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New Hamiltonian for a charged particle in an applied electromagnetic field

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We derive a new Hamiltonian, for a charged particle in a time-dependent applied electromagnetic field, which depends on the fields $\vec{E}$ and $\vec{B}$ directly rather than on the potentials ϕ and $\vec{A}$. The Hamiltonian is "nonlocal" in that it involves $\vec{E}$ and $\vec{B}$ at all points in space, in contrast to the usual "local" Hamiltonian, which only involves ϕ and $\vec{A}$ at the position of the charge. The new and usual Hamiltonians are compared, and the canonical transformation which connects them is presented. We discuss the physical interpretation of the interaction terms appearing in the canonical momentum, angular momentum, and the Hamiltonian. A relativistic generalization is given.

I. INTRODUCTION

The vector potential appears immediately in most discussions of the Aharonov-Bohm effect[1–4] because it is the *canonical* momentum $\vec{p}$ that is replaced by $-i\hbar\vec{\nabla}$ in quantum mechanics, and the canonical and "mechanical" momenta differ by $(e/c)\vec{A}(\vec{q},t)$,

$$\vec{p}=m\dot{\vec{q}}+\frac{e}{c}\vec{A}(\vec{q},t)\,, \tag{1.1}$$

where $\vec{q}$ is the position of the particle with charge e and mass m. Throughout this paper we will consider only the motion of a particle in a classical, applied electromagnetic field. Trammel[5] has shown that, if the Coulomb gauge $[\vec{\nabla}\cdot\vec{A}(\vec{r},t)=0]$ is adopted, Eq. (1.1) is equivalent to

$$\vec{p}=m\dot{\vec{q}}+\frac{1}{4\pi c}\int\vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}\,, \tag{1.2}$$

where $\vec{B}(\vec{r},t)=\vec{\nabla}\times\vec{A}(\vec{r},t)$, and

$$\vec{\mathscr{E}}(\vec{r}-\vec{q})\equiv\frac{e(\vec{r}-\vec{q})}{|\vec{r}-\vec{q}|^3}\,. \tag{1.3}$$

Equation (1.2) does not involve $\vec{A}$ explicitly, and at least to lowest order in $|\dot{\vec{q}}|/c\ll 1$, admits of a simple physical interpretation[5]: Since in that limit the electromagnetic field from a point charge consists only of an electric field equal to $\vec{\mathscr{E}}$, the part of the momentum of the total electromagnetic field due to the field from the charge *in the presence of the applied field* would, in usual electromagnetic theory, be written as

$$\vec{p}_{\rm int}=\frac{1}{4\pi c}\int\vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}\,. \tag{1.4}$$

Referring to Eq. (1.2), the canonical momentum can hence be considered as consisting of a mechanical part $m\dot{\vec{q}}$, plus the "interaction momentum" $\vec{p}_{\rm int}$. Peshkin,[6] working in the Coulomb gauge, has used such a decomposition and intepretation of the canonical *angular* momentum $\vec{J}$ to shed some light on the connection between the Aharonov-Bohm effect and the usual quantization conditions of quantum mechanics (see also Casimir[7] and Kunstatter *et al.*[8]).

Now it is easy to see that, if $\vec{\nabla}\cdot\vec{A}\neq 0$, an extra term appears in Eq. (1.2); Eq. (1.1) is in general equivalent to

$$\vec{p}=m\dot{\vec{q}}+\frac{1}{4\pi c}\int\vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}-\frac{1}{4\pi c}\int\vec{\mathscr{E}}(\vec{r}-\vec{q})\vec{\nabla}\cdot\vec{A}(\vec{r},t)d\vec{r}\,. \tag{1.5}$$

