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Coherence and photon statistics for optical fields generated by Poisson random emissions

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We examine the coherence properties and photon statistics of stationary light obtained by the superposition of nonstationary emissions occurring at random times, in accordance with a homogeneous Poisson point process. The individual emissions are assumed to be in a coherent, chaotic, or n state. The statistical nature of the emission times results in fluctuations of the relative contributions of different emissions at a given time. This is manifested by an additional positive term, exhibiting particlelike properties, in the normalized second-order correlation function. Thus, the photon-counting variance is increased. For coherent emissions, interference between the randomly delayed emissions produces additional wavelike noise. In the limit when the emissions overlap strongly, the field exhibits the correlation properties of chaotic light, regardless of the statistics of the individual emissions. In the opposite limit, when emissions seldom overlap, the light intensity is describable by a shot-noise stochastic process, and the detected photocounts show an enhanced particlelike noise, which has its largest value when the counting time is long. In that limit, the photocounts obey the Neyman type-*A* and generalized Polya-Aeppli distributions, when the individual emissions are coherent and chaotic, respectively. When the individual emissions correspond to the n state, the Poisson emission times result in bunching which reduces or eliminates the inherent antibunching associated with the n state.

I. INTRODUCTION

Since 1956, when Hanbury-Brown and Twiss observed correlation in the fluctuations of two photoelectric currents induced by thermal light[1] and by starlight,[2] the coherence properties of optical fields have been studied intensively, from both a theoretical and an experimental point of view.[3–6] The usual kinds of light that have been investigated are chaotic (thermal) light, coherent (laser) light, and mixtures of both.[4–6] More recently, the fluctuation properties of antibunched light have received considerable attention.[7–9]

Fluctuations in the overall number of active radiators in a source of light can be an important determinant of its coherence properties, as pointed out by Forrester,[10(a)] and discussed by Loudon.[10(b)] This effect is of central importance for scattered light, where the number of active radiators is a stochastic quantity.[11] Such fluctuations also play a role in the generation of antibunched resonance fluorescence, as discussed by Carmichael *et al.*,[12,13] Jakeman *et al.*,[14] and Mandel *et al.*[9,15]

In this paper, we examine the coherence and fluctuation properties of optical fields when the times of emission of the individual radiators are describable by a homogeneous Poisson process. In particular, we calculate the first- and second-order field correlation functions for such light in the framework of semiclassical theory and quantum electrodynamics. The mean and variance of the photon count are also obtained. In certain limits, expressions for the photon-counting distributions are derived. We consider in detail a number of special cases, including individual atomic emissions modeled by coherent, chaotic, and number-state descriptions.

Our model is expected to play an important role

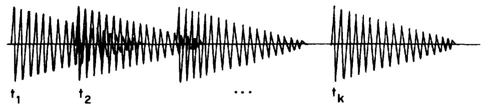

FIG. 1. Schematic representation of $E(t)$ for nonstationary coherent emissions occurring at the Poisson times $\{t_k\}$.

in a number of diverse applications. Consider, for example, the light emitted when a stream of energetic electrons impinges on a radioluminescent material. If the times at which the electrons strike the material occur in accordance with the Poisson process, this randomness will be imparted to the emitted optical field. This effect will be particularly evident when the electron current is low (its fluctuations are then the largest). These fluctuations are in addition to those intrinsic to the optical field.[12]

In previous work,[16,17] we presented a semiclassical analysis of phenomena similar to those described above, under the assumption that the *irradiance* could be modeled as shot noise. In this paper, we provide a more complete analysis of this class of processes when the optical field, rather than the irradiance, is modeled as a shot-noise process. Thus, intrinsic field as well as intensity fluctuations are incorporated into our model.

II. SEMICLASSICAL MODEL

A. Coherent emissions

Consider an optical field composed of a sequence of independent pulses centered around random times $\{t_k\}$. Let each pulse correspond to a quasi-monochromatic field represented by a deterministic, time-decaying, bandpass, complex analytic signal $\epsilon_0(t)$. The complex analytic signal of the total field is then the sum

$$E(t)=\sum_k \epsilon_0(t-t_k)\ , \tag{1}$$

where the $\{t_k\}$ are assumed to be realizations of a homogeneous Poisson point process of rate μ (see Fig. 1). The field $E(t)$ can be regarded as a stationary shot-noise process produced by a linear filter having a complex impulse response function $\epsilon_0(t)$. We are interested in determining the statistics of the field $E(t)$ and its corresponding intensity (irradiance) $I(t)=|E(t)|^2$.

By use of the properties of shot-noise processes,[18,19] it can be shown that the mean value of the field is

$$\langle E(t)\rangle=\mu\int_{-\infty}^{\infty}\epsilon_0(t)dt=0\ , \tag{2}$$

and the mean value of the intensity is

$$\langle I(t)\rangle=\mu\int_{-\infty}^{\infty}h(t)dt\ , \tag{3}$$

where

$$h(t)=|\epsilon_0(t)|^2\ . \tag{4}$$

The function $h(t)$ is decaying and real, and represents the intensity of an individual pulse. In deriving Eq. (2), we have used the fact that $\epsilon_0(t)$ is a bandpass (narrow-band) function.

1. Field correlations

The correlation function of the field $G^{(1)}(\tau)$ can also be determined through the use of the properties of shot-noise processes.[19] Thus,

$$G^{(1)}(\tau)=\langle E^*(t)E(t+\tau)\rangle$$
$$=\mu\int_{-\infty}^{\infty}\epsilon_0^*(t)\epsilon_0(t+\tau)dt\ , \tag{5}$$

which is associated with a normalized (first-order) correlation function

$$g^{(1)}(\tau)=\frac{G^{(1)}(\tau)}{G^{(1)}(0)}=\frac{\int_{-\infty}^{\infty}\epsilon_0^*(t)\epsilon_0(t+\tau)dt}{\int_{-\infty}^{\infty}|\epsilon_0(t)|^2dt}\ . \tag{6}$$

The optical field has a power spectral density determined by

$$S(\omega)=\mu|\tilde{\epsilon}_0(\omega)|^2\ , \tag{7}$$

where $\tilde{\epsilon}_0(\omega)$ is the Fourier transform of $\epsilon_0(t)$. The power spectral density is determined completely by the shape of the decaying individual pulses. For example, if

$$\epsilon_0(t)\propto\exp(j\omega_0 t)\exp(-t/\tau_p),\quad t>0$$

and zero otherwise, $S(\omega)$ displays a Lorentzian spectrum centered about ω_0. It can also be shown that

$$\langle E(t)E(t+\tau)\rangle=0\ .$$

2. Intensity correlations

Again by use of the properties of a squared shot-noise process,[19] we readily determine the normalized intensity (second-order) correlation function

$$g^{(2)}(\tau)=\langle I(t)I(t+\tau)\rangle/\langle I(t)\rangle^2$$
$$=1+|g^{(1)}(\tau)|^2+\frac{1}{\mu\tau_p}\eta(\tau)\ , \tag{8}$$

where

$$\eta(\tau)=\int_{-\infty}^{\infty} h(t)h(t+\tau)dt \Big/ \int_{-\infty}^{\infty} h^2(t)dt\ , \tag{9}$$

and

$$\tau_p=\left[\int_{-\infty}^{\infty} h(t)dt\right]^2 \Big/ \int_{-\infty}^{\infty} h^2(t)dt \tag{10}$$

is a time representing the characteristic decay width of a single pulse. For $\tau=0$ we obtain the normalized mean square of the intensity fluctuations

$$\langle I^2\rangle/\langle I\rangle^2=g^{(2)}(0)=2+1/\mu\tau_p\ . \tag{11a}$$

With the comparison of Eq. (8) and the result for chaotic light,[5]

$$g^{(2)}(\tau)=1+|g^{(1)}(\tau)|^2\ ,$$

it is clear that shot-noise coherent light exhibits chaotic fluctuations, manifested by the first two terms on the right-hand side of Eq. (8), together with particlelike fluctuations manifested by the third term. This latter contribution is directly proportional to $\eta(\tau)$, the normalized autocorrelation function of the intensity of an individual pulse. It is also seen to be inversely proportional to $\mu\tau_p$, the average number of flashes (pulses) per lifetime of a flash. This third term is therefore significant when $\mu\tau_p \ll 1$, i.e., when the flashes are sparse and do not frequently overlap. This result is similar to that obtained by Loudon.[10(b)]

On the other hand, when $\mu\tau_p \gg 1$, i.e., when the pulses overlap strongly,

$$g^{(2)}(\tau)\to 1+|g^{(1)}(\tau)|^2\ ,$$

as in the case of chaotic light.[12] This limit arises because the Poisson arrival times lead to interference between the coherent pulse trains.

