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Direct observation of crises of the chaotic attractor in a nonlinear oscillator

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For a driven nonlinear oscillator we report direct evidence for three cases of an interior crisis of the attractor, as conjectured by Grebogi, Ott, and Yorke. These crises are sudden and discontinuous changes in the attractor, observed directly from bifurcation diagrams and attractor diagrams (Poincaré sections) in real time. The crises arise from intersection of an unstable orbit with the chaotic attractor.

As a nonlinear dissipative system is driven harder, it may follow some recognizable route to chaotic or turbulent behavior.[1] The chaos is belived to be a consequence of a chaotic, or "strange," attractor: a surface in the phase space of the system to which orbits are attracted and on which they move pseudorandomly.[2] Grebogi, Ott, and Yorke[3] have investigated theoretically sudden qualitative changes, called *crises,* of the attractor, using a one-dimensional quadratic map, which has been experimentally shown to model chaos surprisingly well in many physical systems (see Ref. 3). An interesting question is: how does the attractor, which controls the chaotic dynamics, depend on the driving parameter of the map? They conclude that crises can occur at parameter values for which the chaotic attractor is intersected by a coexisting unstable periodic orbit. In this paper we report novel experimental evidence for three types of such crises in a driven nonlinear semiconductor oscillator.

The physical system used is a series connected inductance L, resistance R, and diode C (type 1N953) driven by an external voltage $V_0(t) = |V_0| \times \cos(2\pi t/T)$, of period $T = 12.5$ μsec. This resonant LRC system is described by $LI + RI + V_c(t)$ and two additional differential equations in the diode voltage $V_c(t)$ and series current $I(t)$ describing the nonlinear capacitance and switching characteristics of the diode. Previous experiments[4] showed that this system follows the universal period-doubling route to chaos,[5,6] with bifurcation diagram, universal numbers, and window sequences and patterns[7] explicable by the logistic quadratic map

$$x_{n+1} = \lambda x_n (1 - x_n) \quad , \tag{1}$$

with the correspondences $|V_c| \leftrightarrow x$, $|V_0| \leftrightarrow \lambda$ between experimental quantities and the map. Effects of additive noise[8] and the observation of an intermittency route to chaos[9] can also be understood by theoretical models using Eq. (1). However, higher-dimensional effects, which should vanish as $R \to \infty$, are observable for finite resistance: (1) The system shows hysteresis in the period-3 window (see below). (2) The Poincaré section, or attractor diagram, shows a Henon-like structure,[10] characteristic of a two-dimensional system (see below). (3) The return map shows a quadratic maximum with some folding.[9] We believe the system is well characterized and interesting for the experimental study of crises. Some novel results are presented below.

Crises following the period-3 window. The bifurcation diagram of Fig. 1 plots the iterated value $\{x_n\}$ vs λ, computed from Eq. (1); it shows a period-doubling cascade and onset of the chaotic regime for $\lambda > \lambda_c$. Grebogi *et al.* show that a crisis occurs at points near C, Fig. 1, where the period-3 unstable orbits intersect the three-band chaotic attractor. Figure 2 is an oscillogram of the upper part of a bifurcation diagram (generated from the diode voltage V_c by a scanning window comparator) as the driving voltage V_0 is increased through the value V_3 for the onset of the window at point A, to the value V_{*3} at point B, where the crisis occurs: V_c suddenly takes on a set of values (white dots) between the semiperiodic bands shown. It can be seen that point B is also the con-

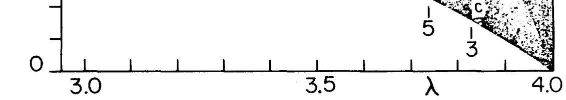

FIG. 1. Attractor $\{x_n\}$ vs λ computed from Eq. (1), showing onset of chaos at λ_c, period-5 and period-3 windows. A crisis occurs at points near C (Ref. 3), and also at points near A and B where an unstable orbit (dashed line) intersects the chaotic attractor.