This may be verified by setting $\vec{B}=\vec{\nabla}\times\vec{A}$ in Eq. (1.5), performing two partial integrations, using

$$\begin{aligned}&\vec{\nabla}\cdot\vec{\mathscr{E}}(\vec{r}-\vec{q})=4\pi e\delta(\vec{r}-\vec{q})\,,\\&\frac{\partial}{\partial r_i}\mathscr{E}_j(\vec{r}-\vec{q})=-\frac{\partial}{\partial q_i}\mathscr{E}_j(\vec{r}-\vec{q})\,,\\&\mathscr{E}_i(\vec{r}-\vec{q})=-\frac{\partial}{\partial r_i}\frac{e}{|\vec{r}-\vec{q}|}\,,\end{aligned} \tag{1.6}$$

and assuming that the applied fields are sufficiently well behaved at infinity to neglect the surface integrals which appear. The last assumption, of course, was also required in deriving Eq. (1.3) in the specific case of the Coulomb gauge. Looking at Eq. (1.5) it is natural to ask if a different canonical momentum $\vec{p}\,'$ can be found which does not involve the vector potential $\vec{A}(\vec{r},t)$ explicitly, but only the magnetic field $\vec{B}(\vec{r},t)$, *regardless* of the gauge chosen for the applied field.

In this paper we show that a whole class of such $\vec{p}\,'$ exists, each $\vec{p}\,'$ related to the usual $\vec{p}$ by a canonical transformation. Any one of these transformations leads to both a canonical momentum $\vec{p}\,'$ and angular momentum $\vec{J}\,'$ which do not depend explicitly on the potentials, but only on the electromagnetic fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$. In addition, this is also true for the new Hamiltonian H'. Thus we can present for the first time a canonical formulation of the motion of a charged particle in a time-dependent applied field involving only the applied electromagnetic fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ and which is hence manifestly gauge invariant.

Of course, since the new quantities $\vec{p}\,'$, $\vec{J}\,'$, and H' are related to the old ones by a canonical transformation no "new results" are found: The new canonical equations still lead to the Lorentz equation of motion

$$m\ddot{\vec{q}}=e\vec{E}(\vec{q},t)+\frac{e}{c}\dot{\vec{q}}\times\vec{B}(\vec{q},t)\ , \tag{1.7}$$

and the correspondence of canonical transformations in classical mechanics to unitary transformations in quantum mechanics guarantees the usual equivalence of H and H' if the particle is treated quantum mechanically.[9] Since the new quantities $\vec{p}\,'$, $\vec{J}\,'$, and H' reduce to the old quantities $\vec{p}$, $\vec{J}$, and H, if a Coulomb gauge is adopted, this equivalence is explicitly confirmed by noting that all measurable quantities are invariant under gauge transformation. Nonetheless, it is interesting to note that a potential-free canonical formulation of the motion of a charged particle in an applied field is possible. This explicitly confirms that reference to the vector potential is not essential in understanding the Aharonov-Bohm effect: in a discussion of the effect using H', which is just as valid a Hamiltonian as the usual H,

$$H=\frac{1}{2m}\left[\vec{p}-\frac{e}{c}\vec{A}(\vec{q},t)\right]^2+e\phi(\vec{q},t)\ , \tag{1.8}$$

only the fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ appear. Further, we extend the idea of Trammel[5] and Peshkin[6] in being able to identify in greater generality the interaction terms in the canonical momentum, angular momentum, and Hamiltonian as resulting from the field of the charge in the presence of the applied field. This identification and its limitations, along with suggestions for future work along these lines, are given in Sec. IV. We begin in Sec. II by deriving perhaps the simplest Lagrangian, for a charged particle in an applied field that depends on $\vec{E}$ and $\vec{B}$ directly. The corresponding Hamiltonian H' is found in Sec. III, where we also present the canonical transformation that leads directly from H to H'. A relativistic generalization is given in Sec. IV.

