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Brief Reports

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Analytic evaluation of an $O(\alpha)$ vertex correction to the decay rate of orthopositronium

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In this report the order-α correction to the lowest-order orthopositronium decay rate due to the two outer-vertex graphs is obtained in analytic form.

Orthopositronium decays electromagnetically to three photons. The lowest-order decay rate, due to Ore and Powell,[1] is[2]

$$\Gamma_{\rm LO}=\frac{2}{9\pi}(\pi^2-9)m\alpha^6 \ . \tag{1}$$

Order-α corrections to this rate have been computed numerically by Pascual and de Rafael[3]; Stroscio and Holt[4]; Caswell, Lepage, and Sapirstein[5]; and the author.[6] Recently Stroscio[7] has obtained an exact analytic value for the $O(\alpha)$ correction due to the self-energy graphs [shown in Fig. 1(b)]. In this report I give an analytic evaluation of the $O(\alpha)$ correction due to the outer-vertex graphs.

The orthopositronium decay rate (in the center of mass frame) is[8]

$$\Gamma=\frac{m}{2^7\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2\,\frac{1}{3!}\overline{|M|^2} \ , \tag{2}$$

where $x_i=\omega_i/m$ is the normalized energy of photon i, and $\overline{|M|^2}$ is the invariant matrix element squared, summed over final polarizations, and averaged over the three orthopositronium spins. The invariant matrix element has contributions from each of the graphs in Fig. 1:

$$M=M_{\rm LO}+2M_{\rm SE}+2M_{\rm OV}+\cdots \tag{3}$$

(the two self-energy graphs contribute equally, as do the two outer-vertex graphs). The lowest-order invariant matrix element is

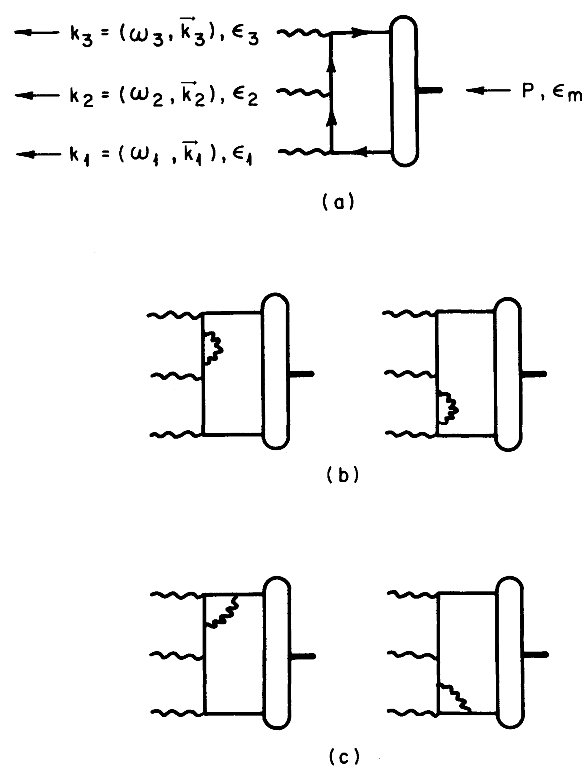

FIG. 1. Orthopositronium decay graphs. Graph (a) is the lowest-order graph. Graphs (b) are the self-energy (SE) correction graphs, and graphs (c) are the outer-vertex (OV) correction graphs. The object to the right in each graph is the orthopositronium wave function.

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$$M_{\mathrm{LO}} \cong -i\pi\alpha^3 \sum_{\sigma\epsilon S_3} \frac{x_{\sigma(2)}}{x_1 x_2 x_3} \operatorname{tr}\left[\gamma\epsilon_{\sigma(3)}(-\gamma R_{\sigma(3)}+1)\gamma\epsilon_{\sigma(2)}(\gamma R_{\sigma(1)}+1)\gamma\epsilon_{\sigma(1)}\begin{bmatrix}0 & \vec{\sigma}\cdot\hat{\epsilon}_m\\ 0 & 0\end{bmatrix}\right] . \tag{4}$$

The sum is over the 3! permutations of the final photons. The four-vectors ϵ_1, ϵ_2, and ϵ_3 are photon polarizations, and $\hat{\epsilon}_m$ is the orthopositronium spin three-vector. The R's are dimensionless momentum vectors: $R_i = N - K_i$ with $N=(1,\vec{0})$ and $K_i = (\omega_i/m, \vec{k}_i/m)$.