Earlier studies[11] on the fluctuations of light composed of a random number N of waves, having constant amplitude and statistically independent, uniformly distributed phases, led to the result

$$g^{(2)}(0)=2+1/\langle N\rangle \tag{11b}$$

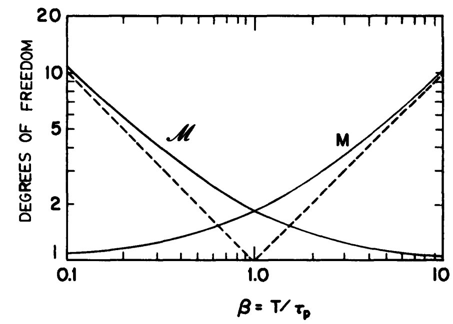

FIG. 2. Dependence of the degrees-of-freedom parameters M and $\mathscr{M}$ on the ratio $\beta=T/\tau_p$ for exponentially decaying coherent emissions. Dashed lines represent unity slope.

when N was assumed to have a Poisson distribution.[10(b),11(a)] Interpreting $\mu\tau_p$ as the average number of independent contributions at a given time, i.e., letting $\langle N\rangle=\mu\tau_p$, we see that Eq. (11a) reproduces Eq. (11b). This is not surprising in view of the strong underlying similarity between the two models.

3. Photon-count mean and variance

Let n be the number of photoelectrons released by such light in the time interval $[0,T]$. The statistics of n may be determined from the statistics of the integrated intensity $W=\int_0^T I(t)dt$, through the use of the usual techniques.[20] The mean value of n is given by

$$\langle n\rangle=\langle W\rangle=T\langle I(t)\rangle=\mu\alpha T\ , \tag{12}$$

where

$$\alpha=\int_{-\infty}^{\infty} h(t)dt \tag{13}$$

is the integrated intensity of a single pulse. Without loss of generality, we have assumed that the quantum efficiency of detection is unity. The effects of nonunity quantum efficiency may be easily included.[21] The variance of n is determined by use of[5,6,20]

$$\mathrm{var}(n)=\langle n\rangle+\mathrm{var}(W)=\langle n\rangle+\frac{\langle n\rangle^2}{T^2}\int_0^T\int_0^T[g^{(2)}(t_2-t_1)-1]dt_1dt_2\ . \tag{14}$$

Using Eq. (8), we obtain

$$\mathrm{var}(n)=\left[1+\frac{\alpha}{\mathscr{M}}\right]\langle n\rangle+\frac{1}{M}\langle n\rangle^2\ , \tag{15}$$

where

$$M^{-1}=\frac{2}{T}\int_0^T(1-\tau/T)|g^{(1)}(\tau)|^2d\tau \tag{16}$$

and

$$\mathscr{M}^{-1}=\frac{2}{\tau_n}\int_0^T(1-\tau/T)\eta(\tau)d\tau\ . \tag{17}$$

The parameter M is the degrees of freedom[22,6,16] for a chaotic field with first-order correlation function $g^{(1)}(\tau)$, whereas the parameter $\mathscr{M}$ is the degrees of freedom for shot-noise-intensity light.[16,17] The dependence of M and $\mathscr{M}$ on the ratio $\beta=T/\tau_p$ is illustrated in Fig. 2 for the case when

$$\epsilon_0(t)=(2\alpha/\tau_p)^{1/2}e^{-t/\tau_p}e^{j\omega_0 t},\quad t>0$$
$$=0,\quad \text{elsewhere}\,. \qquad (18)$$

The quantities M and $\mathscr{M}$ are then given by[16]

$$M=2\beta^2/(e^{-2\beta}+2\beta-1) \qquad (19)$$

and

$$\mathscr{M}=2\beta/(e^{-2\beta}+2\beta-1) \qquad (20)$$

with $\beta=T/\tau_p$. When $T\ll\tau_p$, $M\approx 1$ and $\mathscr{M}\approx\tau_p/T\gg 1$. As T/τ_p increases, M increases while $\mathscr{M}$ decreases. In the limit $T\gg\tau_p$, $M\approx T/\tau_p\gg 1$ and $\mathscr{M}\approx 1$.

The ratio T/τ_p affects the count variance dramatically. For $T\ll\tau_p$,

$$\text{var}(n)=\langle n\rangle+\langle n\rangle^2\,, \qquad (21)$$

as for the Bose-Einstein distribution, which characterizes a single-mode chaotic field. For $T\gg\tau_p$,

$$\text{var}(n)=(1+\alpha)\langle n\rangle\,, \qquad (22)$$

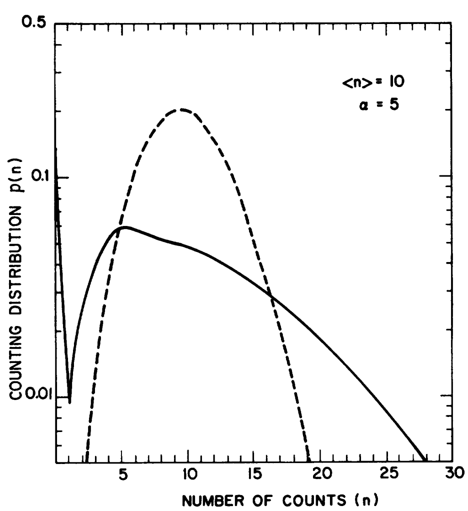

FIG. 3. Photon-counting distribution $p(n)$ vs count number n for the SNDP. Solid curve represents the Neyman type-A counting distribution ($\langle n\rangle=10$, $\alpha=5$) which arises in the limit $T/\tau_p\gg 1$. Dashed curve represents the Poisson counting distribution ($\langle n\rangle=10$) which arises in the limit $T/\tau_p\ll 1$.

as for the Neyman type-A distribution which characterizes shot-noise light.[16] Intermediate values of T/τ_p correspond to light in which the chaotic and the shot-noise behavior are mixed.

4. Photon-counting distribution

In general, it is difficult to determine the probability distribution of the intensity $I(t)$. In certain limits this is possible, however.

In the limit $\mu\tau_p\ll 1$, viz., when pulses do not overlap, we can write Eq. (1) in the form

$$I(t)=\sum_k I_0(t-t_k)\,,$$

where

$$I_0(t)=|\epsilon_0(t)|^2\,.$$

In this case $I(t)$ itself becomes a shot-noise process. Its statistics are then well known,[18,19] as are the corresponding photon-counting statistics which form a shot-noise-driven doubly stochastic Poisson point process (SNDP), and have been recently studied in great detail.[16,17] In Fig. 3, we present a plot of the theoretical photon-counting distribution $p(n)$ versus the count number n. The solid curve represents the Neyman type-A, which is the limiting counting distribution for the SNDP when $\beta=T/\tau_p\gg 1$. The dashed curve is the Poisson, which is the appropriate counting distribution for the SNDP in the limit $\beta\ll 1$. In both cases, the overall mean count $\langle n\rangle=10$. For the Neyman type-A, the multiplication parameter $\alpha=5$. Counting statistics for arbitrary β, as well as time statistics and multifold statistics, are displayed in Ref. 17.