II. THE LAGRANGIAN

The equation of motion (1.7) may be derived by requiring the action S to be stationary $\delta S=0$, where

$$S=\int_{t_1}^{t_2}L\ dt\ , \tag{2.1}$$

and where the usual Lagrangian L is given by[10]

$$\begin{aligned}L&=L_0+\frac{e}{c}\vec{A}(\vec{q},t)\cdot\dot{\vec{q}}-e\phi(\vec{q},t),\\&=L_0+\frac{1}{c}\int\vec{j}(\vec{r})\cdot\vec{A}(\vec{r},t)d\vec{r}-\int\rho(\vec{r})\phi(\vec{r},t)d\vec{r}\ .\end{aligned} \tag{2.2}$$

Here

$$L_0=\tfrac{1}{2}m\dot{\vec{q}}^{\,2} \tag{2.3}$$

is the free-particle Lagrangian, and the charge and current densities are given by

$$\begin{aligned}&\rho(\vec{r})=e\delta(\vec{r}-\vec{q})\ ,\\&\vec{j}(\vec{r})=e\dot{\vec{q}}\delta(\vec{r}-\vec{q}),\end{aligned} \tag{2.4}$$

where we do not explicitly indicate the dependence of ρ and $\vec{j}$ on $\vec{q}$ and $\dot{\vec{q}}$. Now the variation of S does not involve the endpoints of the integral (2.1), so a Lagrangian which differs from L by a total time derivative,

$$L'=L+\dot{G} \tag{2.5}$$

will lead to the same equation of motion: if "polarization and magnetization potentials" $\vec{P}$ and $\vec{M}$ can be found such that

$$\begin{aligned}&\vec{j}(\vec{r})=\dot{\vec{P}}(\vec{r})+c\,\vec{\nabla}\times\vec{M}(\vec{r}),\\&\rho(\vec{r})=-\vec{\nabla}\cdot\vec{P}(\vec{r}),\end{aligned} \tag{2.6}$$

where the dependence of $\vec{P}$ and $\vec{M}$ on $\vec{q}$ and $\dot{\vec{q}}$ is not explicitly indicated, a choice of

$$G=-\frac{1}{c}\int\vec{P}(\vec{r})\cdot\vec{A}(\vec{r},t)d\vec{r} \tag{2.7}$$

in Eq. (2.5) leads to

$$\begin{aligned}L'=L_0&+\int\vec{P}(\vec{r})\cdot\vec{E}(\vec{r},t)d\vec{r}\\&+\int\vec{M}(\vec{r})\cdot\vec{B}(\vec{r},t)d\vec{r}\ ,\end{aligned} \tag{2.8}$$

a Lagrangian involving $\vec{E}$ and $\vec{B}$ rather than ϕ and $\vec{A}$. In obtaining Eq. (2.8) from Eqs. (2.5)–(2.7) we have performed partial integrations, assuming that (ϕ, A) vanish sufficiently rapidly as $|\vec{r}| \to \infty$. We have also used

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\dot{\vec{A}},\quad \vec{B} = \vec{\nabla}\times\vec{A}, \tag{2.9}$$

and therefore in general the Lagrangian (2.8) will not lead to the Lorentz equation (1.7) for arbitrary functions $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$, but only for functions satisfying

$$c\vec{\nabla}\times\vec{E} + \dot{\vec{B}} = 0\,,\quad \vec{\nabla}\cdot\vec{B} = 0, \tag{2.10}$$

and corresponding to physically possible applied fields.

The use of functions of the form (2.7) for the derivation of new Lagrangians was introduced by Goeppert-Mayer,[11] in the course of developing a multipole Hamiltonian for applications in molecular physics (for recent work, see Refs. 12–17); for a neutral molecule, $\vec{P}$ and $\vec{M}$ may be expanded in series involving the Dirac delta function and its derivatives, with coefficients that are identified as the electric and magnetic multipole moments of the charge distribution. In our problem, however, we seek a set $(\vec{P},\vec{M})$ satisfying Eq. (2.6) for the point-charge-current density (2.4). To find such a set, it is useful to note that while only the exact electric and magnetic fields generated by the charge (plus an arbitrary freely propagating field) satisfy all the Maxwell equations

