Classical statistics inherent in pure quantum states

A. K. Rajagopal
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803
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A new coherent state introduced recently by us is shown to lead to a Wigner distribution function which maximizes the Shannon entropy with given mean, variance, and covariance of position and momentum of a particle. This result is discussed in the light of a recent paper of Senitzky. A more general discussion of entropy based on smoothed Wigner functions is also given.

In a very interesting paper, Senitzky[1] has recently proposed to separate the statistics of pure quantum states of the harmonic oscillator into a quantum-mechanical and a classical part by associating a classical analog with each quantum state. Senitzky did not give any indication of how this classical probability is to be found. We here take the natural view point that a classical statistical description is given by a classical phase-space probability density. In this Report we construct this classical probability density by using Shannon's principle of maximum entropy.[2] The classical probability function $\mathcal{P}(\tilde{q},\tilde{p})$ has the normalization

$$\int\int_{-\infty}^{\infty} d\tilde{q}\, d\tilde{p}\, \mathcal{P}(\tilde{q},\tilde{p})=1 \tag{1}$$

and the associated entropy is

$$S=-\int\int_{-\infty}^{\infty}\mathcal{P}(\tilde{q},\tilde{p})\ln\mathcal{P}(\tilde{q},\tilde{p})\, d\tilde{q}\, d\tilde{p} \quad . \tag{2}$$

This is maximized subject to the constraints (using Senitzky's notation)

$$\begin{gathered}\langle\tilde{q}\rangle=q_{\rm cl},\ \ \langle\tilde{p}\rangle=p_{\rm cl},\ \ (\Delta\tilde{p})^2=\xi^2,\ \ (\Delta\tilde{q})^2=\eta^2\ ,\\ V(\tilde{q},\tilde{p})=\tfrac{1}{2}\langle\tilde{p}\,\tilde{q}+\tilde{q}\,\tilde{p}\rangle-\langle\tilde{p}\rangle\langle\tilde{q}\rangle=-\delta\ .\end{gathered} \tag{3}$$

We then compare this function with the Wigner distribution function associated with a new coherent state constructed by Rajagopal and Marshall.[3] This state has the property that (we use units with $\hbar=1$)

$$a|z\rangle=z|z\rangle\ ,\quad z\text{ complex} \tag{4}$$

with

$$\begin{aligned}a&=(\tfrac{1}{2})^{1/2}(\zeta_1\hat{q}+i\zeta_2\hat{p})\ ,\\ a^\dagger&=(\tfrac{1}{2})^{1/2}(\zeta_1^*\hat{q}-i\zeta_2^*\hat{p})\ ,\end{aligned} \tag{5}$$

$$\begin{aligned}&\langle\hat{q}\rangle=q_{\rm cl},\ \ \langle\hat{p}\rangle=p_{\rm cl},\ \ (\Delta\hat{p})^2=\tfrac{1}{2}|\zeta_1|^2\ ,\\ &(\Delta\hat{q})^2=\tfrac{1}{2}|\zeta_2|^2\ ,\\ &V(\hat{q},\hat{p})=-\tfrac{1}{2}(|\zeta_1|^2|\zeta_2|^2-1)^{1/2}\ ,\end{aligned} \tag{6}$$

where ζ_1 and ζ_2 are two complex numbers (may depend on time) introduced in the definition of destruction and creation operators, Eq. (5), with the condition

$$\zeta_1\zeta_2^*+\zeta_1^*\zeta_2=2 \tag{7}$$

to maintain the canonical commutation relations among the operators $\hat{q}$ and $\hat{p}$.