In the opposite limit, $\mu\tau_p\gg 1$, $E(t)$ approaches a complex Gaussian process, characteristic of a chaotic field.[23] Properties of the intensity fluctuations of chaotic light have been studied extensively.[4–6] The photon statistics[6,20,22,24,25] are well described by the negative-binomial distribution.[26]

B. Chaotic emissions

We now consider an optical field composed of a sequence of independent flashes of light, emitted at random times $\{t_k\}$, in accordance with a homogeneous Poisson process of rate μ. We assume that each flash of light is a realization of a time-decaying chaotic optical field. We let $\epsilon_0(t)$ be the complex analytic signal of the optical field of the flash emitted at $t=0$ and take it to be a nonstationary, complex Gaussian, circularly symmetric, random process. We write the total field as

$$E(t)=\sum_k \epsilon_k(t-t_k)\ , \tag{23}$$

where the $\epsilon_k(t)$ are statistically independent realizations of the process $\epsilon_0(t)$. Equation (23) indicates that the field $E(t)$ is a shot-noise process, obtained when a Poisson-distributed random series of impulses is passed through a random linear filter, with a complex-Gaussian impulse response function (see Fig. 4).

The correlation functions of light described by this model can be determined by appropriate averaging over the fluctuations of the field, for the individual flashes, and over the fluctuations of their times of occurrence. Let us first write some of the statistical moments of the field produced by a single flash. Because of circular symmetry, the mean value of $\epsilon_0(t)$ vanishes. The mean intensity is assumed to be a decaying function of time,

$$\langle\, |\epsilon_0(t)|^2 \rangle \equiv h(t)\ . \tag{24}$$

The field correlation function is

$$\langle \epsilon_0^*(t)\epsilon_0(t+\tau)\rangle \equiv R(t,\tau)\ , \tag{25}$$

where, of course, $h(t)=R(t,0)$. By using the Gaussian property of the field, we can expand its higher-order moments so that, e.g.,[27]

$$\langle \epsilon_0^*(t_1)\epsilon_0(t_2)\epsilon_0^*(t_3)\epsilon_0(t_4)\rangle = R(t_1,t_2-t_1)R(t_3,t_4-t_3)+R(t_1,t_4-t_1)R(t_3,t_2-t_3)\ . \tag{26}$$

1. *Field correlations*

We are now in a position to determine the correlation functions of the total field (which is stationary) by using Eqs. (24) and (25) and the properties of Poisson processes.[18–20,23] This readily gives the average intensity

$$\langle I(t)\rangle = \mu\int_{-\infty}^{\infty} R(t,0)dt = \mu\int_{-\infty}^{\infty} h(t)dt\ , \tag{27}$$

the first-order correlation function of the field

$$G^{(1)}(\tau)=\langle E^*(t)E(t+\tau)\rangle = \mu\int_{-\infty}^{\infty} R(t,\tau)dt\ , \tag{28}$$

and the normalized first-order correlation function

$$g^{(1)}(\tau)=\int_{-\infty}^{\infty} R(t,\tau)dt \Big/ \int_{-\infty}^{\infty} R(t,0)dt\ . \tag{29}$$

2. *Intensity correlations*

The intensity correlation function is

$$G^{(2)}(\tau)=\langle I(t)I(t+\tau)\rangle = \mu^2\left[\left[\int_{-\infty}^{\infty} R(t,0)dt\right]^2 + \left|\int_{-\infty}^{\infty} R(t,\tau)dt\right|^2\right] + \mu\left[\int_{-\infty}^{\infty} R(t,0)R(t+\tau,0)dt + \int_{-\infty}^{\infty} |R(t,\tau)|^2 dt\right]\ , \tag{30}$$

whereas its normalized version is

$$g^{(2)}(\tau)=1+|g^{(1)}(\tau)|^2 + \frac{1}{\mu\tau_p}[\eta(\tau)+\xi(\tau)]\ , \tag{31}$$

where

$$\eta(\tau)=\frac{\int_{-\infty}^{\infty} h(t)h(t+\tau)dt}{\int_{-\infty}^{\infty} h^2(t)dt}\ ,$$

$$\xi(\tau)=\frac{\int_{-\infty}^{\infty} |R(t,\tau)|^2 dt}{\int_{-\infty}^{\infty} |R(t,0)|^2 dt}\ , \tag{32}$$

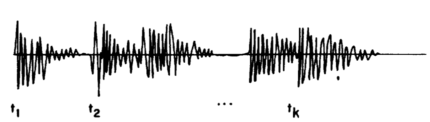

FIG. 4. Schematic representation of $E(t)$ for nonstationary chaotic emissions occurring at the Poisson times $\{t_k\}$.

and

$$\tau_p=\frac{\left[\int_{-\infty}^{\infty} h(t)dt\right]^2}{\int_{-\infty}^{\infty} h^2(t)dt} . \tag{33}$$

This is to be compared with the expression for ordinary chaotic light

$$g^{(2)}(\tau)=1+|g^{(1)}(\tau)|^2 .$$

From Eq. (31) we see that the quantity $g^{(2)}(\tau)$ for shot-noise chaotic light is larger than that for chaotic light by a term which is inversely proportional to the number of flashes per flash lifetime $\mu\tau_p$. This corresponds to an additional bunching comprised of one contribution dependent on the shape of the pulse $[\eta(\tau)]$, as for shot-noise coherent light, and another dependent on the spectral properties of the field $[\xi(\tau)]$. This latter contribution is, of course, absent from shot-noise coherent light.

The normalized mean square of the intensity fluctuations is as follows:

$$\langle I^2\rangle/\langle I^2\rangle=g^{(2)}(0)=2(1+1/\mu\tau_p) . \tag{34}$$

This is to be compared with ordinary chaotic light, for which $g^{(2)}(0)=2$, and with shot-noise coherent light, for which $g^{(2)}(0)=2+1/\mu\tau_p$. Again, for $\mu\tau_p \gg 1$, the results reduce to those for ordinary chaotic light.

An interesting special case is that in which the correlation function of the light for individual pulses factors into the form

$$R(t,\tau)=h(t)g(\tau) . \tag{35}$$

Equations (28)—(32) then yield

$$g^{(1)}(\tau)=g(\tau) , \tag{36}$$

$$\eta(\tau)=\frac{\int_{-\infty}^{\infty} h(t)h(t+\tau)dt}{\int_{-\infty}^{\infty} h^2(t)dt} ,$$
$$\xi(\tau)=|g(\tau)|^2=|g^{(1)}(\tau)|^2 , \tag{37}$$

$$g^{(2)}(\tau)=1+|g^{(1)}(\tau)|^2 + \frac{1}{\mu\tau_p}[\eta(\tau)+|g^{(1)}(\tau)|^2] . \tag{38}$$

A specific example is that of exponentially decaying pulses with Lorenzian spectrum

$$h(t)=\frac{2\alpha}{\tau_p}e^{-2t/\tau_p},\ g(\tau)=e^{-\tau/\tau_c} , \tag{39}$$

where $1/\tau_c$ is the spectral bandwidth. This corresponds to $\eta(\tau)=e^{-2\tau/\tau_p}$ and

