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Frequency dependence of laser light fluctuations

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The equations of motion for an inhomogeneously broadened laser operating off resonance in the neighborhood of threshold are solved. It is shown that the effects of detuning cannot be described just by a frequency-dependent laser pump parameter, but that the laser characteristics are modified in other ways also. The relative intensity fluctuations and the correlation time of the laser are calculated as a function of mean intensity and frequency, and it is shown that both are affected by detuning. These theoretical conclusions are then confirmed quantitatively by direct photoelectric counting and two-time correlation measurements of a He:Ne laser.

I. INTRODUCTION

The behavior of a laser in which the medium is inhomogeneously broadened and the cavity is detuned from resonance, was already discussed in the classic paper of Lamb.[1,2] This first treatment was semiclassical, but fully quantum-mechanical treatments were presented shortly afterwards.[3,4] However, although full solutions of the equations of motion, including various correlation functions of the laser field, were ultimately derived,[5,6] most subsequent treatments seem to have been limited largely to the on-resonance behavior of the laser.

It is well known that the effect of detuning an inhomogeneously broadened laser generally is to lower the gain (except for a possible Lamb dip), as fewer active atoms contribute to the laser action. Indeed, in many experimental investigations of inhomogeneously broadened lasers near threshold it has been common practice to control the working point, or the pump parameter, by varying the cavity tuning. However, detuning the laser has a more significant effect on the properties of the emitted light than is commonly supposed. Although detuning does reduce the pump parameter, it also causes other explicitly frequency-dependent changes in the light, as we have recently demonstrated.[7] The frequency dependence of the optical properties seems to have been largely ignored in the past. Often, the laser working point was varied by detuning, but the magnitude of the detuning was not even stated, and the experimental results were presented simply as a function of laser light intensity or pump parameter. Similar remarks apply to the correlation functions of the laser field, which also depend explicitly on the detuning in addition to depending on the pump parameter.

In the following we derive the Fokker-Planck equation for a single-mode laser off resonance, in order to exhibit the frequency dependence explicitly. We show that if one integrates over all phases and focuses on the amplitude or the intensity of the laser field, the equation of motion can be cast into the same form as on resonance, provided the light intensity, the pump parameter, and the time are all scaled appropriately. This leads to simple predictions of how the relative intensity fluctuations and the two-time intensity correlation function of the laser should vary with detuning. Finally, we confirm these predictions by direct photoelectric counting and correlation measurements on a He:Ne laser that is operated at various working points in the neighborhood of threshold.

II. EQUATIONS OF MOTION

We consider a single-mode laser in which the active medium is a set of gas atoms, with the laser levels having energy separation $\hbar\omega_0$. We assume that the medium is Doppler broadened to a width ku, which is generally larger than the natural atomic linewidth γ for transitions between the laser levels. The cavity is tuned to a frequency ω that may differ from ω_0, and we introduce the parameters $\Delta\omega \equiv \omega - \omega_0$,

$$\eta \equiv \Delta\omega/\gamma, \quad \rho \equiv \gamma/ku \tag{1}$$

to descibe the relative detuning, and the ratio of the natural to the inhomogeneous linewidth.

Our starting point is the semiclassical equation of motion for the laser field $E(\vec{r},t)$ at position $\vec{r}$ at time t within the cavity. If we write

$$E(\vec{r},t)=U(\vec{r})\mathscr{E}(t)e^{-i\omega t}+\text{c.c.}\ , \tag{2}$$

where $U(\vec{r})$ is the cavity mode function, and $\mathscr{E}(t)$ is a slowly varying, dimensionless, complex amplitude, then, provided the laser is not operating too far above threshold, $\mathscr{E}(t)$ obeys the Lamb equation of motion[1,2]

$$\dot{\mathscr{E}}=(A-C-B\,|\,\mathscr{E}\,|^2)\mathscr{E}\ . \tag{3}$$

The parameters A, C, and B are gain, loss, and saturation coefficients, and they are all real on resonance when $\omega=\omega_0$. However, when the laser is detuned from resonance, A and B have to be replaced by more complicated coefficients $\tilde{A}$ and $\tilde{B}$ that are complex in general, and depend on the detuning $\Delta\omega$ or η. The new coefficients can be related to the old ones with the help of the plasma dispersion function $Z(y)$ of complex argument y,[1,2] defined by

$$Z(y)\equiv i\sqrt{\pi}e^{y^2}(1-\text{erf}y)\ . \tag{4}$$

$Z(y)$ is a complex function in general, but it becomes purely imaginary for real argument y, and it is of order unity for $y<<1$. It is shown in the Appendix, with the help of results derived in Ref. 2, that for small ρ, i.e., for relatively large inhomogeneous broadening,

$$\tilde{A}=A\left[\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}-\frac{iZ_r(\rho+i\eta\rho)}{Z_i(\rho)}\right] \tag{5}$$

$$=A\{\,1-\eta^2\rho^2[1-2\rho/\,|\,Z(\rho)\,|\,+2\rho^2-\tfrac{1}{2}\eta^2\rho^2+O(\rho^3)]$$
$$-[2i\eta\rho/\,|\,Z(\rho)\,|\,][1-\rho\,|\,Z(\rho)\,|\,-\tfrac{2}{3}\eta^2\rho^2+\eta^2\rho^3\,|\,Z(\rho)\,|\,+O(\rho^4)]\,\}\ , \tag{6}$$

and

$$\tilde{B}=B\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\left[\left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2}\right]+O(\rho^3)+i\eta\left[\frac{\frac{1}{2}}{1+\eta^2}-\frac{\rho[1-2\rho/\,|\,Z(\rho)\,|\,]}{1+2\rho/\,|\,Z(\rho)\,|\,-2\rho^2}+O(\rho^3)\right]\right]\ . \tag{7}$$

Here $Z_r(y)$ and $Z_i(y)$ stand for the real and imaginary parts of $Z(y)$. When the detuning η is zero, both coefficient $\tilde{A}$ and $\tilde{B}$ become real and equal to A and B, respectively.

