Amplitude equation near a polycritical point for the convective instability of a binary fluid mixture in a porous medium

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An amplitude equation is derived for a binary fluid mixture in a porous medium, in the vicinity of the intersection point of the lines of stationary and oscillatory instabilities. This point represents an experimentally realizable example of a codimension-two bifurcation.
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The investigation of bifurcation phenomena in fluid dynamics has attracted increasing attention during the last decade.[1–3] For various systems in condensed matter physics (such as, e.g., simple fluids,[1] fluid mixtures,[4] or liquid crystals[5]) subjected to external temperature gradients, the first instability which occurs might be either stationary or oscillatory. The behavior of these systems near onset may often be exactly described by amplitude equations,[6] which are differential equations for the amplitudes of the critical modes at the instability. A natural problem which arises in this context is the derivation of an amplitude equation near the point in parameter space where the lines of stationary and oscillatory instabilities intersect.

The purpose of the present paper is to derive such an equation for the convective instability of a binary fluid mixture in a porous medium.[7,8] We thus find an experimentally realizable example of a codimension-two bifurcation, a phenomenon which has attracted recent attention in the mathematical literature.[9–11]

The nonlinear equations for the deviations from the heat-conducting state may be written in the usual dimensionless units,[7] as

$$[(K\kappa/l^2\epsilon\nu)\partial_t+1]\Delta w-\Delta_2\theta-\Psi\Delta_2 c=0 \ ,$$
$$Rw+(-\partial_t-\vec{v}\cdot\vec{\nabla}+\Delta)\theta=0 \ , \qquad (1)$$
$$Rw-(D/\kappa)\Delta\theta+[(D/\kappa)\Delta-\partial_t-\vec{v}\cdot\vec{\nabla}]c=0 \ ,$$

where w is the z component of the velocity $\vec{v}$, R is the Rayleigh number, Ψ is the separation ratio, κ and D are the thermodiffusivity and the diffusion coefficient, respectively, and $K\kappa/l^2\epsilon\nu$, which contains the permeability K, the porosity ϵ, the kinematic viscosity ν, and the height of the layer l, is a coefficient which turns out to be very small, and will be neglected in what follows. The second and third equations above are the dynamic equations for temperature θ and concentration field c and have the same form as in a binary fluid mixture.[12] The velocity field of a porous mixture is no longer a conserved quantity; thus the equation for the averaged velocity $\vec{v}$ changes its structure: It is a differential equation which has no second-order spatial derivatives characteristic of diffusion processes (cf. Ref. 13 for a detailed discussion). To arrive at the first of the Eqs. (1) we have eliminated the pressure making use of the incompressibility condition.

The boundary conditions for the velocity field $\vec{v}=(u,v,w)$ for a rectangular box with sides L_x, L_y, $L_z=l$ are

$$u=0 \ \text{ for } x=0,L_x \ ,$$
$$v=0 \ \text{ for } y=0,L_y \ , \qquad (2)$$
$$w=0 \ \text{ for } z=0,L_z \ .$$

For the temperature and concentration fluctuations we assume

$$\partial_x\theta=\partial_x c=0 \ \text{ for } x=0,L_x \ , \qquad (3a)$$
$$\partial_y\theta=\partial_y c=0 \ \text{ for } y=0,L_y \ , \qquad (3b)$$
$$\theta=c=0 \ \text{ for } z=0,L_z \ . \qquad (3c)$$