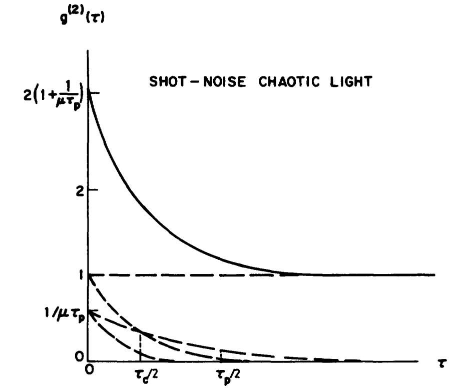

FIG. 5. Intensity correlation function $g^{(2)}(\tau)$ vs τ (solid curve) for shot-noise chaotic light with exponential pulse decay and Lorentzian spectrum. Note that $g^{(2)}(0)=2(1+1/\mu\tau_p)$ and $g^{(2)}(\infty)=1$. Dashed curves represent the various contributions to $g^{(2)}(\tau)$ as indicated in Eq. (40).

$$g^{(1)}(\tau)=e^{-\tau/\tau_c} ,$$
$$g^{(2)}(\tau)=(1+e^{-2\tau/\tau_c}) + \frac{1}{\mu\tau_p}(e^{-2\tau/\tau_p}+e^{-2\tau/\tau_c}) . \tag{40}$$

The intensity correlation function $g^{(2)}(\tau)$ is illustrated as the solid curve in Fig. 5. The individual contributions in Eq. (40) are shown as dashed curves. The Gaussian-Markovian condition, in which $\epsilon_0(t)$ satisfies the stochastic differential equation

$$d\epsilon_0(t)=(1/\tau_c)\epsilon_0(t)dt+dw(t) ,$$

where $w(t)$ is a Wiener process, provides a special case of the above example in which $\tau_c=\tau_p$.

3. *Photon-count mean and variance*

The mean and variance of the number of photoelectrons in the counting time T may be determined through the use of Eqs. (12) and (14). Substituting Eq. (31) into Eq. (14), we obtain

$$\text{var}(n)=\left[1+\alpha\left[\frac{1}{\mathscr{M}}+\frac{1}{\mathscr{M}_\xi}\right]\right]\langle n\rangle+\frac{1}{M}\langle n\rangle^2 , \tag{41}$$

where M and $\mathscr{M}$ are given by Eqs. (16) and (17), respectively, and

$$\mathscr{M}_\xi^{-1}=\frac{2}{\tau_p}\int_0^T(1-\tau/T)\xi(\tau)d\tau . \tag{42}$$

Again, the parameter M is the degrees of freedom for chaotic or wave-fluctuation noise, $\mathcal{M}$ is the degrees of freedom for the particle fluctuation noise, and $\mathcal{M}_\xi$ is a mixed degrees-of-freedom parameter.

For emission in which the spectrum is Lorentzian, and the envelope is decaying exponentially [i.e., when Eqs. (35) and (39) are satisfied], we obtain

$$M=2\theta^2/(e^{-2\theta}+2\theta-1),\quad \theta=T/\tau_c \tag{43}$$

$$\mathcal{M}=2\beta/(e^{-2\beta}+2\beta-1),\quad \beta=T/\tau_p \tag{44}$$

$$\mathcal{M}_\xi=2\theta/(e^{-2\theta}+2\theta-1),\quad \theta=T/\tau_c\ . \tag{45}$$

The parameters M and $\mathcal{M}_\xi$ are determined by the spectrum, whereas $\mathcal{M}$ is determined by the pulse envelope. However, the dependence of $\mathcal{M}$ and $\mathcal{M}_\xi$ on T/τ_p and T/τ_c, respectively, are similar, both being opposite to the dependence of M on T/τ_c. When T/τ_c is large, M is large and $\mathcal{M}_\xi=1$, so that

$$\mathrm{var}(n)=\left[1+\alpha\left[1+\frac{1}{\mathcal{M}}\right]\right]\langle n\rangle\ . \tag{46}$$

When T/τ_p is also large, $\mathrm{var}(n)=(1+2\alpha)\langle n\rangle$, which should be compared with Eq. (22). When *both* T/τ_c and T/τ_p are small, additional particle fluctuations are cut apart and chaotic behavior results, so that

$$\mathrm{var}(n)=\langle n\rangle+\langle n\rangle^2\ . \tag{47}$$

4. Photon-counting distribution

The intensity probability distribution and the photon statistics for light described by this model are difficult to determine. One limit in which the photon-counting statistics can be approximated is that of sparse pulses ($\mu\tau_p \ll 1$). The light intensity can then be written as

$$I(t)=\sum_k I_k(t-t_k)\ , \tag{48}$$

where I_k represents the intensity of a nonstationary chaotic light pulse. If the counting time T is much larger than τ_p, then we have a Poisson-distributed number of pulses.

If the pulses are assumed to have rectangular profiles, and if the field is assumed to be chaotic, the number of photons in a single pulse will be approximately described by the negative-binomial distribution.[6,20,22,26] It has been previously shown[28] that the number of photons in an exponentially decaying pulse of Lorentzian spectrum may also be well approximated by the negative-binomial distribution. The total number of collected photons must then follow the generalized Polya-Aeppli[16,29] counting

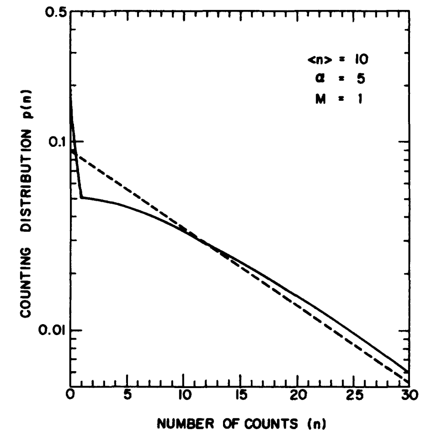

FIG. 6. Generalized Polya-Aeppli photon-counting distribution $p(n)$ vs n. Solid curve represents a special case, the simple Polya-Aeppli, for which $M=1$ ($\langle n\rangle=10$, $\alpha=5$). Dashed curve represents the Bose-Einstein counting distribution ($\langle n\rangle=10$) which arises in the limit of very short counting time.

distribution, which has the moment-generating function

$$\langle e^{-sn}\rangle=\exp\left[\frac{\langle n\rangle}{\alpha}\{[1-\alpha(e^{-s}-1)]^{-M}-1\}\right]\ .$$

The special case of a simple Polya-Aeppli distribution ($M=1$) is illustrated in Fig. 6.

III. QUANTUM-ELECTRODYNAMIC MODEL

In this section we develop a quantum-mechanical model for the optical field with features analogous to those of the semiclassical model described earlier. Consider an optical field generated by a sequence of emissions, at random times $\{t_k\}$, in accordance with a homogeneous Poisson point process. The positive-frequency part[3,4] of the field, which is now an operator, may be written as the sum

$$\hat{E}^+(t)=\sum_k \hat{E}_k^+(t-t_k)\ , \tag{49}$$

where $\hat{E}_k^+(t)$, the positive-frequency part of the field for the kth emission, is itself composed of a number of modes

$$\hat{E}_k^+(t)=\sum_l c_l(t)\hat{a}_{k,l}\ . \tag{50}$$

Here, $\hat{a}_{k,l}$ is the annihilation operator for the lth mode and the kth emission. The time dependence of each mode is determined by the coefficients $c_l(t)$, which are chosen such that the field of a single emission is nonstationary (time decaying).

We wish to determine the first- and second-order correlation functions of the field.[3,4] We shall first calculate these functions, conditioned on a given realization of the emission times $\{t_k\}$, by using the relations

$$G^{(1)}(t,t+\tau\mid\{t_k\})=\mathrm{tr}[\hat{\rho}\hat{E}^{-}(t)\hat{E}^{+}(t+\tau)]\ , \qquad (51)$$

$$G^{(2)}(t,t+\tau\mid\{t_k\})=\mathrm{tr}[\hat{\rho}\hat{E}^{-}(t)\hat{E}^{-}(t+\tau)\hat{E}^{+}(t+\tau)\hat{E}^{+}(t)]\ , \qquad (52)$$

where $\hat{\rho}$ is the density operator of the field and $\hat{E}^{-}(t)$ is the operator representing the negative-frequency part of the field. We shall then average Eqs. (51) and (52) over the fluctuations of $\{t_k\}$ to finally obtain the correlation functions

$$G^{(j)}(\tau)=\langle G^{(j)}(t,t+\tau\mid\{t_k\})\rangle,\ \ j=1,2 \qquad (53)$$

where $\langle\ \rangle$ represents the classical ensemble average over the fluctuations of $\{t_k\}$. The aforementioned classical ensemble average can be thought of as reflecting an indeterminacy of the initial state of the emitter which can be described by a suitable initial density operator for the combined matter-field system. The problem cannot be pursued further unless $\hat{\rho}$ is specified. We therefore consider three cases below.