The derivation of $\mathcal{P}(\tilde{q},\tilde{p})$ is a straightforward exercise in variational calculus and we obtain

$$\mathcal{P}(\tilde{q},\tilde{p})=\frac{1}{2\pi(\xi^2\eta^2-\delta^2)^{1/2}}\exp\left[-\frac{1}{2(\xi^2\eta^2-\delta^2)}[\eta^2(\tilde{p}-p_{\rm cl})^2+\xi^2(\tilde{q}-q_{\rm cl})^2-2\delta(\tilde{p}-p_{\rm cl})(\tilde{q}-q_{\rm cl})]\right]\ . \tag{8}$$

On the other hand, the Wigner distribution function is a quasiprobability (not necessarily positive everywhere) distribution in phase space, which provides a formulation of quantum mechanics of systems described by their canonical coordinates and momenta.[4] The relation of this function to coherent states and many other properties have been investigated extensively in the literature.[5,6] The Wigner distribution function associated with the coherent state $|z\rangle$ is constructed as follows. First note[3]

$$\psi(x)\equiv\langle x|z\rangle=\left[\frac{1}{\pi|\zeta_2|^2}\right]^{1/4}\exp\left[-\frac{1}{2}\frac{\zeta_1}{\zeta_2}(x-q_{\rm cl})^2+ip_{\rm cl}(x-q_{\rm cl})\right] \tag{9}$$

and the Wigner function[4] associated with it is

$$f_W(Q,P)=\int_{-\infty}^{\infty}\frac{dx}{2\pi}e^{-ixP}\psi^*(Q-\tfrac{1}{2}x)\psi(Q+\tfrac{1}{2}x)\ , \tag{10}$$

where Q and P are c-number variables. This calculation is also straightforward and we obtain

$$f_W(Q,P)=\frac{1}{\pi}\exp[-|\zeta_2|^2(P-p_{\rm cl})^2-|\zeta_1|^2(Q-q_{\rm cl})^2-2(|\zeta_1|^2|\zeta_2|^2-1)^{1/2}(P-p_{\rm cl})(Q-q_{\rm cl})] \ . \tag{11}$$

Note that this function is positive, which is not the case in general for a Wigner function associated with an arbitrary state.

It is a simple exercise to verify that

$$\begin{aligned}&\int\int_{-\infty}^{\infty}dQ\,dP\,f_W(Q,P)=1 \ ,\\ &\langle\hat q\rangle=\int\int_{-\infty}^{\infty}dQ\,dP\,Qf_W(Q,P)=q_{\rm cl} \ , \text{ etc.}\end{aligned} \tag{12}$$

We now make the identification that

$$\begin{aligned}&\xi^2=\tfrac{1}{2}K_0^2|\zeta_1|^2,\quad \eta^2=\tfrac{1}{2}K_0^2|\zeta_2|^2 \ ,\\ &\delta=-\tfrac{1}{2}K_0^2(|\zeta_1|^2|\zeta_2|^2-1)^{1/2} \ ,\end{aligned} \tag{13}$$

and make the change of variables

$$\begin{aligned}&\tilde p-p_{\rm cl}=K_0(P-p_{\rm cl}) \ ,\\ &\tilde q-q_{\rm cl}=K_0(Q-q_{\rm cl}) \ ,\end{aligned} \tag{14}$$

to obtain a complete equivalence of the classical probability function, Eq. (8), with the quantum-mechanical Wigner distribution function, Eq. (11).

We now make several observations. The above analysis is general and has no explicit mention of the Hamiltonian or the system to which the above analysis applies. All that is demanded is that the system possess specified average, dispersion, and covariance of the coordinate and momentum. In Ref. 3, we have outlined a method of determining the equations obeyed by ζ_1, ζ_2 when the Hamiltonian is specified, by means of the quantum action principle. Senitzky[1] used three criteria and we have here used his first two, given in the present context, by Eq. (13), except that our K_0 is just a scale factor while Senitzky gives it as a special value, $K^2=1-[(\Delta\hat q)^2+(\Delta\hat p)^2]^{-1}$. The reason for this choice is not obvious nor was it given by Senitzky. We also observe that $\xi^2\eta^2-\delta^2=\frac{1}{4}K_0^4$ and by the Schwartz inequality, this quantity is always greater than or equal to zero. Moreover when $\delta=0$, and $\xi\eta=\frac{1}{2}$, we have the minimum uncertainty product, for which $K_0=1$. Thus K_0 has some of the features that Senitzky's K has but clearly it is different. In the present note, we have given an explicit construction of the classical probability function given certain conditions and show its equivalence to a certain Wigner distribution function associated with a quantum-mechanical system described by a wave function.