The semiclassical equation of motion (3), or the more general version that holds off resonance, does not contain the effects of spontaneous atomic emission. Spontaneous emission contributions are, of course, included automatically in a fully quantized treatment of the laser.[3,4] However, it has been shown that one can incorporate these effects to a good approximation in a semiclassical treatment by introducing quantum noise terms into the equation of motion.[4–6] The equation then takes the form

$$\dot{\mathscr{E}}=(\tilde{A}-C-\tilde{B}\,|\,\mathscr{E}\,|^2)\mathscr{E}+q\ , \tag{8}$$

where $q(t)$ is a complex random process representing the spontaneous emission fluctuations, which is generally taken to be Gaussian and of zero mean. If the lifetime for spontaneous emission is short compared with the time in which the laser amplitude $\mathscr{E}(t)$ is changing, and this is generally the case for a "good" resonator cavity, then $q(t)$ may be regarded as effectively δ correlated. We shall therefore write for the correlation function of $q(t)$

$$\langle q^*(t_1)q(t_2)\rangle=4S[Z_i(\rho+i\eta\rho)/Z_i(\rho)]\delta(t_1-t_2)\ , \tag{9}$$

where S represents the strength of the random process $q(t)$ on resonance. The factor $Z_i(\rho+i\eta\rho)/Z_i(\rho)$ has been introduced in order to ensure that the strength of the quantum noise, which is related to the spontaneous emission rate, has the same dependence on frequency η as the stimulated emission rate, which is represented by the real part of A. The ratio of the stimulated to the spontaneous emission probabilities then does not change with cavity frequency.

Because of the quantum fluctuations introduced by $q(t)$, the laser field is no longer deterministic, but becomes a random variable with a certain probability density $p(\mathscr{E},t)$. At this stage it becomes more convenient to regard the field amplitude as a two-dimensional real vector $\vec{\mathscr{E}}$ with components $\mathscr{E}_1$, $\mathscr{E}_2$, which are the real and imaginary parts of the complex amplitude $\mathscr{E}$. Corresponding to the Langevin equation of motion for $\vec{\mathscr{E}}(t)$, $p(\vec{\mathscr{E}},t)$ obeys a Fokker-Planck equation. If we introduce new dimensionless variables $\vec{\mathscr{E}}\,',q',t',a$ defined by

$$\begin{aligned} \vec{\mathscr{E}} &\equiv (S/B)^{1/4}\vec{\mathscr{E}}\,' , \\ t &\equiv (SB)^{-1/2}t' , \\ q &\equiv S^{3/4}B^{1/4}q' , \\ A-C &\equiv (SB)^{1/2}a , \end{aligned} \tag{10}$$

in terms of which

$$\langle q'^{*}(t_1')q'(t_2')\rangle = 4[Z_i(\rho+i\eta\rho)/Z_i(\rho)]\delta(t_1'-t_2') , \tag{11}$$

then the Fokker-Planck equation takes the simple form

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{2}\frac{\partial}{\partial \mathscr{E}_j}(\mathscr{A}_j p) + \frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\sum_{j=1}^{2}\frac{\partial^2 p}{\partial \mathscr{E}_j^2} . \tag{12}$$

For simplicity we have dropped the primes on the new variables with the understanding that we shall be dealing with these dimensionless, scaled variables from now on. From Eqs. (5), (7), and (8) the two-component drift vector $\vec{\mathscr{A}}$ is given by

$$\mathscr{A}_1 \equiv \left[a - \frac{C}{\sqrt{SB}}\left[\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)} - 1\right] - \left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2} + O(\rho^3)\right]\vec{\mathscr{E}}^{\,2}\right]\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\mathscr{E}_1 - \left[\frac{\widetilde{A}_i}{\sqrt{SB}} - \frac{\widetilde{B}_i}{B}\vec{\mathscr{E}}^{\,2}\right]\mathscr{E}_2 , \tag{13}$$

$$\mathscr{A}_2 \equiv \left[a - \frac{C}{\sqrt{SB}}\left[\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)} - 1\right] - \left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2} + O(\rho^3)\right]\vec{\mathscr{E}}^{\,2}\right]\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\mathscr{E}_2 + \left[\frac{\widetilde{A}_i}{\sqrt{SB}} - \frac{\widetilde{B}_i}{B}\vec{\mathscr{E}}^{\,2}\right]\mathscr{E}_1 . \tag{14}$$

We have written $\widetilde{A}_i$, $\widetilde{B}_i$ for the real and the imaginary parts of $\widetilde{A}$ and $\widetilde{B}$, which are given explicitly by Eqs. (6) and (7), respectively. The parameter a is the dimensionless laser pump parameter on resonance, which is proportional to the difference between the gain and the loss on resonance. It can be shown that in the neighborhood of threshold[3]

$$A/(SB)^{1/2} \approx C/(SB)^{1/2} = \sqrt{\pi}n_0 , \tag{15}$$

where n_0 is the absolute photon number present in the laser cavity at threshold ($A=C$) and on resonance ($\eta=0$). This allows us to express the pump parameter in the form

$$a = \sqrt{\pi}n_0(A-C)/C , \tag{16}$$

from which we can readily estimate the largest value of a for which our treatment is appropriate. If we take the third-order equation of motion (8) to be adequate so long as the gain A does not exceed the loss C by more than about 1%, and if n_0 is of order 10 000 (Ref. 8), then a is limited to values below about 100.