A. Coherent state

Assuming that the system is in a coherent state

$$\hat{\rho}=\mid\{\alpha_{k,l}\}\rangle\langle\{\alpha_{k,l}\}\mid\ ,$$

and choosing $\alpha_{k,l}=\alpha_l$ (i.e., the state of different emissions are identical), we obtain

$$G^{(1)}(t,t+\tau\mid\{t_k\})=\mathscr{E}^{*}(t)\mathscr{E}(t+\tau)\ , \qquad (54)$$

$$G^{(2)}(t,t+\tau\mid\{t_k\})=\mid\mathscr{E}(t)\mid^{2}\mid\mathscr{E}(t+\tau)\mid^{2}\ , \qquad (55)$$

where

$$\mathscr{E}(t)=\sum_k \epsilon_0(t-t_k)\ , \qquad (56)$$

$$\epsilon_0(t)=\sum_l \alpha_l c_l(t)\ . \qquad (57)$$

Equation (56) has the same form as Eq. (1); the result of averaging Eqs. (54) and (55), as specified in Eq. (53), should therefore be identical to the classical results [See Eqs. (5), (6), and (8)–(11)].

B. Thermal state

Let the system be in a thermal (chaotic) state described in terms of the P representation[3,4] as

$$\hat{\rho}=\int P(\{\alpha_{k,l}\})\mid\{\alpha_{k,l}\}\rangle\langle\{\alpha_{k,l}\}\mid\prod_{k,l}d^2\alpha_{k,l} \qquad (58)$$

with

$$P(\{\alpha_{k,l}\})=\prod_{k,l}(1/\pi\bar{n}_l)\exp(-\mid\alpha_{k,l}\mid^{2}/\bar{n}_l)\ . \qquad (59)$$

Note that the average occupation numbers $\bar{n}_l$ are independent of k (i.e., emissions are assumed to be statistically identical). This results in[4]

$$G^{(1)}(t,t+\tau\mid\{t_k\})=\sum_k R(t-t_k,\tau)\ , \qquad (60)$$

$$\begin{aligned}G^{(2)}(t,t+\tau\mid\{t_k\})&=G^{(1)}(t,t\mid\{t_k\})G^{(1)}(t+\tau,t+\tau\mid\{t_k\})+\mid G^{(1)}(t,t+\tau\mid\{t_k\})\mid^{2}\\&=\sum_k R(t-t_k,0)\sum_k R(t+\tau-t_k,0)+\left|\sum_k R(t-t_k,\tau)\right|^{2}\ ,\end{aligned} \qquad (61)$$

where

$$\begin{aligned}R(t,\tau)&=\mathrm{tr}[\hat{\rho}\hat{E}_k^{-}(t)\hat{E}_k^{+}(t+\tau)]\\&=\sum_l \bar{n}_l c_l^{*}(t)c_l(t+\tau)\ .\end{aligned} \qquad (62)$$

After averaging over $\{t_k\}$ by using the known properties of shot noise, we reproduce Eqs. (27)–(33), which were obtained in the semiclassical model; here $R(t,\tau)$ is given by Eq. (62). The function $R(t,\tau)$ can be thought of as the correlation function of a single emission. If a single emission is also a single mode, then the summations in Eqs. (50) and (62) collapse to one term, for example, l_0. The first-order correlation function [Eq. (28)] is then simply an autocorrelation of the amplitude

$$G^{(1)}(\tau)=\mu\bar{n}_{l_0}\int_{-\infty}^{\infty}c_{l_0}^{*}(t)c_{l_0}(t+\tau)dt\ . \qquad (63)$$

The functions $\eta(t)$ and $\xi(t)$, which determine the second-order optical correlation [Eq. (31)], are similarly determined in terms of $c_{l_0}(t)$.

C. Number state

Finally, we consider a state which does not have a classical analog—the number state (Fock state[6])

$$\hat{\rho}=|\{n_{k,l}\}\rangle\langle\{n_{k,l}\}| \ . \tag{64}$$

We further assume that $n_{k,l}=n_l$ (independent of k). Equation (51) gives

$$G^{(1)}(t,t+\tau\,|\,\{t_k\})=\sum_k R(t-t_k,\tau) \ , \tag{65}$$

where

$$R(t,\tau)=\sum_l n_l c_l^*(t)c_l(t+\tau) \ , \tag{66}$$

and Eq. (52) yields

$$G^{(2)}(t,t+\tau\,|\,\{t_k\})=\sum_k R(t-t_k,0)\sum_k R(t+\tau-t_k,0)+\left|\sum_k R(t-t_k,\tau)\right|^2-\sum_k R_1(t-t_k,\tau) \ , \tag{67}$$

where

$$R_1(t,\tau)=\sum_l (n_l^2+n_l)\,|c_l(t)c_l(t+\tau)|^2 \ . \tag{68}$$

When the Poisson times of occurrences are averaged out, we have

$$G^{(1)}(\tau)=\mu\int_{-\infty}^{\infty}R(t,\tau)dt \ , \tag{69}$$

$$G^{(2)}(\tau)=\mu^2\left[\int_{-\infty}^{\infty}R(t,0)dt\right]^2+\mu^2\left|\int_{-\infty}^{\infty}R(t,\tau)dt\right|^2+\mu\int_{-\infty}^{\infty}R(t,0)R(t+\tau,0)dt$$
$$+\mu\int_{-\infty}^{\infty}|R(t,\tau)|^2dt-\mu\int_{-\infty}^{\infty}R_1(t,\tau)dt \ . \tag{70}$$

The normalized first- and second-order correlation functions are then

$$g^{(1)}(\tau)=\frac{\int_{-\infty}^{\infty}R(t,\tau)dt}{\int_{-\infty}^{\infty}R(t,0)dt} \ , \tag{71}$$

$$g^{(2)}(\tau)=1+|g^{(1)}(\tau)|^2+\frac{1}{\mu\tau_p}[\eta(\tau)+\xi(\tau)-\zeta(\tau)] \ , \tag{72}$$

where $\eta(\tau)$, $\xi(\tau)$, and τ_p are determined from $R(t,\tau)$ by the use of Eqs. (32), (33), and where

$$\zeta(\tau)=\frac{\int_{-\infty}^{\infty}R_1(t,\tau)dt}{\int_{-\infty}^{\infty}|R(t,0)|^2dt} \ . \tag{73}$$

It is interesting to observe that $g^{(2)}(\tau)$ for the shot-noise n state, Eq. (72), is identical to $g^{(2)}(\tau)$ for shot-noise thermal light, Eq. (31), except for the fifth term in Eq. (72), which is negative. For $\tau=0$,

$$g^{(2)}(0)=2+\frac{1}{\mu\tau_p}[2-\zeta(0)] \ , \tag{74}$$

where $\zeta(0)$ can be determined from Eqs. (73), (68), and (66). For the example in which the individual emissions are single modes $l=l_0$, $\zeta(0)=1+1/n_{l0}$, and

$$g^{(2)}(0)=2+\frac{1}{\mu\tau_p}(1-1/n_{l0}) \ . \tag{75}$$

This is to be contrasted with the result $g^{(2)}(0)=1-1/n_{l0}$ for simple n-state light with $n=n_{l0}$.