Inserting the ansatz

$$\begin{pmatrix} w\\ \theta\\ c\end{pmatrix}=\begin{pmatrix} W(t)\\ \Theta(t)\\ C(t)\end{pmatrix}\sin\frac{\pi z}{L_z}\cos\frac{\pi x}{L_x} \ , \qquad (4)$$

the linearized equations are solved by setting $W(t) = W_0\exp\sigma t$, etc., which leads to a cubic polynomial[7]

$$\gamma\sigma^3+\sigma^2-\alpha\sigma-\beta=0 \ . \qquad (5)$$

This equation allows for an oscillatory instability for $\beta<0$ when $\alpha\to 0$, which occurs at

$$R=R_{co}(\Psi)=4\pi^2(1+D/\kappa)(1+\Psi)^{-1} \ , \qquad (6)$$

and a stationary instabiltiy for $\alpha<0$ when $\beta\to 0$, which occurs at

$$R = R_{cs}(\Psi) = 4\pi^2(1 + \Psi + \Psi\kappa/D)^{-1} \quad . \tag{7}$$

At the "polycritical point" ($\alpha = \beta = 0$)

$$\psi = \psi_{pc} = -(1 + \kappa/D + \kappa^2/D^2)^{-1} \quad , \tag{8}$$

we have

$$R_{co} = R_{cs} = R_{pc} = 4\pi^2(1 + D/\kappa + D^2/\kappa^2) \quad , \tag{9}$$

and the two instability lines in the (R, ψ) plane intersect. It is easy to see that the first instability is stationary for $\psi > \psi_c$ and oscillatory for $\psi < \psi_c$, and that the oscillation frequency ω_0 vanishes at the polycritical point, $\alpha = \beta = 0$. Furthermore, it turns out[7] that near this point the third eigenvalue of the cubic polynomial is negative, so we may drop the cubic term for sufficiently small σ.

The nonlinear equation satisfied by the real function $W(t)$ near the polycritical point will thus have the form

$$\ddot{W} - \alpha\dot{W} - \beta W + f(W, \dot{W}) = 0 \quad , \tag{10}$$

where the unknown function f is to be determined. Taking into account the symmetry of the basic equations[14] we are left with the ansatz $f = f_1 W^3 + f_2 W^2 \dot{W}$, i.e., with two undetermined constants. To find these we can either perform a direct expansion of the system (1) around the polycritical point, or else we can match Eq. (10) to the previously derived[8] amplitude equations valid along the oscillatory and stationary instability lines separately. We have carried out both calculations[15] and find the same answer, which is given by

$$\ddot{W} - (\alpha - f_2 W^2)\dot{W} - (\beta + f_1 W^2) W = 0 \quad , \tag{11}$$

$$\alpha = 2\pi^2(1 + D/\kappa)[(R - R_{co})/R_{co}] \quad , \tag{12}$$

$$\beta = 4\pi^4(D/\kappa)[(R - R_{cs})/R_{cs}] \quad , \tag{13}$$

$$f_1 = \pi^2/4 \quad , \tag{14}$$

$$f_2 = (1 + \kappa/D)/4 \quad , \tag{15}$$

where $R_{co}(\psi)$ and $R_{cs}(\psi)$ are given in Eqs. (6) and (7).

Equation (10) is one of the standard forms for a codimension-two bifurcation.[9–11] It displays a stable fixed point in quadrant III of Fig. 1, a stationary (pitchfork) bifurcation for fixed $\alpha < 0$ when $\beta \to 0$ to quadrant IV, and an oscillatory (Hopf) bifurcation for fixed $\beta < 0$ when $\alpha \to 0$ to quadrant II. Because of the signs of f_1 and f_2 in Eq. (11), the stationary bifurcation is inverted (i.e., the fixed point in quadrant IV is unstable) and the oscillatory bifurcation is direct, (i.e., the limit cycle in II b is stable). It may be shown by a nonlinear analysis[16,17] that the limit cycle disappears along the line L_1 given by $\alpha = -(f_2/5f_1)\beta$, where the limit cycle has infinite