The invariant matrix element M_{OV} for the outer-vertex graph is obtained from (4) by the replacement $\gamma\epsilon_{\sigma(1)} \rightarrow \Lambda(x_{\sigma(1)})\epsilon_{\sigma(1)}$, where

$$\Lambda^\lambda(x_i) = \frac{\alpha}{4\pi}[\gamma^\lambda f(2x_i) + (\gamma R_i - 1)\gamma^\lambda g(2x_i) + (\gamma R_i - 1)N^\lambda h(2x_i) + N^\lambda j(2x_i)] , \tag{5}$$

with

$$f(2x) = D + \frac{6-2x}{1-2x}\ln(2x) + \frac{1}{x}\eta(x) , \tag{6a}$$

$$g(2x) = \frac{2}{1-2x}\ln(2x) , \tag{6b}$$

$$h(2x) = \frac{1}{x}\left[\frac{2-10x+8x^2}{(1-2x)^2}\ln(2x) - \frac{2-2x}{1-2x} + \frac{1}{x}\eta(x)\right] , \tag{6c}$$

$$j(2x) = \frac{-2}{1-2x}\left[1 + \frac{2x}{1-2x}\ln(2x)\right] . \tag{6d}$$

The factor $\eta(x)$ contains a dilogarithm[9]

$$\eta(x) = \frac{\pi^2}{6} - Li_2(1-2x) . \tag{7}$$

The vertex function was calculated in Feynman gauge in $d=2\omega$ dimensions, and

$$D = \frac{1}{2-\omega} - \gamma_E + \ln(4\pi) . \tag{8}$$

There is no infrared divergence.

Order-α corrections to the decay rate come from cross terms in the square of the invariant matrix element. The $O(\alpha)$ correction due to the outer-vertex graphs is

$$\Gamma_{\mathrm{OV}} = \frac{m}{2^7\pi^3}\int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \sum_{\epsilon_1,\epsilon_2,\epsilon_3} \frac{1}{3} \sum_{\epsilon_m} \frac{1}{3!}(2)2\,\mathrm{Re}[(M_{\mathrm{LO}})^* M_{\mathrm{OV}}] . \tag{9}$$

There is no need to symmetrize in both $(M_{\mathrm{LO}})^*$ and M_{OV}, so the vertex correction can be taken to act on photon 1 only. Performing the polarization sums, the spin sum, and the resulting trace (using REDUCE[10]) one obtains

$$\Gamma_{\mathrm{OV}} = \frac{m\alpha^6}{6\pi}\frac{\alpha}{\pi}\int_0^1 dx \frac{1}{x^2}[P_{\mathrm{LO}}(x)f(2x) + P_{\mathrm{SE}}(x)(-2x)g(2x) + P_{\mathrm{VA}}(x)(-2x)h(2x) + P_{\mathrm{VB}}(x)j(2x)] . \tag{10}$$

The P factors are

$$P_{\mathrm{LO}}(x) = \int_{1-x}^1 dx_2 \frac{1}{(x_2x_3)^2}(-x^{41}+4x^{40}-5x^{32}+7x^{31}-8x^{30}+9x^{22}-14x^{21}+4x^{20}+4x^{11})$$
$$= \left[\left[\frac{-8}{2-x}+12-5x+x^2\right]\ln(1-x) + 8x - 3x^2\right] , \tag{11a}$$

$$P_{\mathrm{SE}}(x) = \int_{1-x}^1 dx_2 \frac{1}{(x_2x_3)^2}(x^{31}-2x^{30}+4x^{22}-14x^{21}+6x^{20}+9x^{11}-4x^{10})$$
$$= \left[\left[\frac{8}{(2-x)^2} - \frac{18}{2-x} + 10 + x\right]\ln(1-x) + \frac{4x}{2-x} + x\right] , \tag{11b}$$

$$P_{\mathrm{VA}}(x) = \int_{1-x}^1 dx_2 \frac{1}{(x_2x_3)^2}\frac{1}{2}(-2x^{32}-3x^{31}+6x^{30}-x^{22}+12x^{21}-10x^{20}-6x^{11}+4x^{10})$$
$$= \frac{1}{2}\left[\left[\frac{-8}{(2-x)^2} + \frac{12}{2-x} - 3x\right]\ln(1-x) - \frac{4x}{2-x} + 6x + x^2\right] , \tag{11c}$$

$$P_{\mathrm{VB}}(x) = \int_{1-x}^1 dx_2 \frac{1}{(x_2x_3)^2}(-2x^{41}+x^{40}-2x^{32}+2x^{31}+2x^{30}-x^{22}+10x^{21}-7x^{20}-7x^{11}+4x^{10})$$
$$= \left[\left[\frac{-8}{(2-x)^2} + \frac{14}{2-x} - 8x + 2x^2\right]\ln(1-x) - \frac{4x}{2-x} + 7x - 3x^2\right] , \tag{11d}$$

where $x^{nm}=(x_2)^n(x_3)^m$, $x=x_1$, and $x_3=2-x-x_2$ by conservation of energy.