In Sec. II we demonstrated that coherent pulses of light [for each of which $g^{(2)}(\tau)=1$] emitted at random (Poisson) times, in the limit of dense emission, manifest chaotic behavior for which

$$g^{(2)}(\tau)=1+|g^{(1)}(\tau)|^2 \ .$$

Here, too Eq. (72) shows that antibunched pulses of light [for each of which $g^{(2)}(\tau)<1$], in the same limit ($\mu\tau_p \gg 1$), manifest identical chaotic behavior.

D. Comparison of results

We conclude that in the limit of dense emissions at random times, we cannot distinguish between thermal, coherent, and antibunched n-state light by means of the second-order correlation function. The shot-noise fluctuations are such that all exhibit chaotic behavior in this limit. This result expresses a fundamental difficulty in attempts to generate antibunched light by excitations at random times, as pointed out by Jakeman *et al.*[14]

In the opposite limit, however, when pulses seldom overlap ($\mu\tau_p \ll 1$), the intensity correlation functions for the three states are given by

$$g^{(2)}(\tau)\approx\frac{1}{\mu\tau_p}[\eta(\tau)],\ \text{coherent}$$
$$\approx\frac{1}{\mu\tau_p}[\eta(\tau)+\xi(\tau)],\ \text{thermal} \quad (76)$$
$$\approx\frac{1}{\mu\tau_p}[\eta(\tau)+\xi(\tau)-\zeta(\tau)]\ ,\ \text{number}\ .$$

The differences are obviously measurable. In particular, for $\tau=0$,

$$g^{(2)}(0)\approx\frac{1}{\mu\tau_p},\ \text{coherent}$$
$$\approx\frac{2}{\mu\tau_p},\ \text{thermal} \quad (77)$$
$$\approx\frac{2}{\mu\tau_p}[1-\zeta(0)/2],\ \text{number}\ .$$

IV. EXAMPLES OF THE GENERATION OF SHOT-NOISE LIGHT

A. Radiation by a classical shot-noise current distribution

Consider a current distribution $j(t)$ radiating a field described by the state $|\Psi(t)\rangle$. It is known[4,30] that if $|\Psi(0)\rangle$ is the vacuum state, then at time t the field is in a coherent state

$$|\Psi(t)\rangle=e^{i\Phi(t)}|\{\alpha_m(t)\}\rangle\ , \quad (78)$$

where $\alpha_m(t)$ is a linear combination of $j(t')$, $t'=[0,t]$,

$$\alpha_m(t)=\int_0^t j(t')\gamma_m(t')dt'\ , \quad (79)$$

$\gamma_m(t)$ and $\Phi(t)$ being some functions of time.[4,30] If the current distribution is stochastic, then the radiated field is no longer coherent. In particular, if $j(t)$ is the shot-noise process

$$j(t)=\sum_k j_0(t-t_k)\ , \quad (80)$$

where the $\{t_k\}$ are random times, and $j_0(t)$ is a deterministic function,

$$\alpha_m(t)=\sum_k \alpha_{m0}(t-t_k)\ , \quad (81)$$

where

$$\alpha_{m0}(t)=\int_0^t j_0(t')\gamma_m(t')dt'\ . \quad (82)$$

If the radiated field is written in the usual form

$$\hat{E}^{+}(t)=i\sum_m(\hbar\omega_m/2)^{1/2}\hat{a}_m(t)e^{-i\omega_m t}\ , \quad (83)$$

then the conditional first- and second-order correlation functions, given a realization of times $\{t_k\}$, may be cast in the form of Eqs. (54) and (55) with

$$\epsilon_0(t)=i\sum(\hbar\omega_m/2)^{1/2}\alpha_{m0}(t)e^{-i\omega_m t}\ . \quad (84)$$

The result is shot-noise coherent light which has been studied in Secs. II and III, and whose correlation functions are given by Eqs. (6) and (8).

B. Interaction between radiation and matter in the presence of a classical shot-noise driving force

In this example we consider fluctuations of radiation from an atomic system driven by a classical force which fluctuates in accordance with a shot-noise process. To begin, we assume that the optical field is a single mode, harmonic oscillator of frequency ω_a. The field is coupled to a radiator, a single atom which is also represented by a harmonic oscillator, of frequency ω_b. The atom is excited by a driving force $F(t)$ which is a random sequence of impulses following a Poisson point process. The driving force results in random excitations of the atom, and hence the radiation of random flashes of light.

The Hamiltonian of the overall system may be written as a sum

$$\hat{H}=\omega_a\hat{a}^{\dagger}\hat{a}+\omega_b\hat{b}^{\dagger}\hat{b}+\rho(\hat{a}^{\dagger}\hat{b}+\hat{a}\hat{b}^{\dagger})$$
$$+g(\hat{b}+\hat{b}^{\dagger})F(t) \quad (85)$$

of the Hamiltonians of the field and atom ($\hat{a}$ and $\hat{b}$ being their lowering operators, respectively), a Hamiltonian of atom-field interaction in the dipole approximation (ρ being the coefficient of coupling), and a Hamiltonian for the interaction between the driving force and atom (g being the coefficient of coupling). The driving force is treated classically, and can be thought of as a sequence of exciting particles which impinge on the atom at random.

The dynamics of the system can be determined by solving the Heisenberg equations for the operators $\hat{a}(t)$ and $\hat{b}(t)$. We have undertaken this by applying a transformation which decouples the system, solving the decoupled equations, and transforming back to the original operators. The derivation is presented in Appendix A. The results are

$$\hat{a}(t)=D(t)\hat{a}+C(t)\hat{b}+\psi(t)\ , \tag{86}$$

where $D(t)$ and $C(t)$ are the time-decaying functions given in Appendix A, and $\psi(t)$ is related to the driving force by

$$\psi(t)=ig\int_0^t F(t')C(t-t')dt'\ . \tag{87}$$

If the driving force is a random process of Poisson impulses

$$F(t)=\sum_k \delta(t-t_k)\ , \tag{88}$$

then the function $\psi(t)$ is a shot-noise process[19]

$$\psi(t)=ig\sum_k C(t-t_k)\ . \tag{89}$$

We can now write the positive-frequency part of the optical field as

$$\hat{E}^+(t)\propto D(t)\hat{a}+C(t)\hat{b}+\psi(t)\ . \tag{90}$$

This is the field radiated when a sequence of impulses drives a *single* atom interacting with a single field mode.

If we assume that we have, instead, a large number of atoms and that each incoming impulse interacts with a different atom, we can rewrite Eq. (90) in the form

$$\hat{E}^+(t)\propto D(t)\hat{a}+\sum_k C(t-t_k)\hat{b}_k+\psi(t)\ , \tag{91}$$

where $\hat{b}_k$ is the lowering operator of the harmonic oscillator representing the kth atom.

We can, furthermore, allow the field to be multimode by writing

$$\hat{E}^+(t)\propto \sum_l D_l(t)\hat{a}_l+\sum_{k,l} C_l(t-t_k)\hat{b}_{k,l}+\psi(t)\ , \tag{92}$$

$$\psi(t)=i\sum_{k,l} g_l C_l(t-t_k)\ . \tag{93}$$

Here $\hat{a}_l$ represents the lowering operator of the lth field mode and $\hat{b}_{k,l}$ the lowering operator of an atom which is driven by the kth impulse to interact with the lth field mode. The functions $C_l(t)$ and $D_l(t)$ are obtained from $C(t)$ and $D(t)$ by replacing ω_a and ω_b with ω_{al} and ω_{bl} and by multiplication with the normalization constants $\sqrt{\omega_{al}}$ and $\sqrt{\omega_{bl}}$, respectively. We readily observe that the second term of $\hat{E}^+(t)$ in Eq. (92) is identical to the field described by Eqs. (49) and (50), except that $\hat{b}_{k,l}$ replaces $\hat{a}_{k,l}$. We note that if the initial state of the field is the vacuum state, the first term of Eq. (92) does not contribute to the first- and second-order correlation functions, and we can ignore it. To examine the effect of the third term, we consider two examples.