III. STEADY-STATE SOLUTION

The Fokker-Planck equation (12) is more complicated than the corresponding equation on resonance, for which the drift vector simplifies to

$$\mathscr{A}_j=(a-\vec{\mathscr{E}}^2)\mathscr{E}_j,\ \ j=1,2\ . \tag{17}$$

Moreover, $\vec{\mathscr{A}}$ given by Eqs. (13) and (14) does not satisfy the so-called potential condition.[9] Nevertheless, it is not difficult to obtain the steady-state solution in general. If we proceed in the usual way, by equating the probability current to zero,

$$-\mathscr{A}_j p+\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\frac{\partial p}{\partial\mathscr{E}_j}=0,\ \ j=1,2$$

and integrating the resulting equation, we readily obtain the steady-state solution $p_s(\vec{\mathscr{E}})$ of Eq. (12). With the help of Eqs. (13) and (14) this takes the form

$$p_s(\vec{\mathscr{E}})=\text{const}\exp\frac{1}{2}\left[a-\sqrt{\pi}n_0\left[\frac{Z_i(\rho)}{Z_i(\rho+i\eta\rho)}-1\right]\vec{\mathscr{E}}^2-\frac{1}{4}\left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2}+O(\rho^3)\right]\vec{\mathscr{E}}^4\right]. \tag{18}$$

It is interesting to observe that the imaginary contributions $\widetilde{A}_i$, $\widetilde{B}_i$ to $\widetilde{A},\widetilde{B}$ play no role here, because these terms have opposite signs in Eqs. (13) and (14) and effectively cancel. As the solution depends only on the light intensity $I\equiv\vec{\mathscr{E}}^2$, and not on the separate components of $\vec{\mathscr{E}}$, it is easier to work with the probability density $\mathscr{P}_s(I)$ of I, which can be put in the compact form

$$\mathscr{P}_s(I)=\text{const}\,e^{-(1/4)(\widetilde{I}-\widetilde{a})^2}\ . \tag{19}$$

The modified variables $\widetilde{I}$ and $\widetilde{a}$ are related to I and a by

$$\widetilde{I}\equiv I(1+\tfrac{1}{2}\eta^2)^{1/2}/(1+\eta^2)^{1/2}\ , \tag{20}$$

$$\widetilde{a}\equiv a-\sqrt{\pi}n_0\left[\frac{Z_i(\rho)}{Z_i(\rho+i\eta\rho)}-1\right]\left[\frac{1+\eta^2}{1+\frac{1}{2}\eta^2}\right]^{1/2}$$

$$=\{a-\sqrt{\pi}n_0\eta^2\rho^2[1-2\rho/|Z(\rho)|+2\rho^2+\tfrac{1}{2}\eta^2\rho^2+\cdots]\}(1+\eta^2)^{1/2}/(1+\tfrac{1}{2}\eta^2)^{1/2}\ , \tag{21}$$

and the modified variables $\widetilde{I}$ and $\widetilde{a}$ coincide with I and a, respectively, on resonance. $\widetilde{a}$ may be looked on as an "effective" pump parameter for the modified intensity $\widetilde{I}$. Equation (21) follows from the previous line with the help of Eq. (A10). However, we note that detuning not only affects $\widetilde{a}$, but also the scale factor associated with the light intensity. The effect of detuning therefore cannot be accounted for entirely in terms of a change of effective pump parameter. Whereas $\widetilde{a}$ generally falls below a for moderate values of a as η increases from zero, the opposite may happen at first when a is large, because of the factor $(1+\eta^2)^{1/2}/(1+\frac{1}{2}\eta^2)^{1/2}$. This behavior can be regarded as a vestigial Lamb dip, and it does not occur in the neighborhood of threshold.

The probability distribution $\mathscr{P}_s(I)$ given by Eq. (19) has a well-known structure for a laser operating in the threshold region,[1,6] and we readily obtain from it

$$\langle I\rangle=\left[\frac{1+\eta^2}{1+\frac{1}{2}\eta^2}\right]^{1/2}\left[\widetilde{a}+\frac{2e^{-(1/4)\widetilde{a}^2}}{\sqrt{\pi}(1+\text{erf}\frac{1}{2}\widetilde{a})}\right], \tag{22}$$

$$\frac{\langle(\Delta I)^2\rangle}{\langle I\rangle^2}=\left[2-\frac{2\widetilde{a}e^{-(1/4)\widetilde{a}^2}}{\sqrt{\pi}(1+\text{erf}\frac{1}{2}\widetilde{a})}-\frac{4e^{-(1/2)\widetilde{a}^2}}{\pi(1+\text{erf}\frac{1}{2}\widetilde{a})^2}\right]\Bigg/\left[\widetilde{a}+\frac{2e^{-(1/4)\widetilde{a}^2}}{\sqrt{\pi}(1+\text{erf}\frac{1}{2}\widetilde{a})}\right]^2 \tag{23}$$

$$\equiv F[\langle I\rangle(1+\tfrac{1}{2}\eta^2)^{1/2}/(1+\eta^2)^{1/2}]\ , \tag{24}$$

where $F(x)$ is a monotonic, transcendental function defined by Eqs. (22)–(24). Equation (24) implies that the relationship between $\langle(\Delta I)^2\rangle/\langle I\rangle^2$ and $\langle I\rangle$ varies with detuning η, and is not a unique relationship, as has sometimes been assumed. Figure 1 shows some curves relating the relative intensity fluctuations of the light intensity to $\langle I\rangle$ for two different detunings η. The rms intensity fluctuations are always of order $\langle I\rangle$

