\hat{k}+\vec{v}_\perp \; ,$$
$$\vec{E}=E_{\parallel}\hat{k}+\vec{E}_\perp \; , \tag{9}$$

and

$$\vec{\Omega}(t)=\omega_s(t)\hat{k}+\vec{\Omega}_\perp(t) \; . \tag{10}$$

As is usual in discussing spin resonance, it proves convenient to pass to a coordinate system which rotates about the magnetic field axis $\hat{k}$ with the time-dependent angular frequency $\omega_S(t)$. In this new coordinate frame the spin is rotated only in perpendicular directions with

$$\frac{d}{dt}\underline{\vec{S}}=\underline{\vec{\Omega}}_\perp\times\underline{\vec{S}} \; , \tag{11}$$

where

$$\underline{\vec{\Omega}}_\perp = -\frac{e}{2mc}[(1+2a)(v_{\|}\hat{k}\times\underline{\vec{E}}_\perp + \underline{\vec{v}}_\perp\times\hat{k}E_{\|})/c + av_{\|}B\underline{\vec{v}}_\perp/c^2] . \tag{12}$$

Here we have underlined vectors to indicate that they are referred to the axes of the rotating frame.

The spin resonance occurs when the frequency of $\mathscr{E}(t)$ is adjusted to make some piece of $\underline{\vec{\Omega}}_\perp$ time independent. Thus we can neglect terms in $\underline{\vec{\Omega}}_\perp$ which are total time derivatives as they do not give rise to a persistent, secular perturbation. With this in mind, we note that $\underline{\vec{v}}_\perp$ obeys the equation of motion

$$\frac{d}{dt}\underline{\vec{v}}_\perp = -\omega_S\hat{k}\times\underline{\vec{v}}_\perp - \frac{e}{m}\left[\underline{\vec{E}}_\perp + \underline{\vec{v}}_\perp\times\frac{\vec{B}}{c}\right]$$

$$= -a\omega_c\hat{k}\times\underline{\vec{v}}_\perp - \frac{e}{m}\underline{\vec{E}}_\perp . \tag{13}$$

To obtain the second equality we have neglected the relativistic corrections and used

$$\omega_S = \tfrac{1}{2}g\omega_c = (a+1)\omega_c , \tag{14}$$

where

$$\omega_c = \frac{eB}{mc} \tag{15}$$

is the cyclotron frequency. This approximation is permissible since it will be employed in a term that is itself a small relativistic correction. Using the axial equation of motion

$$\frac{d}{dt}v_{\|} = -\frac{e}{m}E_{\|} , \tag{16}$$

we now obtain

$$\underline{\vec{\Omega}}_\perp = -\frac{1}{2c^2}\frac{d}{dt}(v_{\|}\hat{k}\times\underline{\vec{v}}_\perp) + \underline{\vec{\Omega}}_1 \tag{17}$$

with

$$\underline{\vec{\Omega}}_1 = -\frac{e}{2mc^2}(gv_{\|}\hat{k}\times\underline{\vec{E}}_\perp + 2a\underline{\vec{v}}_\perp\times\hat{k}E_{\|}) . \tag{18}$$

The total time derivative in Eq. (17) can be neglected. Note that the $1/c^2$ relativistic correction in Eq. (12) has canceled the major part of the motional magnetic field effect involving $\underline{\vec{v}}_\perp\times\hat{k}E_{\|}/c$, leaving a coefficient involving a factor of the anomaly a.

The field $\underline{\vec{E}}_\perp$ arises from the electric quadrupole potential, and it may be expressed as

$$\underline{\vec{E}}_\perp = -\frac{m}{2e}\omega_z^2\underline{\vec{\rho}} . \tag{19}$$

Here $\underline{\vec{\rho}}$ is the particle's perpendicular position whose components are measured in the rotating frame. Hence

$$\frac{d}{dt}\underline{\vec{\rho}} = -\omega_s\hat{k}\times\underline{\vec{\rho}} + \underline{\vec{v}}_\perp . \tag{20}$$