$$\vec{\nabla}\cdot\vec{e} = 4\pi\rho\,, \tag{2.11a}$$

$$c\vec{\nabla}\times\vec{b} - \dot{\vec{e}} = 4\pi\vec{j}\,, \tag{2.11b}$$

$$\vec{\nabla}\cdot\vec{b} = 0\,, \tag{2.11c}$$

$$c\vec{\nabla}\times\vec{e} + \dot{\vec{b}} = 0\,, \tag{2.11d}$$

with Eq. (2.4), *many* sets of fields $(\vec{e},\vec{b})$ satisfy Eqs. (2.4), (2.11a), and (2.11b) alone. In particular, any $\vec{e} = \vec{e}(\vec{r}-\vec{q})$ satisfying Eqs. (2.4) and (2.11a) and a $\vec{b}$ given by

$$\vec{b} = \frac{1}{c}\dot{\vec{q}}\times\vec{e}\,, \tag{2.12}$$

together satisfy Eqs. (2.4), (2.11a), and (2.11b). Thus if we take $\vec{\mathscr{E}}(\vec{r}-\vec{q})$ given by Eq. (1.3) and

$$\vec{\mathscr{B}}(\vec{r}-\vec{q};\dot{\vec{q}}) \equiv \frac{1}{c}\dot{\vec{q}}\times\vec{\mathscr{E}}(\vec{r}-\vec{q})\,, \tag{2.13}$$

we find

$$\vec{\nabla}\cdot\vec{\mathscr{E}}(\vec{r}-\vec{q}) = 4\pi\rho\,,\quad c\vec{\nabla}\times\vec{\mathscr{B}}(\vec{r}-\vec{q};\dot{\vec{q}}) - \dot{\vec{\mathscr{E}}}(\vec{r}-\vec{q}) = 4\pi\vec{j} \tag{2.14}$$

with $(\rho,\vec{j})$ given by Eq. (2.4); $\vec{\mathscr{E}}$, and $\vec{\mathscr{B}}$ are, of course, the electric and magnetic fields that would result, in the zeroth and first order of $|\dot{\vec{q}}|/c$, respectively, from a charge moving with a velocity $\dot{\vec{q}}$. However, the terms defined by Eqs. (1.3), (2.4), and (2.13) satisfy Eq. (2.14) exactly, for any $\vec{q}(t)$, regardless of the velocity or acceleration of the charge.

Comparing Eqs. (2.6) and (2.14) we see that we should choose

$$\vec{P} = -\frac{1}{4\pi}\vec{\mathscr{E}}\,,\quad \vec{M} = \frac{1}{4\pi}\vec{\mathscr{B}}\,, \tag{2.15}$$

in Eq. (2.8); we then find a Lagrangian

$$L' = L_0 - \frac{1}{4\pi}\int \vec{\mathscr{E}}(\vec{r}-\vec{q})\cdot\vec{E}(\vec{r},t)d\vec{r} + \frac{1}{4\pi}\int \vec{\mathscr{B}}(\vec{r}-\vec{q};\dot{\vec{q}})\cdot\vec{B}(\vec{r},t)d\vec{r}\,, \tag{2.16}$$

involving only the electric and magnetic fields, and not the potentials. Using Eqs. (2.14) and the Maxwell equations (2.10) for the applied field, along with the assumption that $\vec{E}$ and $\vec{B}$ vanish sufficiently rapidly as $|\vec{r}| \to \infty$, the Lorentz equation of motion (1.7) can be recovered from Lagrange's equation

$$\frac{d}{dt}\left[\frac{\partial L'}{\partial \dot{\vec{q}}}\right] - \frac{\partial L'}{\partial \vec{q}} = 0\,, \tag{2.17}$$

with L' given by Eq. (2.16). The last-mentioned assumption [previously used, see comment after Eq. (2.8)] is crucial, since $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$ only drop off with increasing distance as $|\vec{r}-\vec{q}|^{-2}$; $\vec{E}$ and $\vec{B}$ must drop off fast enough that the integrals in Eq. (2.16) are well defined. As an example, we cannot use Eq. (2.16) for a charge in an electric field which is supposed to be uniform over all of space, for the first integral is then poorly defined; however, that equation *can* be used for a charge inside a parallel-plate capacitor finite in at least one direction, regardless of the size of the capacitor.