To make the discussion of the Shannon entropy more general than given above for arbitrary Wigner distributions, denoted by $f(Q,P)$, we may proceed as follows. It has been known for sometime that one may smooth the Wigner function so as to make it positive. Let this smoothed function be denoted by $f_s(Q,P)$. A natural way of doing this was introduced by Husimi[7] which has been shown by Mehta and Sudarshan[8] to be the "antinormal ordered distribution function" and the usual Wigner distribution is the "symmetrically ordered distribution function," with the former being a Gaussian weighted integral over the latter. We suggest here that one use Eq. (11) for $f_W(Q,P)$ as the Gaussian weight for smoothing an arbitrary Wigner function $f(Q,P)$:

$$\begin{aligned}f_s(Q,P)&=\int\int_{-\infty}^{\infty}dQ'\,dP'\,f_W(Q-Q',P-P')f(Q',P')\\ &=\int\int_{-\infty}^{\infty}\frac{dQ'\,dP'}{\pi}\exp[-|\zeta_2|^2(P-P')^2-|\zeta_1|^2(Q-Q')^2-2(|\zeta_1|^2|\zeta_2|^2-1)^{1/2}(P-P')(Q-Q')]f(Q',P') \ .\end{aligned} \tag{15}$$

The reason for suggesting this will be evident presently. It should be pointed out that Mehta and Sudarshan[8] used the minimum uncertainty Gaussian, corresponding to $|\zeta_1|^2|\zeta_2|^2=1$ in Eq. (15). Cartwright[9] used a more general Gaussian smoothing to demonstrate the positivity of f_s. The one given above is the most general Gaussian smoothing.

We note here a few important properties of f_s given by Eq. (15) which will be useful for our present discussion. These properties may be verified easily by direct computation: (a)

$$\int\int_{-\infty}^{\infty}f_s(Q,P)\,dQ\,dP=\int\int_{-\infty}^{\infty}f(Q,P)\,dQ\,dP=1 \ , \tag{16a}$$

(b)

$$\begin{aligned}\int_{-\infty}^{\infty}f_s(Q,P)\,dP&=\int\int_{-\infty}^{\infty}f(Q',P')\frac{1}{\pi^{1/2}|\zeta_2|}\exp[-(Q-Q')^2/|\zeta_2|^2]\,dQ'\,dP'\\ &=\int_{-\infty}^{\infty}\mathcal{P}(Q')\frac{1}{\pi^{1/2}|\zeta_2|}\exp[-(Q-Q')^2/|\zeta_2|^2]\,dQ'\end{aligned} \tag{16b}$$

and

$$\int_{-\infty}^{\infty} f_s(Q,P)\,dQ = \int\int_{-\infty}^{\infty} f(Q',P')\frac{1}{\pi^{1/2}|\zeta_1|}\exp[-(P-P')^2/|\zeta_1|^2]\,dQ'\,dP'$$

$$= \int_{-\infty}^{\infty} \tilde{\mathcal{P}}(P')\frac{1}{\pi^{1/2}|\zeta_1|}\exp[-(P-P')^2/|\zeta_1|^2]\,dP' \;, \tag{16c}$$

(c)

$$\langle Q\rangle_s = \int\int_{-\infty}^{\infty} Qf_s(Q,P)\,dQ\,dP = \int\int_{-\infty}^{\infty} Qf(Q,P)\,dQ\,dP = \langle Q\rangle \tag{16d}$$

and

$$\langle P\rangle_s = \int\int_{-\infty}^{\infty} Pf_s(Q,P)\,dQ\,dP = \int\int_{-\infty}^{\infty} Pf(Q,P)\,dQ\,dP = \langle P\rangle \;, \tag{16e}$$