1. Coherent emissions

Let the field modes and atoms initially be in coherent states $|\{\alpha_{al}\}\rangle$ and $|\{\alpha_{bk,l}\}\rangle$, respectively. Conditioned on a given driving force [deterministic $\psi(t)$], it can be shown (see Appendix B) that the field remains in a coherent state.

In the special case in which $\alpha_{bk,l}=0$, $\alpha_{al}=0$, i.e., at $t=0$, neither the field nor the atoms are excited and we obtain the first- and second-order conditional correlation functions of Eqs. (54) and (55) with $\mathscr{E}(t)=\psi(t)$. Averaging over the times $\{t_k\}$ results in correlation functions reproducing Eqs. (5)–(11), with $\epsilon_0(t)=\sum_l c_l(t)$.

2. Mixture of coherent and thermal emissions

In this example we assume that at $t=0$ the field is in its ground state, while the atom is in a thermal state with a density operator as in Eqs. (58) and (59). Through the use of Eq. (91), we can determine the conditional correlations

$$G^{(1)}(t,t+\tau\,|\,\{t_k\})=\sum_k R(t-t_k,\tau)+\psi^*(t)\psi(t+\tau)\ , \tag{94}$$

$$G^{(2)}(t,t+\tau\,|\,\{t_k\})=\sum_k R(t-t_k,0)\sum_k R(t+\tau-t_k,0)+\left|\sum_k R(t-t_k,\tau)\right|^2+|\psi(t)|^2|\psi(t+\tau)|^2+2\,\mathrm{Re}\left[\psi(t)\psi^*(t+\tau)\sum_k R(t-t_k)\right]\ , \tag{95}$$

where $R(t,\tau)$ is given by Eq. (62).

Averaging over the fluctuations of the times $\{t_k\}$, and using Eqs. (92) and (62), we finally obtain

$$G^{(1)}(\tau)=\mu\int_{-\infty}^{\infty}R(t,\tau)dt+\mu\int_{-\infty}^{\infty}\epsilon_0^*(t)\epsilon_0(t+\tau)dt\ , \tag{96}$$

$$\begin{aligned}G^{(2)}(\tau)=\mu^2\Bigg[&\left[\int_{-\infty}^{\infty}R(t,0)dt\right]^2+\int_{-\infty}^{\infty}|R(t,\tau)|^2dt+\left[\int_{-\infty}^{\infty}h(t)dt\right]^2+\left|\int_{-\infty}^{\infty}\epsilon_0^*(t)\epsilon_0(t+\tau)dt\right|^2\\&+2\,\mathrm{Re}\left[\int_{-\infty}^{\infty}\epsilon_0(t)\epsilon_0^*(t+\tau)dt\int_{-\infty}^{\infty}R(t,0)dt\right]\Bigg]\\&+\mu\left[\int_{-\infty}^{\infty}R(t,0)R(t+\tau,0)dt+\int_{-\infty}^{\infty}|R(t,\tau)|^2dt+\int_{-\infty}^{\infty}h(t)h(t+\tau)dt\right.\\&\left.+\int_{-\infty}^{\infty}\epsilon_0^*(t+\tau)\epsilon_0(t)R(t,0)dt\right],\end{aligned} \tag{97}$$

where

$$\epsilon_0(t)=-i\sum_l g_l c_l(t),\quad h(t)=|\epsilon_0(t)|^2\ , \tag{98}$$

and, as before,

$$R(t,\tau)=\sum_l \bar{n}_l c_l^*(t)c_l(t+\tau)\ . \tag{99}$$

The first-order correlation function is the sum of contributions of a thermal part and a coherent part. By examining Eqs. (98) and (99), we see that the thermal part dominates if the average occupation number of the atoms $\bar{n}_l$ is much greater than $|g_l|^2$, the coefficient of coupling between the driving force and the atoms. In this case, we have a thermal field with underlying shot-noise fluctuations as discussed in Sec. III B. The second-order correlation function is given by Eq. (97). It is the sum of two contributions. The first dominates for dense emission (large μ). It appears to be the sum of two interacting effects. The underlying shot noise has "thermalized" the originally coherent component. The second term of Eq. (97), which contains components proportional to μ, dominates at low-density emission. It contains contributions from the two fields as well as from an interference term.

V. CONCLUSION

We have investigated the coherence properties and photon statistics of light generated by a random number of radiators emitting at random times, in accordance with a homogeneous Poisson point process. We have shown that such underlying randomness of the emission times imparts additional fluctuations to the radiated field. When the optical field at a given time is a result of contributions from a large average number of radiators, the light becomes chaotic, whether the individual emissions are themselves coherent or chaotic. This is a consequence of the central limit theorem. When the average number of radiators contributing to the optical field at a certain time is not large, the deviations from chaotic behavior have been calculated. These deviations are characterized by an increase in the normalized second-order correlation function, which corresponds to an increase in the variance of the number of photons counted in a fixed time interval, and to additional photon bunching.

The excess variance is proportional to the mean number of photons, indicating that these excess fluctuations are particlelike in nature. Furthermore, we have shown that the corresponding excess fluctuations of the number of photons in a given time interval are enhanced by an increase of the time interval. This is unlike the excess noise due to wave fluctuations, which are known to be averaged by an increase in the counting time.

It has also been shown that, when the lifetime of the individual emissions is so short, or when their rate is so low, that overlap is unlikely, no interemission interference takes place, and the light intensity is described by a shot-noise stochastic process. The photons are then described by the shot-noise-driven doubly stochastic Poisson point process, which also exhibits excess bunching of a particlelike nature. If counted over a counting time longer than the lifetime of a single emission, the excess fluctuations due to random emission times exhibit themselves fully. In this case, when each single emission is coherent, and when it is chaotic, the photons are described by the Neyman type-A and the generalized Polya-Aeppli counting distributions, respectively.

We have also formulated a general quantum-electrodynamic model for an optical field generated by a sequence of emissions at random times. This enabled us to examine the effect of random emission times in cases when the emissions cannot be described classically. For example, when the individual emissions are in the highly antibunched Fock state, we find that the randomness of the emission times results in particlelike bunching which quickly overpowers the inherent quantum-mechanical antibunching, as the rate of emissions increases and the emissions overlap. In the limit of a large number of overlapping emissions, the second-order correlation function eventually exhibits the usual (bunched) chaotic behavior. The important role played by the randomness in the number of radiators for the observation of antibunched light in resonance fluorescence is well recognized.[14]

While the analysis presented in this paper has been limited to the more common statistical models which individual emissions may satisfy (coherent, chaotic, coherent-chaotic mixture), other statistical models may be analyzed through the use of the same methods.

Spatial effects have also not been considered here. These effects may be included by assigning positions to the radiators, and by making the radiated field a function of position as well as time. By assuming that the positions of the radiators in the source volume, and their emission times, are random points in four-dimensional time-position space, in accordance with a 4D Poisson point process, we may proceed to determine the temporal and spatial coherence properties of the radiated field through the use of generalizations of the methods employed in this paper.