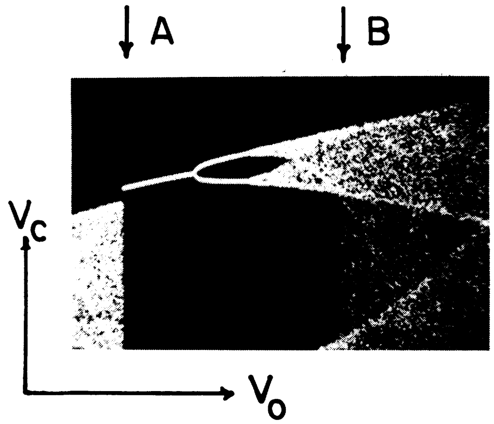

FIG. 2. Upper section of bifurcation diagram observed for nonlinear oscillator. At point A the driving voltage $V_0 = V_3$ has the threshold value for the period-3 window. At point B, $V_0 = V_{*3}$ and a crisis of the attractor occurs (cf. Ref. 3, Fig. 2).

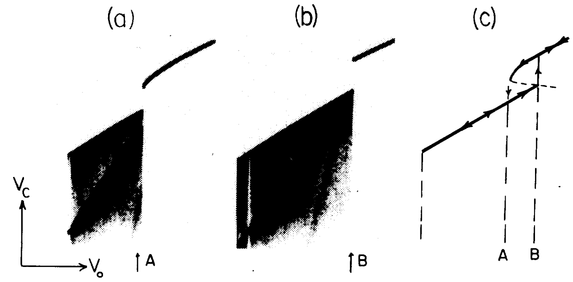

FIG. 4. Reversed oscillogram of bifurcation diagram observed for nonlinear oscillator at period-3 window. (a) With driving voltage V_0 decreasing; there is a sudden jump down at $V_0 = A$. (b) With V_0 increasing; there is a sudden jump up at $V_0 = B$. (c) Scale drawing overlay of envelopes of (a) and (b) together with an unstable orbit (dashed line) intersecting the chaotic attractor—this gives rise to the hysteresis. Experimental values are $A = 3.1054$ V rms and $B = 3.1469$ V rms for the system used.

junction of two "veils," i.e., high iterates of the critical point of the map. We note the close resemblance between our experiment, Fig. 2, and the computed behavior (Ref. 3, Fig. 2), including the veils. Figure 3 is an oscillogram of the diode current $I(t)$ vs $V_c(t)$ with the oscilloscope intensity strobed by short pulses at period T, synchronous with $V_0(t)$. If we make the correspondences $I(t) \leftrightarrow \dot{q}$, $V_c(t) \leftrightarrow q$, then this figure is a real-time Poincaré section in the phase space $(\dot{q},q)$ of the system. Figure 3(a), for $V_3 < V_0 < V_{*3}$, shows a three-band chaotic attractor; in Fig. 3(b), for $V_0 = V_{*3}$, the crisis occurs: the attractor suddenly becomes a one-band attractor, with increased transverse width. The threshold voltage V_0 for this was observed to be exactly the same as for point B, Fig. 2. The shape of the attractor is found to be Henon-like,[10] except for the vertical section at small values of V_c where the diode voltage is clamped under forward conduction.

Hysteresis crisis of period-3 window. The hysteresis at point A in Fig. 2 is shown under high resolution in Fig. 4(a) for V_0 decreasing, and in Fig. 4(b) for V_0 increasing. A hysteresis is clearly displayed in the composite drawing, Fig. 4(c). We note the parabolic shape of the (stable) periodic attractor in Fig. 4(a), which is redrawn in Fig. 4(c), together with the associated unstable orbit (dashed line). We explain the hysteresis crises by the intersection of this unstable period-3 orbit with the chaotic attractor. We note the veils in Fig. 4 do not show hysteresis, nor is it expected. Although hysteresis does not occur for Eq. (1), it is predicted by Huberman and Crutchfield[11] in an integration of the second-order differential equation of a driven nonlinear oscillator. It is also predicted by Gibson[12] for the Henon map. Figure 5 is an oscillogram of the attractor diagram observed under the conditions of Fig. 4. The three dots are stable period-3 fixed point attractors for $V_0 > V_3$. As