III. THE HAMILTONIAN

Since the Lagrangian (2.16) leads to the desired equation of motion (1.7), we may obtain the desired Hamiltonian from it in the usual fashion; we have

$$\vec{p}\,' \equiv \frac{\partial L'}{\partial \dot{\vec{q}}} = m\dot{\vec{q}} + \frac{1}{4\pi c}\int \vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}\,, \tag{3.1}$$

and we find

$$H' = \vec{p}\,'\cdot\dot{\vec{q}} - L' = T + \frac{1}{4\pi}\int \vec{\mathscr{E}}(\vec{r}-\vec{q})\cdot\vec{E}(\vec{r},t)d\vec{r}\,, \tag{3.2}$$

where

$$T = \frac{1}{2m}\left[\vec{p}\,' - \frac{1}{4\pi c}\int \vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}\right]^2 \tag{3.3}$$

is numerically equal to the kinetic energy of the particle, as is clear from Eq. (3.1). As expected, the new Hamiltonian (3.2) involves only the applied electric and magnetic fields, and not their potentials.

The Hamiltonians H' and H [Eq. (1.8)] may easily be compared by noting that H' can be obtained *directly* from H by a canonical transformation. Recall that in general, given a Hamiltonian $H(\vec{p},\vec{q},t)$, new coordinates and momenta $(\vec{q}\,',\vec{p}\,')$ and a new Hamiltonian $H'(\vec{p}\,',\vec{q}\,',t)$ may be obtained from a generating function $F(\vec{q},\vec{p}\,',t)$ through the equations[10]

$$\vec{p} = \frac{\partial F}{\partial \vec{q}}, \quad \vec{q}\,' = \frac{\partial F}{\partial \vec{p}\,'}\,, \tag{3.4}$$

and

$$H' = H + \frac{\partial F}{\partial t}\,. \tag{3.5}$$

In our instance we choose

$$F = \vec{q}\cdot\vec{p}\,' - \frac{1}{4\pi c}\int \vec{\mathscr{E}}(\vec{r}-\vec{q})\cdot\vec{A}(\vec{r},t)d\vec{r}\,, \tag{3.6}$$

and Eqs. (3.4) yield

$$\begin{aligned} \vec{q}\,' &= \vec{q}\,, \\ \vec{p}\,' &= \vec{p} - \frac{e}{c}\vec{A}(\vec{q},t) + \frac{1}{4\pi c}\int \vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}\,, \end{aligned} \tag{3.7}$$

while Eq. (3.5) gives

$$H' = H - \frac{1}{4\pi c}\int \vec{\mathscr{E}}(\vec{r}-q)\cdot\dot{\vec{A}}(\vec{r},t)d\vec{r}\,. \tag{3.8}$$

Using Eqs. (1.8), (2.9) and (3.7) in Eqs. (3.8) we recover Eqs. (3.2) and (3.3). We note that the numerical value of $\vec{p}\,'-\vec{p}$ depends on the choice of gauge in H, as does that of $H'-H$. Using the third equation of (1.6) in Eq. (3.8), and performing a partial integration, we see that H' and H are numerically equal if the Coulomb gauge, or any gauge with $\dot{\vec{A}}=0$, is chosen in H. Only in the case of Coulomb gauge will $\vec{p}$ and $\vec{p}\,'$ be identical, as is clear from comparing Eqs. (1.5) and (3.1); the same is true for the canonical angular momenta $\vec{J} = \vec{q}\times\vec{p}$ and

$$\begin{aligned} \vec{J}\,' &= \vec{q}\times\vec{p}\,' \\ &= \vec{q}\times m\dot{\vec{q}} + \frac{1}{4\pi c}\int \vec{r}\times[\vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)]d\vec{r}\,, \end{aligned} \tag{3.9}$$

where the second line of Eq. (3.9) follows from Eq. (3.1) and the fact that

$$0 = \int (\vec{r}-\vec{q})\times[\vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)]d\vec{r}\,. \tag{3.10}$$

Equation (3.10) may be verified by using Eq. (1.3), performing a partial integration, and noting that $\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0$.