(d)

$$\langle Q^2\rangle_s = \langle Q^2\rangle + \tfrac{1}{2}|\zeta_2|^2 \tag{16f}$$

and

$$\langle P^2\rangle_s = \langle P^2\rangle + \tfrac{1}{2}|\zeta_1|^2 \;, \tag{16g}$$

(e)

$$f_s(Q,P) \equiv \exp\frac{1}{4}\left\{|\zeta_1|^2\frac{\partial^2}{\partial P^2} + |\zeta_2|^2\frac{\partial^2}{\partial Q^2} - 2(|\zeta_1|^2|\zeta_2|^2-1)^{1/2}\frac{\partial^2}{\partial Q\,\partial P}\right\}f(Q,P) \;. \tag{16h}$$

The above relations are generalizations of those given by Mehta and Sudarshan[8] and the operator identity relating f_s to f is a generalization of a similar result given by McKenna and Frisch.[10] The first relationship (a) shows that the smoothed Wigner function has the same normalization as the usual distribution functions. The positivity of f_s is proved easily[8] either by following Cartwright[9] or by using an identity due to Wigner recently discussed again in the literature.[11] The second set or relationships (b) shows that the smoothed function does not directly lead to distributions of momenta, $\tilde{\mathcal{P}}(P)$, and coordinates, $\mathcal{P}(Q)$, upon integration over the other variable as was the case with the unsmoothed Wigner function, but is now smoothed by an appropriate Gaussian. As a consequence we obtain the mean values of coordinate and momentum to be the same for both the smoothed and unsmoothed functions. Since $\langle Q\rangle$ and $\langle P\rangle$ obey Ehrenfest relations, i.e., classical equations, so do the averages $\langle Q\rangle_s$ and $\langle P\rangle_s$. The difference between the two functions is most usefully displayed by (d) which exhibits the Gaussian smearing explicitly. It may not be out of place here to point out that Baker[4] showed the uniqueness of the structure of the Wigner function $f(Q,P)$ from arguments based on spectral theory, which was proved again recently by more elaborate analysis by O'Connell and Wigner.[12] Baker also showed how $f(Q,P)$ can be used to completely formulate quantum mechanics including the theory of measurement. In this respect, some of the comments of Senitzky[1] concerning collapse of wave functions can also be discussed in the present framework.

Now, having constructed a positive, normalizable distribution function with desirable properties, and relations of averages of Q,P, etc. to the fully quantum-mechanical averages, we may apply the information theoretic scheme to construct an entropy functional of the original Wigner distribution function:

$$S_s = -\int\int_{-\infty}^{\infty} f_s(Q,P)\ln f_s(Q,P)\,dQ\,dP \;. \tag{17}$$

From Eqs. (16f) and (16g) $\langle Q^2\rangle \equiv \langle Q^2\rangle_s - \frac{1}{2}|\zeta_2|^2$ and $\langle P^2\rangle \equiv \langle P^2\rangle_s - \frac{1}{2}|\zeta_1|^2$ imply that in the classical limit $\hbar \to 0$, $\langle Q^2\rangle \to \langle Q^2\rangle_{\rm cl}$ and $\langle P^2\rangle \to \langle P^2\rangle_{\rm cl}$, so that by suitably choosing $|\zeta_1|$ and $|\zeta_2|$ we may make the smoothed Wigner function have the desired property that Senitzky was looking for, namely to discover the classical probability aspect in a fully quantum problem. Clearly $f(Q,P)$ contains the entire quantum-mechanical information and with the aid of the smoothing Gaussian function, we may construct $f_s(Q,P)$ which delineates the classical counterpart from their quantum partners.

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