In the original laboratory frame, $\vec{\rho}$ executes an epicyclic motion which is the sum of a slow magnetron motion plus a fast cyclotron motion. The slow laboratory magnetron motion appears as a fast oscillation in the rotating frame so it does not contribute to the spin resonance. Thus we have, effectively,[6]

$$\omega_c\hat{k}\times\underline{\vec{\rho}} = \underline{\vec{v}}_\perp \tag{21}$$

and

$$\frac{d}{dt}\underline{\vec{\rho}} = -a\underline{\vec{v}}_\perp . \tag{22}$$

Writing $v_{\|} = (d/dt)z$ and omitting another total time derivative now gives the replacement

$$-\frac{eg}{2mc^2}v_{\|}\hat{k}\times\underline{\vec{E}}_\perp \to \frac{eg}{2mc^2}z\hat{k}\times\frac{d}{dt}\underline{\vec{E}}_\perp$$

$$= \frac{ag}{4c^2}\omega_z^2 z\hat{k}\times\underline{\vec{v}}_\perp . \tag{23}$$

This puts Eq. (14) in the form

$$\underline{\vec{\Omega}}_1 = \frac{ea}{4mc^2}\left[g\frac{m}{e}\omega_z^2 z + 4E_{\|}\right]\hat{k}\times\underline{\vec{v}}_\perp , \tag{24}$$

which exhibits an overall factor of the anomaly a.

There remains one final effect which is of the same order as are the terms in Eq. (24). The alternating electric field $\vec{\mathscr{E}}(t)$ must be accompanied by a magnetic field in order to satisfy the Maxwell equation

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} . \tag{25}$$

Since $\vec{\mathscr{E}}(t)$ is spatially uniform, the solution of this Maxwell equation is given by[7]

$$\vec{\mathscr{B}}(\vec{\rho},t) = -\frac{1}{2c}\vec{\rho}\times\frac{d}{dt}\vec{\mathscr{E}}(t) . \tag{26}$$

This alternating magnetic field gives a contribution

$$\vec{\Omega}_2 = \frac{ge}{2mc}\vec{\mathscr{B}} \tag{27}$$

to the spin rotation frequency. Passing to the rotating frame and neglecting a total time derivative we have

$$\underline{\vec{\Omega}}_2 = \frac{g}{4mc^2}\frac{d\underline{\vec{\rho}}}{dt}\times\vec{\mathscr{E}}(t) . \tag{28}$$

In view of Eq. (22), this may be expressed as

$$\underline{\vec{\Omega}}_2 = -\frac{gae}{4mc^2}\underline{\vec{v}}_\perp \times \vec{\mathscr{E}}(t) . \tag{29}$$

The total spin rotation frequency is given by

$$\begin{aligned} \underline{\vec{\Omega}} &= \underline{\vec{\Omega}}_1 + \underline{\vec{\Omega}}_2 \\ &= \frac{ea}{4mc^2}(g+4)E_{\parallel}\hat{k}\times\underline{\vec{v}} , \end{aligned} \tag{30}$$

where E is the total electric field

$$E_{\parallel} = \frac{m}{e}\omega_z^2 z(t) + \mathscr{E}(t) . \tag{31}$$

This is the result quoted in Eq. (1).

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[1]Reviews of this work are given by R. S. Van Dyck, Jr., P. B. Schwinberg, and H. G. Dehmelt, in *New Frontiers in High Energy Physics*, edited by B. Kursunoglu, A. Perlmutter, and L. Scott (Plenum, New York, 1978); H. G. Dehmelt, in *1982/3 Yearbook of Science and Technology* (McGraw-Hill, New York, 1982); in *Atomic Physics 7*, edited by D. K. Kleppner and F. T. Pipkin (Plenum, New York, 1981); P. Ekstrom and D. Wineland, Sci. Am. 243, 105 (1980).

[2]The line shape has been derived by L. S. Brown (unpublished).

[3]H. G. Dehmelt and P. Ekstrom, Bull. Am. Phys. Soc. 18, 727 (1973).

[4]See, for example, J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), Sec. 11.11, especially Eq. (11.170).

[5]L. S. Brown (unpublished).

[6]Here we have neglected the very small shift of the cyclotron motion caused by the electric quadrupole potential. The exact effective equality is given by

$$(\omega_c - \omega_m)\hat{k}\times\underline{\vec{\rho}} = \underline{\vec{v}}_\perp ,$$

where ω_m is the magnetron frequency. Since

$$\omega_m \simeq \frac{\omega_z^2}{2\omega_c} \ll \omega_c ,$$

this is a negligible correction.

[7]This magnetic field produces a spatially varying correction to the electric field $\mathscr{E}(t)$ by virtue of the Maxwell equation

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} ,$$

but this is a higher-order effect which is negligible.