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APPENDIX A: SOLUTION OF THE HEISENBERG EQUATIONS OF MOTION

In this appendix, we outline the solution of the Heisenberg equation of motion for the field operator $\hat{a}(t)$, for the system described by the Hamiltonian in Eq. (85). We do the calculation in two stages. First we ignore the external driving force $F(t)$ that is coupled to the atomic oscillator in Eq. (85). We write the Hamiltonian as

$$\hat{H}=\hat{H}_0+\hat{H}_1 \ , \tag{A1}$$

where

$$\hat{H}_0=\omega_a\hat{a}^\dagger\hat{a}+\omega_b\hat{b}^\dagger\hat{b}+\rho(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger) \tag{A2}$$

and

$$\hat{H}_1=g(\hat{b}+\hat{b}^\dagger)F(t) \ . \tag{A3}$$

We shall determine the two normal modes of the Hamiltonian $\hat{H}_0$. Both of these normal modes will be driven by the external force $F(t)$.

Defining a pair of lowering operators, $\hat{a}'$ and $\hat{b}'$, by means of the relation

$$\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix}\hat{a}'\\ \hat{b}'\end{bmatrix} \tag{A4}$$

and substituting Eq. (A4) into $\hat{H}_0$, we determine the quantities $\alpha,\beta,\gamma,\delta$, such that $\hat{H}_0$ is uncoupled in the new variables $\hat{a}',\hat{b}'$. The condition for decoupling is given by

$$\omega_b\gamma^*\delta+\omega_a\alpha^*\beta+\rho(\alpha^*\delta+\beta\gamma^*)=0 \ . \tag{A5}$$

The necessity for the commutation relations among the various operators $\hat{a},\hat{a}^\dagger,\hat{b},\hat{b}^\dagger$ to be preserved, i.e., to be satisfied also by the operators, $\hat{a}',\hat{a}'^\dagger,\hat{b}',\hat{b}'^\dagger$, leads to the constraints

$$|\alpha|^2+|\beta|^2=1 \ , \quad |\gamma|^2+|\delta|^2=1 \ , \tag{A6}$$

$$\alpha^*\gamma+\beta^*\delta=0 \ . \tag{A7}$$

In terms of the primed operators, $\hat{H}_0$ becomes

$$\hat{H}_0=\Omega_a\hat{a}'^\dagger\hat{a}'+\Omega_b\hat{b}'^\dagger\hat{b}' \ , \tag{A8}$$

where

$$\Omega_a=\omega_b|\gamma|^2+\omega_a|\alpha|^2+2\rho\,\mathrm{Re}(\alpha^*\gamma) \ , \quad \Omega_b=\omega_b|\delta|^2+\omega_a|\beta|^2+2\rho\,\mathrm{Re}(\beta^*\delta) \ . \tag{A9}$$

The external driver part of the Hamiltonian becomes

$$\hat{H}_1=[g(\gamma\hat{a}'+\gamma^*\hat{a}'^\dagger)+g(\delta\hat{b}'+\delta^*\hat{b}'^\dagger)]F(t) \ . \tag{A10}$$

Therefore, both normal modes are driven by the external force, as previously stated. The parameters $\alpha,\beta,\gamma,\delta$ may be found by solving Eqs. (A5)–(A7). We can satisfy Eqs. (A6) and (A7) identically with the choice

$$\alpha=\cos\theta \ , \quad \beta=\sin\theta \ , \quad \gamma=-\sin\theta\, e^{iu} \ , \quad \delta=\cos\theta e^{iu} \ . \tag{A11}$$

Substituting this into Eq. (A5), and recalling that we take ω_a to be real and $\omega_b=\omega_{b,0}-i\Gamma$, we get two equations that determine θ and u,

$$\tan(2\theta)=2\rho\cos(u)/(\omega_{b,0}-\omega_a)\ ,\qquad \sin(2\theta)=-2\rho\sin(u)/\Gamma\ . \tag{A12}$$

We solve the dynamical problem in terms of the primed operators, and transform the solution back to the original variables. The Heisenberg equation of motion for the operator $\hat{a}'(t)$ is

$$i\frac{d\hat{a}'}{dt}=[\hat{a}',\hat{H}]=\Omega_a\hat{a}'+g\gamma^*F(t)\ . \tag{A13}$$

This is the equation for a forced harmonic oscillator whose solution is easily arrived at and is given by

$$\hat{a}'(t)=[\hat{a}'(0)+\phi_a(t)]e^{-i\Omega_a t}\ ,$$

where (A14)

$$\phi_a(t)=-ig\gamma^*\int_0^t F(t')e^{i\Omega_a t'}dt'\ .$$

The equation for $\hat{b}'(t)$, and its solution, follow in similar fashion. The result is

$$\hat{b}'(t)=[\hat{b}'(0)+\phi_b(t)]e^{-i\Omega_b t}$$

with (A15)

$$\phi_b(t)=-ig\delta^*\int_0^t F(t')e^{i\Omega_b t'}dt'\ .$$

Using Eq. (A4), and its inverse, as well as Eqs. (A11), (A14), and (A15), we finally obtain Eq. (86),

$$\hat{a}(t)=D(t)\hat{a}+C(t)\hat{b}+\psi(t)\ , \tag{A16}$$

where

$$D(t)=e^{-i\Omega_a t}\cos^2\theta+e^{-i\Omega_b t}\sin^2\theta\ , \tag{A17}$$

$$C(t)=\sin\theta\cos\theta\, e^{-iu}(e^{-i\Omega_b t}-e^{-i\Omega_a t})\ , \tag{A18}$$

$$\psi(t)=\phi_a(t)\cos\theta\, e^{-i\Omega_a t}+\phi_b(t)\sin\theta\, e^{-i\Omega_b t}\ . \tag{A19}$$

Substituting from Eqs. (A14) and (A15) in Eq. (A19) and using Eqs. (A11) and (A18), we obtain Eq. (87).

APPENDIX B: PERMANENCE OF THE COHERENT STATE

We prove here that if the field and atom of the quantum system described by the Hamiltonian of Eq. (85) are in coherent states initially, then they will remain in coherent states (not necessarily the same ones) at all later times. This result rests on the fact that the relation between the operators $(\hat{a}',\hat{b}')$, defined in Appendix A, and $(\hat{a},\hat{b})$ is a linear one. This leads to the result that the ground state for the $(\hat{a}',\hat{b}')$ modes, i.e., $|0\rangle_{a'}\otimes|0\rangle_{b'}$, is also the ground state for the $(\hat{a},\hat{b})$ modes.

The coherent states can be written in the form of the Weyl operator $D_{\hat{a}}(\alpha)=\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})$ acting on the ground state $|0\rangle_a$. Since the $(\hat{a},\hat{b})$ modes, and the $(\hat{a}',\hat{b}')$ modes, share a common ground state and since the generator of the Weyl operator is linear in the raising and lowering operators, it follows that the product of two Weyl operators in the $(\hat{a}',\hat{b}')$ basis is equal to a product of two Weyl operators in the $(\hat{a},\hat{b})$ basis, i.e.,

$$D_{\hat{a}'}(z_{a'})D_{\hat{b}'}(z_{b'})=D_{\hat{a}}(\mu_a)D_{\hat{b}}(\mu_b)\ , \tag{B1}$$

where

$$z_{a'}=\alpha^*\mu_a+\gamma^*\mu_b\ ,\qquad z_{b'}=\beta^*\mu_a+\delta^*\mu_b\ . \tag{B2}$$

From Eq. (B1) we see that the initial coherent state of the $(\hat{a},\hat{b})$ modes implies that the $(\hat{a}',\hat{b}')$ modes are also initially in a coherent state. It only remains to show that the Hamiltonian $\hat{H}=\hat{H}_0+\hat{H}_1$ given in Eq. (A1) will cause the initially coherent state to evolve into states that are also coherent. This need not be done explicitly here because $\hat{H}$, when written in terms of $(\hat{a}',\hat{b}')$, is of the form that is known to preserve the coherence of a coherent initial state. The only effect of such Hamiltonians on coherent states is to cause the complex amplitude of the state to evolve in time. In fact, it evolves in time the way the complex amplitude of the corresponding classical system does. A discussion of this class of Hamiltonians and related issues has been presented elsewhere.[31]

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