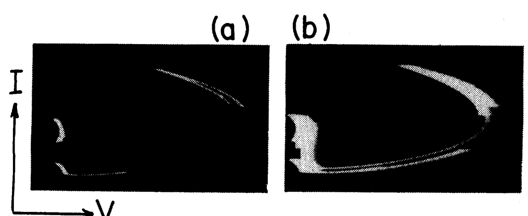

FIG. 3. Oscillogram of attractor diagram (Poincaré section) observed for nonlinear oscillator. (a) $V_3 < V_0 < V_{*3}$, showing three-band attractor. (b) $V_0 = V_{*3}$, showing onset of crisis: sudden change to one-band attractor.

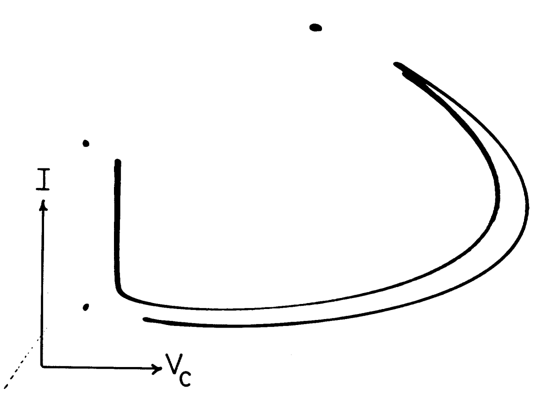

FIG. 5. Reversed oscillogram of two attractor diagrams observed for nonlinear oscillator at period-3 window: three dots, for $V_0 > V_3$, stable fixed point attractors; solid line, for $V_0 < V_3$, in the chaotic region. For $V_0 = V_3$, the system jumps discontinuously between the two diagrams in a hysteresis crisis.

V_0 is reduced below window threshold, the attactor jumps discontinuously to the Henon-like solid lines, which do not lie on the fixed points but extrapolate to them. This jump in size and shape of the attractor is an interior crisis.[3]

Crisis at 4⟷2 band merging. Figure 1 shows unstable period-2 orbits intersecting the chaotic attractor at points near A and B, were there is $4 \rightarrow 2$ band merging. This should give rise to a crisis, although, like hysteresis, it does not seen to be predicted by the logistic map. However, the observed attractor diagram, Fig. 6, clearly shows a sudden change as $V_0 \rightarrow V_{0m}$, the voltage at band merging. Figure 6(a) for $V_0 < V_{0m}$ shows a four-band chaotic attractor imbedded in the fixed points of a period-24 window. Figures 6(b) and 6(c), as V_0 is increased, show a transverse expansion (i.e., along V_c) of the attractor near band merge. In Fig. 6(d), $V_0 = V_{0m}$, 4 bands $\rightarrow$ 2 bands, and there is a crisis: observation of the oscilloscope in real time shows a fountainlike transverse *motion* along the V_c axis of successive crossings of the orbit through the Poincaré section; we call this a transverse explosion. We believe this to be an interior crisis of the attractor.

To summarize, directly from bifurcation and attractor diagrams for a real physical sytsem in chaos, we present novel results for three cases of an interior crisis of the strange attractor: a sudden and discontinuous change as defined by Grebogi *et al.*[3] We also have experimental evidence for crises arising from two attractors with separate basins of attraction.[13] Our experiments support the conjecture of Gregogi *et al.* that these crises arise from the intersection of a coexisting unstable orbit with the chaotic attractor.