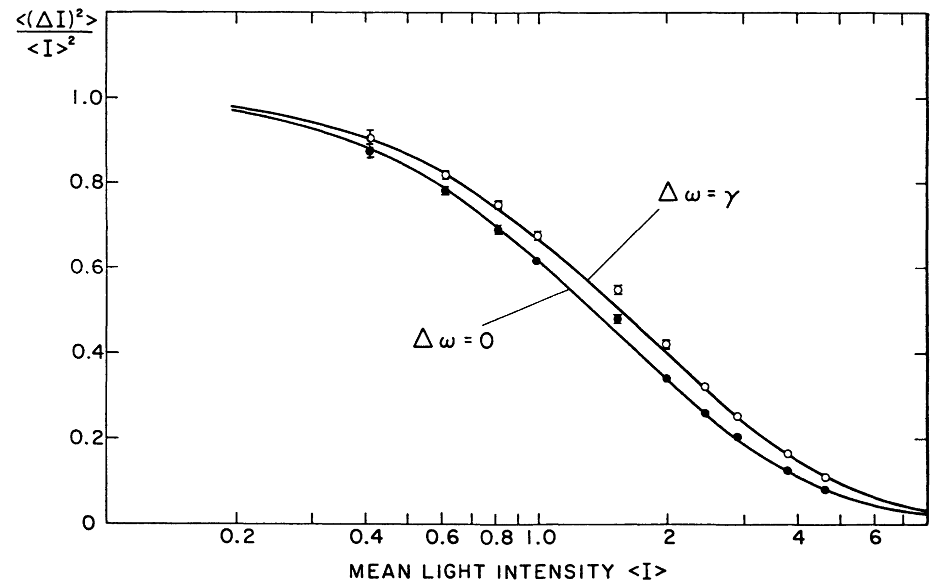

FIG. 1. Variation of the relative intensity fluctuations $\langle(\Delta I)^2\rangle/\langle I\rangle^2$ with mean intensity $\langle I\rangle$ on and off resonance. The full curves are theoretical and the experimental results are shown superimposed. Where the standard deviations are larger than the spot size, they are indicated.

well below threshold, and close to zero well above threshold, as the state of the laser field changes from the thermal to the coherent. But it will be seen that when η is of order 1 or greater, the effect of detuning is clearly significant and should not be ignored.[7]

Equation (21) offers an interesting possibility for determining the absolute photon number at threshold n_0 from values of the effective pump parameter $\tilde{a}$ for various detunings η. $\tilde{a}$ can be obtained from measurements of the mean light intensity with the help of Eq. (22). The method is interesting in that it requires no absolute but only relative intensity measurements, and its effectiveness has recently been demonstrated.[8]

IV. TIME-DEPENDENT SOLUTION

As the drift vector given by Eqs. (13) and (14) does not satisfy the potential condition $\partial\mathscr{A}_1/\partial\mathscr{E}_2 = \partial\mathscr{A}_2/\partial\mathscr{E}_1$, we shall not attempt to solve the Fokker-Planck equation (12) in general. However, most photoelectric measurements relate only to the light intensity $I(t)\equiv\vec{\mathscr{E}}^2(t)$, and the solution of the general time-dependent problem for $I(t)$ turns to be no more difficult than for the laser on resonance. It is convenient to transform Eq. (12) to polar coordinates by putting

$$\mathscr{E}_1 = r\cos\theta\,,\qquad \mathscr{E}_2 = r\sin\theta\,. \tag{25}$$

The joint probability density $P(r,\theta,t)$ is then related to $p(\mathscr{E}_1,\mathscr{E}_2,t)$ by

$$P(r,\theta,t) = rp(r\cos\theta, r\sin\theta, t)\,. \tag{26}$$

With the help of the transformations

$$\begin{aligned}\partial/\partial\mathscr{E}_1 &= \cos\theta\,\partial/\partial r - (\sin\theta/r)\partial/\partial\theta\,,\\ \partial/\partial\mathscr{E}_2 &= \sin\theta\,\partial/\partial r + (\cos\theta/r)\partial/\partial\theta\,,\\ \partial^2/\partial\mathscr{E}_1^2 + \partial^2/\partial\mathscr{E}_2^2 &= \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + (1/r^2)(\partial/\partial\theta^2)\,,\end{aligned} \tag{27}$$

we readily obtain from Eq. (12)

$$\frac{1}{r}\frac{\partial P(r,\theta,t)}{\partial t}=-\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2}\right]^{1/2}\left[\frac{\partial}{\partial r}+\frac{1}{r}\right]\left[\tilde{a}-r^2\left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2}\right]^{1/2}\right]P(r,\theta,t)$$
$$-\frac{1}{r}\left[\frac{\tilde{A}_i}{\sqrt{SB}}-\frac{\tilde{B}_i}{B}r^2\right]\frac{\partial P(r,\theta,t)}{\partial\theta}+\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\frac{P(r,\theta,t)}{r}\ . \quad (28)$$

We now integrate each term with respect to θ over the whole range 0 to 2π, and put

$$\int_0^{2\pi}d\theta\, P(r,\theta,t)\equiv P(r,t)\ . \quad (29)$$

Then the integrals of $\partial P(r,\theta,t)/\partial\theta$ and $\partial^2 P(r,\theta,t)/\partial\theta^2$ vanish, and with the help of the rescaled variables

$$r\equiv\left[\frac{1+\eta^2}{1+\frac{1}{2}\eta^2}\right]^{1/2}\tilde{r}\ ,$$
$$t\equiv\frac{Z_i(\rho)}{Z_i(\rho+i\eta\rho)}\left[\frac{1+\eta^2}{1+\frac{1}{2}\eta^2}\right]^{1/2}\tilde{t}\ , \quad (30)$$

the equation (28) reduces to

$$\frac{\partial P(\tilde{r},\tilde{t})}{\partial\tilde{t}}=-\frac{\partial}{\partial\tilde{r}}\left[\tilde{r}(\tilde{a}-\tilde{r}^2)+\frac{1}{\tilde{r}}-\frac{\partial}{\partial\tilde{r}}\right]P(\tilde{r},\tilde{t})\ . \quad (31)$$

This is exactly the same equation in the variables $\tilde{r},\tilde{t},\tilde{a}$ as the laser obeys on resonance, when, needless to say, $\tilde{r},\tilde{t},\tilde{a}$ coincide with r,t,a. The time-dependent solution can therefore be obtained in exactly the same manner, and, of course, has already been found.