Integration of Eq. (10) yields

$$\Gamma_{OV}=\Gamma_{LO}\frac{\alpha}{\pi}\left\{D+\frac{3}{4(\pi^2-9)}\left[-26-\tfrac{115}{3}\ln 2+\tfrac{91}{18}\zeta(2)+\tfrac{443}{54}\zeta(3)+\tfrac{3419}{108}\ln 2\,\zeta(2)-R\right]\right\} , \tag{12}$$

where $\zeta(n)$ is the Riemann ζ function: $\zeta(2)=\pi^2/6$ and $\zeta(3)=1.202\,056\,903\,2$. Note that

$$\frac{3}{4(\pi^2-9)}\int_0^1 dx\frac{1}{x^2}P_{LO}(x)=1 \ . \tag{13}$$

This integral is related to the lowest-order rate.[11] The quantity

$$R=\int_0^1 dx\frac{\ln(1-x)}{2-x}\left[\frac{\pi^2}{6}-Li_2(1-2x)\right] \tag{14}$$

has the value $R=-1.743\,033\,8(4)$, so the outer-vertex contribution to the orthopositronium decay rate is[12]

$$\Gamma_{OV}=\Gamma_{LO}\frac{\alpha}{\pi}[D+2.971\,138\,5(4)] \ . \tag{15}$$

The number in (15) can be compared with 2.9707(7), obtained numerically in Ref. 6. The two numbers are consistent. Unfortunately Stroscio and Holt[4] and Caswell, Lepage, and Sapirstein[5] do not list the contribution of the outer-vertex graphs separately. A direct integration of Eq. (10), using the adaptive Monte Carlo integration routine VEGAS,[13] yields 2.9712(2).

The contribution of the self-energy graphs to the orthopositronium decay rate (in Feynman gauge) is obtained from Eq. (10) by the replacements

$$f(2x)\to -D-4-\frac{2}{1-2x}\left[-2+3x+\frac{2-4x-2x^2}{1-2x}\ln(2x)\right] , \tag{16a}$$

$$g(2x)\to\frac{1}{1-2x}\left[1-\frac{2-6x}{1-2x}\ln(2x)\right] , \tag{16b}$$

$$h(2x)\to 0 , \tag{16c}$$

$$j(2x)\to 0 . \tag{16d}$$

The resulting integral has the value

$$\Gamma_{SE}=\Gamma_{LO}\frac{\alpha}{\pi}\left\{-D-4+\frac{3}{4(\pi^2-9)}\left[-7+\tfrac{67}{3}\ln 2+\tfrac{805}{36}\zeta(2)-\tfrac{1049}{54}\zeta(3)-\tfrac{775}{54}\ln 2\,\zeta(2)\right]\right\} . \tag{17}$$

If this result is renormalized by subtracting

$$\Gamma_{LO}\frac{\alpha}{\pi}[-D-4-2\ln(\lambda^2/m^2)] , \tag{18}$$

where λ is an unphysical photon mass,[14] one obtains the result of Stroscio.[7]

The outer-vertex graphs are the second set of $O(\alpha\Gamma_{LO})$ orthopositronium decay graphs to be evaluated analytically. Hopefully the rest will follow. Especially important would be an analytic result for the binding (or ladder) graph, which is the most difficult graph to evaluate numerically.

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[8] See Ref. 6. When considering $O(\alpha)$ corrections to Γ_{LO} it is permissible to approximate the positronium mass P^0 by $2m$, since $P^0=2m[1+O(\alpha^2)]$.
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[11] The lowest-order rate is obtained from Eq. (10) by the replacements $f(2x)\to(4\pi/\alpha)$, $g(2x)\to 0$, $h(2x)\to 0$, $j(2x)\to 0$; followed by a division by 4 [the two factors of 2 in Eq. (9) do not appear in the lowest-order rate].
[12] The result for Γ_{OV} is not analytic according to the strictest definition of the word. However, the word is appropriate because the problem has been reduced to the evaluation of one relatively simple integral, R, whose numerical value can be obtained easily and with great accuracy.
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[14] The photon mass λ is introduced so that the theory can be renormalized on shell. It is not needed for the regularization of physical infrared divergences, since there are none. Note that in Feynman gauge in the mass shell renormalization scheme, $Z_2=1-(\alpha/4\pi)\,[D+4+2\ln(\lambda^2/m^2)]+O(\alpha^2)$.