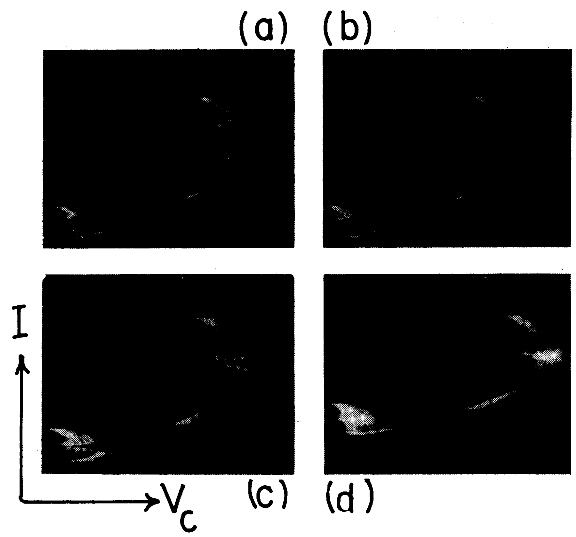

FIG. 6. Oscillograms of attractor diagrams observed for nonlinear oscillator as $V_0 \rightarrow V_{0m}$ for $4 \rightarrow 2$ band merge. (a) $V_0 < V_{0m}$. (b) $V_0 < V_{0m}$. (c) $V_0 \lesssim V_{0m}$. (d) $V_0 = V_{0m}$, a crisis (transvere explosion) of the attractor occurs.

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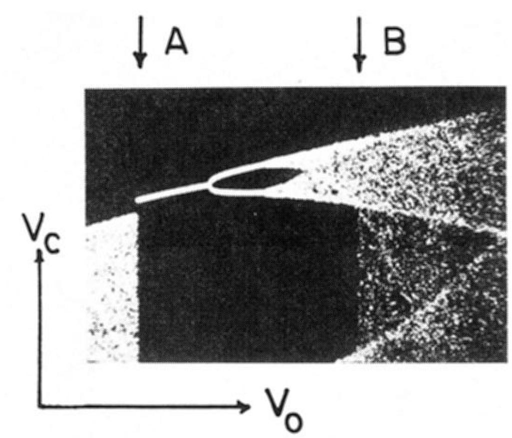

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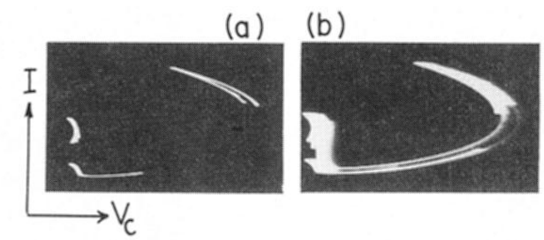

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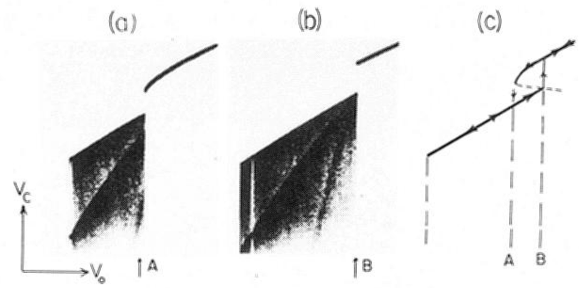

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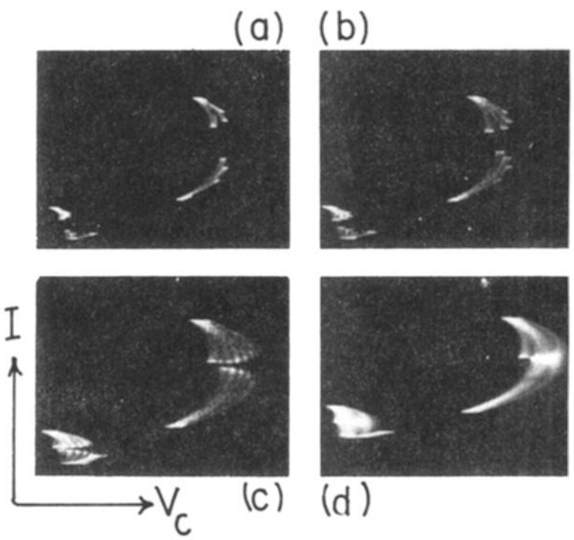

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