The steady-state solution $P_s(\tilde{r})$ of Eq. (31) is obtained immediately if we equate the probability current to zero

$$[\tilde{r}(\tilde{a}-\tilde{r}^2)+1/\tilde{r}-\partial/\partial\tilde{r}]P_s(\tilde{r})=0 \quad (32)$$

and integrate, and it yields

$$P_s(\tilde{r})=\text{const}\,\tilde{r}e^{-(1/4)(\tilde{r}^2-\tilde{a})^2}\ , \quad (33)$$

which is equivalent to Eq. (19). The general time-dependent solution is obtained by writing[5,6,9]

$$P(\tilde{r},\tilde{t})=\sum_{m=0}^{\infty}c_m\sqrt{P_s(\tilde{r})}\psi_{mo}(\tilde{r})e^{-\lambda_{mo}\tilde{t}}\ , \quad (34)$$

where the coefficients c_m are determined by the initial conditions, and the $\psi_{mo}(\tilde{r})$ and λ_{m0} are eigenfunctions and eigenvalues of the Schrödinger equation[5,6]

$$-d^2\psi_{m0}(\tilde{r})/d\tilde{r}^2+V_0(\tilde{r},\tilde{a})\psi_{m0}(\tilde{r})=\lambda_{m0}\psi_{m0}(\tilde{r})\ , \quad (35)$$

with the potential $V_0(\tilde{r},\tilde{a})$ given by

$$V_0(\tilde{r},\tilde{a})=-1/4\tilde{r}^2+\tilde{a}+(\tfrac{1}{4}\tilde{a}^2-2)\tilde{r}^2-\tfrac{1}{2}\tilde{a}\tilde{r}^4+\tfrac{1}{4}\tilde{r}^6\ . \quad (36)$$

These are the same eigenfunctions and eigenvalues that one encounters in the solution of the laser problem on resonance. The only difference is that the general solution of the two-dimensional Fokker-Planck equation leads to a two-dimensional array of eigenfunctions $\psi_{mn}(\tilde{r})$ and eigenvalues λ_{mn}, whereas in our case the second suffix is zero because we have suppressed the phase of the solution. The lowest eigenvalue $\lambda_{00}=0$, and the lowest eigenfunction $\psi_{00}(\tilde{r})=\sqrt{P_s(\tilde{r})}$, so that $P(\tilde{r},\tilde{t})\to P_s(\tilde{r})$ in the long-time limit.

The Green function of the Fokker-Planck equation is a special case of the general solution (34), and we can use it to calculate any two-time correlation function. If we proceed exactly as for the laser on resonance, we obtain formally the same expression for two-time intensity correlation function in the steady state[5,6]

$$\langle\Delta\tilde{I}(\tilde{t})\Delta\tilde{I}(\tilde{t}+\tilde{\tau})\rangle=\langle(\Delta\tilde{I})^2\rangle\sum_{m=1}^{\infty}M_m(\tilde{a})e^{-\lambda_{mo}(\tilde{a})|\tilde{\tau}|}\ , \quad (37)$$

with the coefficients $M_m(\tilde{a})$ given by

$$M_m(\tilde{a})=\left|\int_0^{\infty}d\tilde{r}\,\tilde{r}^2\psi_{00}(\tilde{r},\tilde{a})\psi_{mo}(\tilde{r},\tilde{a})\right|^2\ . \quad (38)$$

In the neighborhood of threshold the series is usually dominated by the first term, and the correlation function is approximately exponential.

Despite the formal structural identity between this solution and that for the laser on resonance, Eq. (37) implies that the shape of the correlation function is different on and off resonance, and so is the intensity correlation time. The difference arises, in part, because the scale factor [Eq. (30)] that relates $\tilde{\tau}$ to the measured time τ depends on detuning, and, in part, because the effective pump parameter $\tilde{a}$ also varies with the detuning. It is convenient to define a normalized correlation function

$$\lambda(\tau)\equiv\langle\Delta I(t)\Delta I(t+\tau)\rangle/\langle I\rangle^2\ , \qquad (39)$$

which is independent of the scale of I or $\tilde{I}$, and an effective intensity correlation time T_c by[10]

$$T_c\equiv\int_0^\infty d\tau\frac{\lambda(\tau)}{\lambda(0)}\ . \qquad (40)$$

From Eq. (37) we find that the rescaled time $\tilde{T}_c$ is given by

$$\tilde{T}_c=\sum_{m=1}^{\infty}M_m(\tilde{a})/\lambda_{mo}(\tilde{a})$$

so that with the help of Eq. (30),

$$T_c=\frac{Z_i(\rho)}{Z_i(\rho+i\eta\rho)}\left[\frac{1+\eta^2}{1+\frac{1}{2}\eta^2}\right]^{1/2}\sum_{m=1}^{\infty}\frac{M_m(\tilde{a})}{\lambda_{m0}(\tilde{a})}\ . \qquad (41)$$

The ratio $Z_i(\rho)/Z_i(\rho+i\eta\rho)$ is derived in the Appendix and is shown to be of the form [cf. Eq. (A10)]

$$Z_i(\rho)/Z_i(\rho+i\eta\rho)=1+\eta^2\rho^2[1-2\rho/|Z(\rho)|+2\rho^2+\tfrac{1}{2}\eta^2\rho^2+O(\rho^3)]\ , \qquad (42)$$

while $\tilde{a}$ is given by Eq. (21). The dependence of the effective correlation time T_c on detuning and on the mean light intensity $\langle I\rangle$ can therefore be calculated from Eqs. (41) and (22).

The calculation is greatly simplified by the table of eigenvalues λ_{mn} and coefficients M_m provided by Hempstead and Lax.[5] With the help of these numbers we have calculated the relation between T_c and the mean light intensity $\langle I\rangle$ for a laser on resonance, and for one detuned by one natural linewidth ($\eta=1$). The results are shown by the two curves in Fig. 2. The maximum of the correlation time always occurs just above the threshold, and this is related to the slowing down of fluctuations near the phase transition. Detuning evidently lengthens T_c, and moreover its effect on T_c is somewhat larger than on the relative intensity fluctuations $\langle(\Delta I)^2\rangle/\langle I\rangle^2$, although both are increased.