IV. CONCLUDING REMARKS

We first summarize our main results: For an applied electromagnetic field $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ satisfying Eq. (2.10), the canonical equations

$$\dot{\vec{q}} = \frac{\partial H'}{\partial \vec{p}\,'}, \quad \dot{\vec{p}}\,' = -\frac{\partial H'}{\partial \vec{q}}\,, \tag{4.1}$$

when used with the Hamiltonian

$$H'(\vec{p}\,',\vec{q},t) = \frac{1}{2m}\left[\vec{p}\,' - \frac{1}{4\pi c}\int \vec{\mathscr{E}}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}\right]^2 + \frac{1}{4\pi}\int \vec{\mathscr{E}}(\vec{r}-\vec{q})\cdot\vec{E}(\vec{r},t)d\vec{r}\,, \tag{4.2}$$

where $\vec{\mathscr{E}}$ is given by Eq. (1.3), lead to the Lorentz equation of motion (1.7). Equations (4.1) and (4.2) thus constitute a potential-free, canonical formulation of the dynamics of a charged particle in an arbitrarily time-dependent applied electromagnetic field. The only assumption is that $\vec{E}$ and $\vec{B}$ are sufficiently well behaved at infinity so that the surface integrals which result when Eq. (4.1) is applied with Eq. (4.2), and the necessary partial integrations are performed, vanish. These conditions *include* such extended fields as those present in a parallel-plate capacitor (infinite in two directions), and in the infinite solenoid appearing in discussions of the Aharonov-Bohm effect.[1–4] Subtleties arising in passing to the limit of such geometries are discussed by, e.g., Peshkin.[6]

The L' [Eq. (2.16)] and H' [Eq. (4.2)] given here are not the only potential-free Lagrangian and Hamiltonian that can be constructed, since Eqs. (2.6) do not uniquely determine $\vec{P}$ and $\vec{M}$. In fact, a whole class of such Lagrangians and Hamiltonians exists, since if fields $(\vec{P},\vec{M})$ satisfy Eqs. (2.6), then fields $(\vec{P}',\vec{M}')$ given by

$$\vec{P}'=\vec{P}+\vec{\nabla}\times\vec{g}\ ,\qquad \vec{M}'=\vec{M}-\frac{1}{c}\dot{\vec{g}}+\vec{\nabla}h\ , \tag{4.3}$$

where $\vec{g}$ is any vector function and h any scalar function, also satisfy Eqs. (2.6). Thus the gauge freedom of $(\phi,\vec{A})$ in Eqs. (2.2) and (1.8) has in some sense been replaced by the gauge freedom of $(\vec{P},\vec{M})$ in Eqs. (2.16) and (4.2).

For the Hamiltonian H' of Eq. (4.2), let us separate the "mechanical" and "interaction" contributions to $\vec{p}\,'$, $\vec{J}\,'$, and H'. Setting

$$\vec{p}\,'_{\text{mech}}=m\dot{\vec{q}}\ ,\qquad \vec{J}\,'_{\text{mech}}=\vec{q}\times m\dot{\vec{q}}\ ,\qquad H'_{\text{mech}}=\tfrac{1}{2}m\dot{\vec{q}}^{\,2}\ , \tag{4.4}$$

from Eqs. (3.1), (3.2), and (3.9) we have

$$\vec{p}\,'-\vec{p}\,'_{\text{mech}}=\frac{1}{4\pi c}\int\mathscr{E}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}\ ,$$
$$\vec{J}\,'-\vec{J}\,'_{\text{mech}}=\frac{1}{4\pi c}\int\vec{r}\times[\mathscr{E}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)]d\vec{r}\ , \tag{4.5}$$
$$H'-H'_{\text{mech}}=\frac{1}{4\pi}\int\mathscr{E}(\vec{r}-\vec{q})\cdot\vec{E}(\vec{r},t)d\vec{r}\ .$$