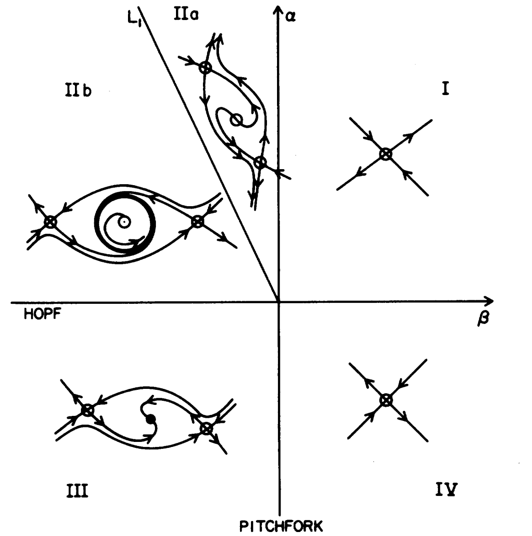

FIG. 1. Schematic phase portrait of the behavior of Eq. (11) in the vicinity of the polycritical point $\alpha = \beta = 0$. The parameter space is divided into sectors with different characteristic behavior. In each sector we show typical phase-space orbits and the stability of various attractors (the stable fixed point is shown as a solid circle, the unstable ones as open circles, and the stable limit cycle is a thick solid line; orbits are drawn as thin solid lines). Quadrant III has one stable and two unstable fixed points, while quadrants I and IV have one unstable fixed point, with an inverted stationary bifurcation along the line $\beta = 0$, $\alpha < 0$. On the line $\alpha = 0$, $\beta < 0$ the system makes a forward oscillatory bifurcation to a stable limit cycle, which disappears along the line L_1 where the oscillation period diverges. In sector II a there are three unstable fixed points joined by a heteroclinic orbit.

period and its orbit intersects the unstable fixed points. In the sectors II a, I, and IV, Eq. (11) only displays unstable fixed points (since $f_1 > 0$), so it would be necessary to go to higher order to have a complete description of the dynamics near the polycritical point.

The results presented here are not restricted to this system, since intersections of stationary and oscillatory bifurcation lines occur in the thermohaline problem[16,17] and in convection in magnetic fields[16] or under rotation.[1] Such intersections also occur in ordinary binary mixtures[4] (not in porous media), but the calculations for that case are more complicated, and have so far only been carried out for the unrealistic case of free-slip boundary conditions.[18] An important advantage of Soret driven instabilities in binary mixtures[4,7] over the thermohaline systems[16,17] is that in the former the polycritical point may be reached by varying physical parameters, such as the temperature and the concentration, independently. In the thermohaline case, on the other hand, the control parameters are temperature and salinity *gradients* and these cannot be varied independently in realistic experiments.

An obvious generalization of our work is the inclusion of slow spatial variation[6] and consideration of more realistic boundary conditions than used in Eq. (3). Note, however, that in contrast to other hydrodynamic instabilities, the rigid velocity boundary conditions (2) are easily satisfied in porous media. It is only our concentration boundary condition (3c) which is idealized. The proper boundary condition is $J=(k_T/T)\partial_z\theta+\partial_z c=0$, but its implementation requires numerical computations even for the linear problem.

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[14]For the present system, due to its up-down symmetry, the equations must be invariant under the replacement $W \leftrightarrow -W$, so that terms in $\dot{W}W$, W^2, or $\dot{W}^2$ will not appear. A similar symmetry occurs in nematics and in the thermohaline problem, but not in cholesterics. Terms of the form $\dot{W}^2 W$ or $\dot{W}^3$, which are allowed by symmetry, do not appear because the critical eigenspace of the linearized adjoint operator has dimension two [cf. H. Brand, P. C. Hohenberg, and V. Steinberg (unpublished)].
[15]H. Brand, P. C. Hohenberg, and V. Steinberg, Ref. 14.
[16]See, for example, E. Knobloch and M. R. E. Proctor, J. Fluid Mech. 108, 291 (1981). These authors analyze the thermohaline system in a region equivalent to II b of Fig. 1.
[17]P. H. Coullet and E. A. Spiegel (unpublished). These authors give a general discussion of systems near polycritical points, and work out the codimension-two example of thermohaline convection as a special case. Note that Eq. (5.106) of this paper is equivalent to Eq. (2.35) of Ref. 16, in sector II b of our Fig. 1.
[18]H. Brand, Ph.D. thesis (University of Essen, West Germany, 1981) (unpublished); (unpublished).