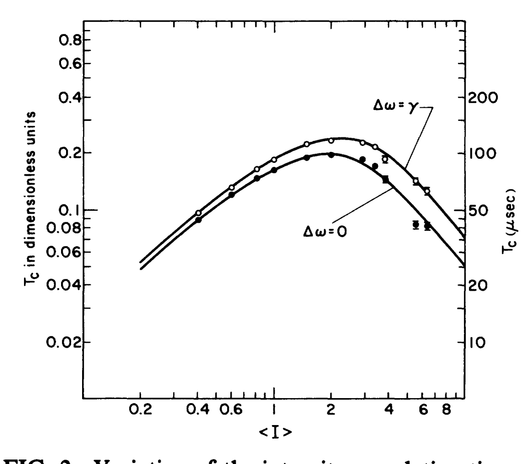

FIG. 2. Variation of the intensity correlation time T_c with mean intensity $\langle I\rangle$ on and off resonance. The full curves are theoretical and the experimental results are shown superimposed. Where the standard deviations are larger than the spot size, they are indicated.

V. EXPERIMENTAL

These theoretical conclusions have been put to the experimental test by photoelectric counting and two-time correlation measurements of a He:Ne laser. Figure 3 shows an outline of the apparatus. The plasma tube is located inside a 20.5-cm long optical cavity, whose axial mode spacing of 730 MHz is large enough to ensure that only one mode is excited up to the largest detuning. The output mirror has about 99% reflectivity and the other, flat mirror is mounted on a piezoelectric cylinder that allows the cavity frequency to be varied. The mirror mounts and the mounts for the plasma tube and aperture are attached to an invar base, and the whole laser is enclosed in a Plexiglass box to ensure acoustic isolation. It is found that once thermal equilibrium has been established the cavity resonance is normally quite stable, and the frequency drift is usually no more than a few MHz in the course of a measurement.

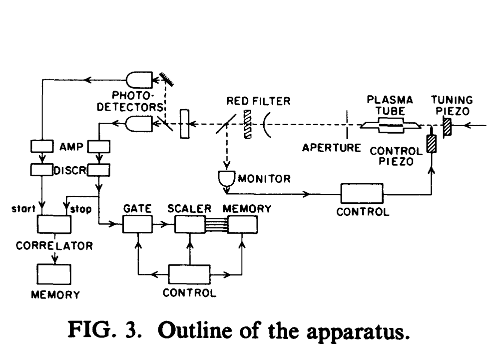

FIG. 3. Outline of the apparatus.

In order to control the working point of the laser, or the pump parameter a, a variable loss element in the form of a piezoelectrically driven knife edge is inserted into the beam. Its displacement by a few microns can change the working point from $a=-10$ below threshold to $a=10$ above threshold. The movable knife edge together with a monitor phototube and control amplifier are made part of a feedback loop, which allows the pump parameter to be controlled and held constant in the range $a=-10$ to 10. The main laser output beam passes through a red filter, an attenuator if necessary, and then falls either on a single photomultiplier, or it is split into two with the aid of a beam splitter and strikes two photomultipliers. The former arrangement is used in photon counting measurements, and the latter in two-time correlation measurements. In both cases the photomultiplier pulses are amplified and converted to standard form by discriminators.

In the photon counting experiments the number n of pulses registered in a standard interval of order 1 μsec is counted by a scaler. This number is then transferred to a computer memory, and the cycle is repeated at intervals of several hundred μsec. The counting interval is made sufficiently short compared with the intensity correlation time of the light, to ensure that the light intensity is effectively constant during the interval. After many thousands of such counting cycles the number of times that n counts are registered provides a measure of the probability $p(n)$ that n photoelectric pulses are produced in a standard measurement interval. After correction for background counts and dead-time effects, the probability $p(n)$ can be used to determine the moments of n. In particular, the mean of n is related to the mean light intensity $\langle I\rangle$ by

$$\langle n\rangle=\alpha\langle I\rangle\ , \tag{43}$$

where α is a scale constant, and

$$\langle n(n-1)\rangle=\alpha^2\langle I^2\rangle\ ,$$

so that

$$\langle n(n-1)\rangle/\langle n\rangle^2-1=\langle(\Delta I)^2\rangle/\langle I\rangle^2\ . \tag{44}$$

The constant α that relates the number of photoelectric counts to the dimensionless light intensity is the only unkown, and the only adjustable parameter in these experiments. As indicated by Eq. (44), the relative intensity fluctuations do not depend on α.

The background counting rate, which is attributable partly to photomultiplier dark current and partly to stray light from the gas discharge tube, was measured separately after effectively extinguishing the laser. This can be achieved either by insertion of a piece of glass into the cavity, or by very large cavity detuning. The background counting rate was never more than about 10% of the total counting rate, and usually very much less, and it varied only by about 1 or 2% as the working point of the laser was varied. The counting dead time was determined essentially by the discriminator pulse width; it was measured electronically and found to be 26 nsec. The procedure for making background and dead-time corrections has already been described several times,[11-13] and will not be repeated here.

In the photoelectric correlation measurements the laser output beam is split into two, as shown in Fig. 3, and the two beams fall on two photodetectors. The standardized output pulses are fed to the start and the stop inputs of a digital correlator,[14] that registers the number of stop pulses $n(\tau_r)$ arriving within a resolution interval $\delta\tau$ after a delay $\tau_r=r\delta\tau$ following the start pulse. The numbers $n(\tau_r)$ are accumulated in 256 memory channels labeled $r=0$ to 255. After many repetitions the average $\langle n(\tau_r)\rangle$ provides a measure of the probability that a photoelectric pulse appears in an interval τ_r to $\tau_r+\delta\tau$ following a start pulse, and this is proportional to the intensity correlation function $\langle I(t)I(t+\tau_r)\rangle$. More precisely, at sufficently low start rates[14]

$$\langle n(\tau)\rangle=N_sR_s\delta\tau[1+\theta_1\theta_2\lambda(\tau)]\ , \tag{45}$$

where N_s is the number of times the correlator is initiated by start pulses, R_2 is the average counting rate at the stop input, and $\lambda(\tau)$ is the normalized intensity correlation function defined by Eq. (39). $1-\theta_1$ and $1-\theta_2$ are the fractions of the photoelectric counting rates in the start and stop channels contributed by background. Details of other corrections which are applicable to the correlator data at high counting rates are given in Ref. 14.