All the "interaction" terms on the right-hand sides of Eqs. (4.5) can be interpreted physically, in the limit $|\dot{\vec{q}}|/c\ll 1$, as discussed in Sec. I: They may be considered due to the usual momentum, angular momentum, and energy densities arising from the field of the particle in the presence of the applied field. It would be interesting if a canonical formulation could be found in which this physical interpretation would hold for arbitrary $\dot{\vec{q}}$, but this does not seem possible, at least in a simple way. First, note that in our formulation the form of the interaction terms (4.5) do not arise simply because we started from a nonrelativistic Lagrangian (2.2), (2.3); if in place of L_0 we substituted

$$L_{0r}=-mc^2(1-\dot{\vec{q}}^{\,2}c^{-2})^{1/2} \tag{4.6}$$

in Eq. (2.2) and carried on the derivation of Secs. II and III as before, we would arrive at a Lagrangian

$$L_r'=L_{0r}-\frac{1}{4\pi}\int\mathscr{E}(\vec{r}-\vec{q})\cdot\vec{E}(\vec{r},t)d\vec{r}+\frac{1}{4\pi}\int\mathscr{B}(\vec{r}-\vec{q};\dot{\vec{q}})\cdot\vec{B}(\vec{r},t)d\vec{r}\ , \tag{4.7}$$

and a Hamiltonian

$$H_r'=\left[m^2c^4+c^2\left[\vec{p}\,'-\frac{1}{4\pi c}\int\mathscr{E}(\vec{r}-\vec{q})\times\vec{B}(\vec{r},t)d\vec{r}\right]^2\right]^{1/2}+\frac{1}{4\pi}\int\mathscr{E}(\vec{r}-\vec{q})\cdot\vec{E}(\vec{r},t)d\vec{r}\ . \tag{4.8}$$

Both of these lead to the *exact* relativistic equation of motion for a charged particle in an applied electromagnetic field; nonetheless, we still find Eqs. (4.5), with only Eqs. (4.4) replaced by their corresponding relativistic counterparts. Second, since the electromagnetic field from a moving charge depends, if the $|\dot{\vec{q}}|/c\ll 1$ limit is removed, not only on $\vec{q}$ and $\dot{\vec{q}}$ but also on $\ddot{\vec{q}}$, and at retarded times, it is hard to see how this exact field could be made, by even a clever choice of G [Eq. (2.5)], to appear in a Lagrangian of the usual kind depending only on $\vec{q}$, $\dot{\vec{q}}$, and t. The possibility of such a formulation also seems unlikely on physical grounds, since the exact field from a moving charge contains terms associated with the radiation reaction of the charge on itself; this effect clearly takes us beyond the simple model of a charge moving in a specified electromagnetic field.

Despite the difficulty of interpreting physically the interaction terms of Eqs. (4.5), we stress that the Hamiltonian (4.8), which is free of potentials and involves the electric and magnetic fields, does lead to the correct relativistic equation of motion for a charged particle in an applied electromagnetic field; such a Hamiltonian has not been written down before. The fact that the gauge-independent interaction terms (4.5) can be interpreted physically, although only in the limit $|\dot{\vec{q}}|/c\ll 1$, is interesting, since it establishes a connection between canonical expressions for momentum, angular momentum, and energy with the expressions one would write down for those terms from usual electromagentic theory, interpreting the interaction as between the field of the particle and the applied field. It would be interesting to see how far in what direction this connection could be extended.

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