The same basic measurement procedure was followed in both the photon counting and in the two-time correlation experiments. The cavity frequency was first adjusted to line center, as indicated by the fact that the laser output intensity was a maximum. The working point of the laser was set at some level, the feedback loop was closed, and measurements were made. The laser was then detuned by 260 MHz from line center, corresponding to one natural linewidth at a plasma tube pressure of

3.5 torr,[15] and the measurement was repeated. The detuning was performed by changing the voltage across a calibrated piezoelectric crystal. The working point of the laser was then readjusted to various different levels in the neighborhood of threshold, and measurements were carried out on resonance and off resonance each time.

VI. EXPERIMENTAL RESULTS

The experimental values of $\langle(\Delta I)^2\rangle/\langle I\rangle^2$ and $\langle I\rangle$ are shown superimposed on the theoretical curves in Fig. 1. For this purpose the scale constant α in Eq. (43) was chosen for best fit with the data on resonance. α is the only adjustable parameter in these results, and variation of α corresponds to a translation of the data points along the logarithmic $\langle I\rangle$ axis. It will be seen that there is generally good agreement between theory and experiment, except for the two somewhat high points at $\langle I\rangle \simeq 1.5$. This may be the result of a frequency drift in the course of the measurement brought about by some thermal disturbance. Such a frequency drift would cause the dispersion of the light intensity to be larger than expected.

A typical form of the normalized correlation function derived from the correlation measurements is shown in Fig. 4. As in all cases, it is approximately exponential, and the effective correlation time T_c can be extracted by a least-squares procedure. The results of a determination of T_c for each mean light intensity $\langle I\rangle$ on and off resonance are shown superimposed on the theoretical curves in Fig. 2. Fitting of the data involves two adjustable scale constants, one for the light intensity $\langle I\rangle$, and one that converts measured values of T_c to the dimensionless values given by Eq. (41). We have used a double logarithmic plot to facilitate the curve fitting. With one or two exceptions which may again be connected with thermal disturbances, there is generally reasonable agreement between theory and experiment.

VII. CONCLUSIONS

It follows that the effect of detuning an inhomogeneously broadended laser is to change its characteristics in a way that goes beyond a change of the effective pump parameter. Although the laser theory predicts definite relationships between the mean light intensity $\langle I\rangle$, the relative fluctuations $\langle(\Delta I)^2\rangle/\langle I\rangle^2$, and the correlation time T_c, these

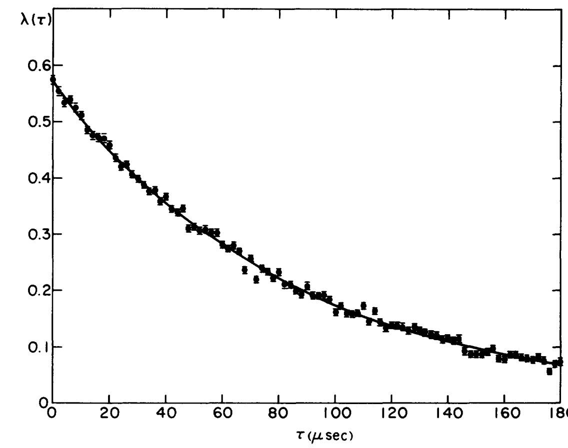

FIG. 4. Results of a two-time correlation measurement. The laser was operated on resonance near the threshold with $\langle I\rangle \approx 1$. The full curve is the best fitting exponential function.

relationships are frequency dependent. The frequency-dependent characteristics should evidently be taken into account whenever the working point of a laser is varied by detuning.

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APPENDIX: FREQUENCY DEPENDENCE OF THE GAIN AND SATURATION COEFFICIENTS

We start from the relations given in Refs. 1 and 2 for the frequency dependence of the gain coefficient $\tilde{A}$ and the saturation coefficient $\tilde{B}$, which are complex functions of frequencies in general. The dependence on detuning η is expressible in terms of the plasma dispersion function $Z(y)$ of complex argument y [Eq. (4)] in the form[2]

$$\tilde{A} \propto Z(\rho+i\eta\rho)\,, \tag{A1}$$

$$\tilde{B} \propto \left[\frac{Z(\rho+i\eta\rho)}{\rho+i\eta\rho} - \frac{dZ(\rho+i\eta\rho)}{d(\rho+i\eta\rho)} + \frac{iZ_r(\rho+i\eta\rho)}{\eta\rho} + \frac{iZ_i(\rho+i\eta\rho)}{\rho}\right]. \tag{A2}$$

$Z_r(y)$ and $Z_i(y)$ stand for the real and the imaginary parts of $Z(y)$, and we observe that for real ρ, $Z(\rho)=iZ_i(\rho)$, and $Z_r(\rho)=0$. Now

$$Z(\rho+i\eta\rho)=Z_r(\rho+i\eta\rho)+iZ_i(\rho+i\eta\rho)$$
$$=iZ_i(\rho)\left[\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}-i\frac{Z_r(\rho+i\eta\rho)}{Z_i(\rho)}\right],$$

and the expression in large parentheses becomes unity when $\eta=0$. It follows from Eq. (A1) that we must have

$$\tilde{A}=A\left[\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}-\frac{iZ_r(\rho+i\eta\rho)}{Z_i(\rho)}\right], \tag{A3}$$

where A is the gain coefficient on resonance ($\eta=0$).

We proceed in a similar manner for B. From Eq. (4),

$$dZ(y)/dy=2yZ(y)-2i\ , \tag{A4}$$

so that

$$\tilde{B}\propto\left[\frac{Z(\rho+i\eta\rho)}{\rho+i\eta\rho}-\frac{dZ(\rho+i\eta\rho)}{d(\rho+i\eta\rho)}+\frac{iZ_r(\rho+i\eta\rho)}{\eta\rho}+\frac{iZ_i(\rho+i\eta\rho)}{\rho}\right]$$

$$=\frac{iZ_i(\rho+i\eta\rho)+Z_r(\rho+i\eta\rho)}{\rho(1+\eta^2)}(1-i\eta)-2\rho(1+i\eta)[iZ_i(\rho+i\eta\rho)+Z_r(\rho+i\eta\rho)]+2i+\frac{iZ_r(\rho+i\eta\rho)}{\eta\rho}+\frac{iZ_i(\rho+i\eta\rho)}{\rho}$$

$$=\frac{iZ_i(\rho+i\eta\rho)}{\rho}\left\{1+\frac{2\rho}{Z_i(\rho+i\eta\rho)}-2\rho^2-2\rho^2\eta\frac{Z_r(\rho+i\eta\rho)}{Z_i(\rho+i\eta\rho)}+\left[\frac{1}{1+\eta^2}\right]\left[1+\frac{1}{\eta}\frac{Z_r(\rho+i\eta\rho)}{Z_i(\rho+i\eta\rho)}\right]\right.$$
$$\left.-i\left[2\rho^2\left[\eta-\frac{Z_r(\rho+i\eta\rho)}{Z_i(\rho+i\eta\rho)}\right]+\left[\frac{1}{1+\eta^2}\right]\left[\eta+\frac{Z_r(\rho+i\eta\rho)}{Z_i(\rho+i\eta\rho)}\right]\right]\right\}$$

$$=2i\frac{Z_i(\rho)}{\rho}\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\left\{\left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2}\right]\left[1+\frac{2\rho}{Z_i(\rho+i\eta\rho)}\right]+\frac{Z_r(\rho+i\eta\rho)/2\eta-\rho}{(1+\eta^2)Z_i(\rho+i\eta\rho)}\right.$$
$$-\rho^2\left[1+\frac{\eta Z_r(\rho+i\eta\rho)}{Z_i(\rho+i\eta\rho)}\right]-i\eta\left[\rho^2\left[1-\frac{Z_r(\rho+i\eta\rho)}{\eta Z_i(\rho+i\eta\rho)}\right]\right.$$
$$\left.\left.+\left[\frac{\frac{1}{2}}{1+\eta^2}\right]\left[1+\frac{Z_r(\rho\ i\eta+\rho)}{\eta Z_i(\rho+i\eta\rho)}\right]\right]\right\}. \tag{A5}$$

In order to determine the effect of a small detuning such that $\eta\rho<1$, we now use the definition (4) to make a Taylor series expansion of $Z(\rho+i\eta\rho)$ about ρ. We then find that

$$Z(\rho+i\eta\rho)=Z(\rho)+i\eta\rho\left[\frac{dZ(y)}{dy}\right]_{y=\rho}+\frac{(i\eta\rho)^2}{2!}\left[\frac{d^2Z(y)}{dy^2}\right]_{y=\rho}+\cdots$$
$$=Z(\rho)(1+2i\eta\rho^2-\eta^2\rho^2-2\eta^2\rho^4-2i\eta^3\rho^4+\tfrac{1}{2}\eta^4\rho^4)+2\eta\rho+2i\eta^2\rho^3-\tfrac{4}{3}\eta^3\rho^3+O(\rho^5)$$
$$=Z(\rho)\{1-\eta^2\rho^2[1-2\rho/|Z(\rho)|+2\rho^2-\tfrac{1}{2}\eta^2\rho^2]\}$$
$$+2\eta\rho[1-\rho|Z(\rho)|-\tfrac{2}{3}\eta^2\rho^2+\eta^2\rho^3|Z(\rho)|]+O(\rho^5)\ .$$

The real and imaginary parts of $Z(\rho+i\eta\rho)$ are therefore given by

$$Z_r(\rho+i\eta\rho)=2\eta\rho[1-\rho|Z(\rho)|-\tfrac{2}{3}\eta^2\rho^2+\eta^2\rho^3|Z(\rho)|]+O(\rho^5)\ ,$$
$$Z_i(\rho+i\eta\rho)=|Z(\rho)|\{1-\eta^2\rho^2[1-2\rho/|Z(\rho)|+2\rho^2-\tfrac{1}{2}\eta^2\rho^2]\}+O(\rho^5)\ . \tag{A6}$$

Both square brackets contain polynomials in η and ρ that tend to unity as $\rho \to 0$.

We now insert Eq. (A6) in Eq. (A5), and obtain after some manipulation

$$\widetilde{B} \propto 2i\frac{Z_i(\rho)}{\rho}\left[1+\frac{2\rho}{|Z(\rho)|}-2\rho^2\right]\left\{\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2}+O(\rho^3)-i\eta\left[\frac{\frac{1}{2}}{1+\eta^2}+\rho^2+O(\rho^3)\right]\right]\right\}. \quad \text{(A7)}$$

We observe that the term in curly braces becomes 1 when $\eta=0$, and that the factor outside the braces is independent of η. The term in braces therefore describes the frequency dependence of $\widetilde{B}$, and it follows from Eqs. (A2) and (A7) that

$$\widetilde{B}=B\frac{Z_i(\rho+i\eta\rho)}{Z_i(\rho)}\left[\frac{1+\frac{1}{2}\eta^2}{1+\eta^2}-i\eta\left[\frac{\frac{1}{2}}{1+\eta^2}+\rho^2\right]+O(\rho^3)\right], \quad \text{(A8)}$$

with the ratio $Z_i(\rho+i\eta\rho)/Z_i(\rho)$ given by Eq. (A6). Also from Eqs. (A3) and (A6) we have

$$\widetilde{A}=A\left[1-\eta^2\rho^2\left[1-\frac{2\rho}{|Z(\rho)|}+2\rho^2-\frac{1}{2}\eta^2\rho^2+\cdots\right]\right.$$
$$\left.-\frac{2i\eta\rho}{|Z(\rho)|}\left[1-\rho|Z(\rho)|-\frac{2}{3}\eta^2\rho^2+\eta^2\rho^3|Z(\rho)|+\cdots\right]\right] \quad \text{(A9)}$$

and

$$\frac{Z_i(\rho)}{Z_i(\rho+i\eta\rho)}=1+\eta^2\rho^2\left[1-\frac{2\rho}{|Z(\rho)|}+2\rho^2+\frac{1}{2}\eta^2\rho^2+O(\rho^3)\right]. \quad \text{(A10)}$$

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